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Retraction

Retracted: Quasilinear Inner Product Spaces and Hilbert Quasilinear Spaces

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At the request of the authors, the article titled "Quasilinear Inner Product Spaces and Hilbert Quasilinear Spaces" [1] has been retracted. The article was found by the authors to contain a number of mathematical errors, undefined expressions, and spelling errors, which mean that the conclusions cannot be relied upon.

References

[1] H. Bozkurt, S. Çakan, and Y. Yılmaz, "Quasilinear inner product spaces and Hilbert quasilinear spaces," *International Journal of Analysis*, vol. 2014, Article ID 258389, 7 pages, 2014.

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Research Article

Quasilinear Inner Product Spaces and Hilbert Quasilinear Spaces

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Aseev launched a new branch of functional analysis by introducing the theory of quasilinear spaces in the framework of the topics of norm, bounded quasilinear operators and functionals (Aseev (1986)). Furthermore, some quasilinear counterparts of classical nonlinear analysis that lead to such result as Frechet derivative and its applications were examined deal with. This pioneering work causes a lot of results in such applications such as (Rojas-Medar et al. (2005), Talo and Başar (2010), and Nikol'skiĭ (1993)). His work has motivated us to introduce the concept of quasilinear inner product spaces. Thanks to this new notion, we obtain some new theorems and definitions which are quasilinear counterparts of fundamental definitions and theorems in linear functional analysis. We claim that some new results related to this concept provide an important contribution to the improvement of quasilinear functional analysis.

1. Introduction

The theory of quasilinear space was introduced by Aseev [1]. He proceeds, in a similar way to linear functional analysis on quasilinear spaces by introducing the notions of norm, with quasilinear operators and functionals. We can see in [1] that, as different from linear spaces, Aseev used the partial order relation when he defined quasilinear spaces and so he can give consistent counterparts of results in linear spaces. Further we note that the norm defined in quasilinear space is compatible with the concept of norm on linear space and if each element of normed quasilinear space has an inverse, then the partial order is determined by equality. Consequently, the concept of normed quasilinear spaces coincides with the concept of normed space in classical analysis.

As known, the theory of inner product space and Hilbert spaces play a fundamental role in functional analysis and its applications such as integral and differential equations, approximation theory, linear and nonlinear stability problems, and bifurcation theory.

We know that any inner product space is a normed space and any normed space is a particular class of normed quasilinear space. Hence, this relation and Aseev's work motivated us to examine quasilinear counterparts of inner product space in classical analysis. Thus, we introduce the concept of quasilinear inner product space as a new structure. Moreover, we obtain some definitions and results related to this notion which provide us with improving the elements of the quasilinear functional analysis.

The definition of quasi-inner product function is extended by classical definition of inner product function. It is normal to expect that inner product which is defined by quasilinear space is supposed to be given by means of a partial order relation, just as in the method defining quasilinear normed spaces. Then we clearly observe from these definitions that the concept of quasilinear inner product space is a generalization of inner product space. While working on this new concept, we noticed that there were some differences related to analysis as different from classical case. For example, the convergence inproof ofcontinuity of quasi-inner product function according to the Hausdorff metric of the quasilinear space leads to slight differences in details of the proof.

In another important part of this work, we introduce the notion of Hilbert quasilinear space, inner product Ω-space, and Hilbert Ω-space. Note that one of the most important consequences of having the quasilinear inner product is the possibility of defining orthogonality of elements of quasilinear space. This makes the theory of Hilbert quasilinear spaces very different from the general theory of Banach quasilinear spaces.

In this paper we aim to give a contribution to the studies on quasilinear spaces by introducing the notion of quasilinear inner product spaces.

2. Preliminaries and Some Results on Quasilinear Spaces

Let us start this section by introducing the definition of quasilinear spaces and some of their basic properties given by Aseev [1].

Aseev proceeds in a similar way to linear functional analysis on quasilinear spaces by introducing the notions of the norm and quasilinear operators and functionals. Further, he presented some results which are quasilinear counterparts of fundamental definitions and theorems in linear functional analysis.

A set X is called a *quasilinear space* (*QLS*, for short), if a partial order relation "$\leq$," an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such way that the following conditions hold for any elements $x, y, z, v \in X$ and any real numbers $\alpha, \beta \in E^1$:

(1) $x \leq x$,

(2) $x \leq z$ if $x \leq y$ and $y \leq z$,

(3) $x = y$ if $x \leq y$ and $y \leq x$,

(4) $x + y = y + x$,

(5) $x + (y + z) = (x + y) + z$,

(6) There exists an element $\theta \in X$ such that $x + \theta = x$,

(7) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$,

(8) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$,

(9) $1 \cdot x = x$,

(10) $0 \cdot x = \theta$,

(11) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$,

(12) $x + z \leq y + v$ if $x \leq y$ and $z \leq v$,

(13) $\alpha \cdot x \leq \alpha \cdot y$ if $x \leq y$.

A linear space is a QLS with the partial order relation "=." Perhaps the most popular example which is not a linear space is the set of all closed intervals of real numbers with the inclusion relation "$\subseteq$," algebraic sum operation

$$A + B = \{a + b : a \in A, b \in B\} \tag{1}$$

and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}. \tag{2}$$

We denote this set by $K_C(\mathbb{R})$. Another one is $K(\mathbb{R})$, the set of all compact subsets of real numbers with a slight modification of algebraic sum operation such as

$$A + B = \overline{\{a + b : a \in A, b \in B\}} \tag{3}$$

and the same real-scalar multiplication above and the inclusion relation again. In general, $K_C(E)$ and $K(E)$ stand for the space of all nonempty closed bounded and convex and nonempty closed bounded subsets of some normed linear space E, respectively. Both are QLSs (which is not a linear space) with the inclusion and with a generalization of corresponding operations defined for $K_C(\mathbb{R})$ and $K(\mathbb{R})$. Hence $K_C(E) = \{A \subseteq K(E) : A \textit{ convex}\}$.

An element $x' \in X$ is called an *inverse* of an $x \in X$ if $x + x' = \theta$. If an inverseelement exists, then it is unique. An element x having an inverse is called *regular*; otherwise it is called *singular*.

Lemma 1 (see [1]). *Suppose that any element x in the QLS X has an inverse element $x' \in X$. Then the partial order in X is determined by equality; the distributivity conditions hold, and, consequently, X is a linear space.*

In a real linear space, equality is the only way to define a partial order such that conditions (1)–(13) hold.

It will be assumed in what follows that $-x = (-1) \cdot x$.

Suppose that X is a QLS and $Y \subseteq X$. Y is called a *subspace of a quasilinear space* X if Y is a quasilinear space with the restriction of the partial ordering and the restriction of the operations on X. Y is subspace of a quasilinear space X if and only if for every $x, y \in Y$ and $\alpha \in \mathbb{R}$,

(i) There exists an element $z \in Y$ such that $z \leq x + y$,

(ii) $\alpha x \in Y$.

We note that if $x + y \in Y$ for all $x, y \in Y$, then the condition (i) holds.

Lemma 2 (see [1]). *Let X be a QLS and Y be a subspace of X. Suppose that each element x in Y has an inverse element $x' \in Y$ then the partial order on Y is determined by the equality. In this case the distributivity conditions hold on Y and Y is a linear subspace of X.*

Let X be a QLS. An $x \in X$ is said to be *symmetric* if $(-1) \cdot x = x$, and X_b denotes the set of all such elements. Further, X_r and X_s stand for the sets of all regular and singular elements in X, respectively. X_r, X_b, and $X_s \cup \{0\}$ are subspaces of X. X_r, X_b, and $X_s \cup \{0\}$ are called *regular, symmetric*, and *singular subspaces* of X, respectively [2].

Definition 3 (see [1]). Let X be a quasilinear space. A real function $\|\cdot\|_X : X \to E^1$ is called a *norm* if the following conditions hold:

(i) $\|x\|_X > 0$ if $x \neq 0$,

(ii) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$,

(iii) $\|\alpha \cdot x\|_X = |\alpha| \cdot \|x\|_X$,

(iv) if $x \leq y$, then $\|x\|_X \leq \|y\|_X$,

(v) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that, $x \leq y + x_\varepsilon$ and $\|x_\varepsilon\|_X \leq \varepsilon$ then $x \leq y$.

A quasilinear space X with a norm defined on it is called *normed quasilinear* space. It follows from Lemma 1 that if any

$x \in X$ has an inverse element $x' \in X$, then the concept of a normed quasilinear space coincides with the concept of a real normed linear space.

Let X be a normed quasilinear space. Hausdorff metric on X is defined by the equality

$$h_X(x, y) = \inf\{r \geq 0 : x \leq y + a_1^r, y \leq x + a_2^r, \|a_i^r\| \leq r\}. \quad (4)$$

Since $x \leq y + (x - y)$ and $y \leq x + (y - x)$, the quantity $h_X(x, y)$ is well-defined for any elements $x, y \in X$, and

$$h_X(x, y) \leq \|x - y\|_X. \quad (5)$$

It is not hard to see that this function $h_X(x, y)$ satisfies all of the metric axioms.

Lemma 4 (see [1]). *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is continuous function with respect to the Hausdorff metric.*

Lemma 5 (see [1]). *(i) Suppose that $x_n \to x_0$ and $y_n \to y_0$, and that $x_n \leq y_n$ for any positive integer n. Then $x_0 \leq y_0$.*

(ii) Suppose that $x_n \to x_0$ and $z_n \to x_0$. If $x_n \leq y_n \leq z_n$ for any n, then $y_n \to x_0$.

(iii) Suppose that $x_n + y_n \to x_0$ and $y_n \to \theta$; then $x_n \to x_0$.

Let X be a real complete normed linear space (a real Banach space). Then X is a complete normed quasilinear space with partial order given by equality. Conversely, if X is a complete normed quasilinear space and any $x \in X$ has an inverse element $x' \in X$, then X is a real Banach space, and the partial order on X is equality. In this case $h_x(x, y) = \|x - y\|_X$.

Example 6 (see [1]). Let E be a Banach space. A norm on $K(E)$ is defined by

$$\|A\|_{K(E)} = \sup_{a \in A} \|a\|_E. \quad (6)$$

Then $K(E)$ and $K_c(E)$ are normed quasilinear spaces. In this case the Hausdorff metric is defined as usual:

$$h_{K(E)}(A, B) = \inf\{r \geq 0 : A \subset B + S_r(\theta), B \subset A + S_r(\theta)\}, \quad (7)$$

where $S_r(\theta)$ denotes a closed ball of radius r about $\theta \in X$.

Definition 7 (see [1]). A normed quasilinear space X is called an Ω-space, if there exists an element $B_X \neq \theta$ such that

$$\text{if } \|x\| \leq \|B_X\| \quad \text{then } x \leq B_X. \quad (8)$$

Example 8 (see [1]). (a) If E is a Banach space then $K(E)$ is an Ω-space.

(b) Suppose that S is a compact topological space and X is a complete Ω-space. Denote by $C(S, X)$ the space of all continuous mappings $f : S \to X$. $C(S, X)$ is a normed quasilinear space with the partial order relation

$$f_1 \leq f_2 \quad \text{if } f_1(s) \leq f_2(s) \text{ for any } s \in S \quad (9)$$

the algebraic sum operation

$$(f_1 + f_2)(s) = f_1(s) + f_2(s), \quad (10)$$

the operation of multiplication by real numbers

$$(\alpha f)(s) = \alpha f(s), \quad (11)$$

and the norm

$$\|f\| = \max_{s \in S} \|f(s)\|. \quad (12)$$

Further the normed quasilinear space $C(S, X)$ is an Ω-space.

Theorem 9 (see [1]). *The Ω-space $C(S, X)$ is complete.*

3. The Main Results

3.1. Quasilinear Inner Product Spaces

Definition 10. Let X be a quasilinear space. A mapping $\langle\, , \rangle : X \times X \to \mathbb{R}$ is called a quasi-inner product in X if for any $x, y, z \in X$ and $\alpha \in \mathbb{R}$ the following conditions are satisfied:

$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \quad (13)$$

$$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle, \quad (14)$$

$$\langle x, y\rangle = \langle y, x\rangle, \quad (15)$$

$$\langle x, x\rangle \geq 0, \qquad \langle x, x\rangle = 0 \Longleftrightarrow x = 0, \quad (16)$$

$$\text{if } x \leq y,\ u \leq v \quad \text{then } \langle x, u\rangle \leq \langle y, v\rangle, \quad (17)$$

$$\begin{aligned} &\text{if for any } \varepsilon > 0 \quad \text{there exists an element } x_\varepsilon \in X \\ &\text{such that } x \leq y + x_\varepsilon,\ \langle x_\varepsilon, x_\varepsilon\rangle \leq \varepsilon \quad \text{then } x \leq y. \end{aligned} \quad (18)$$

A quasilinear space with a quasi-inner product is called a quasilinear inner product space.

Example 11. $K_c(\mathbb{R})$ is a quasilinear inner product space with inner product function defined by

$$\langle A, B\rangle = \sup\{ab : a \in A, b \in B\}. \quad (19)$$

Indeed, since it is easy to verify that the conditions (13), (14), (15), and (16) we only prove that the conditions (17) and (18) hold.

Let $A \subseteq B, C \subseteq D$. Then we have (see [3])

$$\begin{gathered} AC \subseteq BD, \\ \sup\{ac : a \in A, c \in C\} \leq \sup\{bd : b \in B, d \in D\} \end{gathered} \quad (20)$$

for the elements $A, B, C, D \in K_c(\mathbb{R})$ and for all $a \in A$, $b \in B$, $c \in C$, $d \in D$. Hence, $\langle A, C\rangle \leq \langle B, D\rangle$.

Suppose that for any $\varepsilon > 0$ there exists $A, B \in K_c(\mathbb{R})$ and $A_\varepsilon \in K_c(\mathbb{R})$ such that

$$A \subseteq B + A_\varepsilon, \qquad \langle A_\varepsilon, A_\varepsilon\rangle \leq \varepsilon. \quad (21)$$

Suppose that $A \subsetneq B$. Then, there exists an element $a \in A$, but $a \notin B$. Since B is closed, the distance from the element a to the set B is

$$d(a, B) = \inf_{b \in B} \|a - b\| \neq 0. \tag{22}$$

By the hypothesis, for

$$\varepsilon = \left[\frac{d(a, B)}{2}\right]^2 \tag{23}$$

there exists an $A_\varepsilon \in K_c(\mathbb{R})$ such that $A \subseteq B + A_\varepsilon$ and $\langle A_\varepsilon, A_\varepsilon \rangle \leq \varepsilon$. Thus $a \in A$ and $a \in B + A_\varepsilon$. Then we have

$$a = b + a_\varepsilon \tag{24}$$

for $b \in B$ and $a_\varepsilon \in A_\varepsilon$. Hence

$$\begin{aligned} 0 = |a - (b + a_\varepsilon)| &\geq \left||a - b| - |a_\varepsilon|\right| \\ &\geq \left|d(a, B) - |a_\varepsilon|\right| \\ &\geq \left|d(a, B) - \frac{d(a, B)}{2}\right| \\ &= \left|\frac{d(a, B)}{2}\right| = \varepsilon^{1/2}. \end{aligned} \tag{25}$$

This is a contradiction. Thus $A \subseteq B$.

Proposition 12. *Every quasilinear inner product space X is a normed quasilinear space with the norm defined by*

$$\|x\| = \sqrt{\langle x, x \rangle} \tag{26}$$

for every $x \in X$.

Proof. Since it can be seen easily that the conditions (i)–(iii) hold we give only the proof of the conditions (iv) and (v).

$$\begin{aligned} x \leq y &\Longrightarrow \langle x, x \rangle \leq \langle y, y \rangle \\ &\Longrightarrow \sqrt{\langle x, x \rangle} \leq \sqrt{\langle y, y \rangle} \Longrightarrow \|x\| \leq \|y\|. \end{aligned} \tag{27}$$

Suppose that for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that

$$x \leq y + x_\varepsilon, \qquad \|x_\varepsilon\| \leq \varepsilon. \tag{28}$$

Since

$$\|x_\varepsilon\| = \sqrt{\langle x_\varepsilon, x_\varepsilon \rangle} \leq \varepsilon \Longrightarrow \langle x_\varepsilon, x_\varepsilon \rangle \leq \varepsilon^2 = \varepsilon' \tag{29}$$

and X is a quasilinear inner product space, we get $x \leq y$. □

It is routine to show that a norm on a quasilinear inner product space satisfies the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right). \tag{30}$$

We conclude that if a norm does not satisfy the parallelogram equality, it cannot be obtained from a quasilinear inner product by the use of (26). So, not all normed quasilinear spaces are quasilinear inner product spaces.

Example 13. $C([0,1], K_c(\mathbb{R}))$ is not a quasilinear inner product space. In fact, the norm defined by

$$\|x\| = \max_{s \in [0,1]} |x(s)| \tag{31}$$

cannot be obtained from an inner product since this norm does not satisfy the parallelogram equality. Indeed, if we take

$$\begin{aligned} x &: [0,1] \longrightarrow K_c(\mathbb{R}), \quad x(t) = [0,1], \\ y &: [0,1] \longrightarrow K_c(\mathbb{R}), \quad y(t) = [0,t] \end{aligned} \tag{32}$$

we have $\|x\| = 1$, $\|y\| = 1$ and

$$x(t) + y(t) = [0, 1 + t], \qquad x(t) - y(t) = [-t, 1]. \tag{33}$$

Hence $\|x + y\| = 2$, $\|x - y\| = 1$ and

$$\|x + y\|^2 + \|x - y\|^2 = 5 \tag{34}$$

but

$$2\left(\|x\|^2 + \|y\|^2\right) = 4. \tag{35}$$

This completes the proof.

Now let us define an inner product function on the quasilinear space $C([0,1], K_c(\mathbb{R}))$.

Example 14. $C([0,1], K_c(\mathbb{R}))$ is a quasilinear inner product space with inner product defined by

$$\begin{aligned} \langle x, y \rangle &= \int_0^1 \langle x(s), y(s) \rangle_{K_c(\mathbb{R})} ds \\ &= \int_0^1 \sup\{ab : a \in x(s), b \in y(s)\} ds \end{aligned} \tag{36}$$

for $x, y \in C([0,1], K_c(\mathbb{R}))$. Indeed, we suppose that for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in C([0,1], K_c(\mathbb{R}))$ such that

$$x \leq y + x_\varepsilon, \qquad \langle x_\varepsilon, x_\varepsilon \rangle \leq \varepsilon. \tag{37}$$

But $x \nleq y$. We have

$$\begin{aligned} \langle x_\varepsilon, x_\varepsilon \rangle &= \int_0^1 \langle x_\varepsilon(s), x_\varepsilon(s) \rangle_{K_c(\mathbb{R})} ds \leq \varepsilon \\ &\Longrightarrow \int_0^1 \sup\{ab : a \in x_\varepsilon(s), b \in x_\varepsilon(s)\} ds \leq \varepsilon \\ &\Longrightarrow \sup\{ab : a \in x_\varepsilon(s), b \in x_\varepsilon(s)\} \leq \varepsilon \\ &\Longrightarrow \langle x_\varepsilon(s), x_\varepsilon(s) \rangle \leq \varepsilon. \end{aligned} \tag{38}$$

Then $x(s_0) \subsetneq y(s_0)$, for an element $s_0 \in [0,1]$. Thus there exists an $a \in x_{s_0}$ such that $a \notin y(s_0)$. Since the set of $y(s_0)$ is closed, the distance from the element of a to the set of $y(s_0)$

$$d(a, y(s_0)) = \inf_{b \in y(s_0)} \|a - b\| \neq 0. \tag{39}$$

By the hypothesis, for

$$\varepsilon = \left[\frac{d(a, y(s_0))}{2}\right]^2 > 0 \tag{40}$$

there exists an $x_\varepsilon \in C([0,1], K_c(\mathbb{R}))$, such that $x \leq y + x_\varepsilon$ and $\langle x_\varepsilon, x_\varepsilon\rangle \leq \varepsilon$. So there exists $x_\varepsilon \in C([0,1], K_c(\mathbb{R}))$, such that

$$\begin{gathered} x(s) \subseteq y(s) + x_\varepsilon(s), \\ \int_0^1 \sup\{aa' : a, a' \in x(s)\}\, ds \leq \epsilon \end{gathered} \tag{41}$$

for every $s \in [0,1]$. Then $x(s_0) \subseteq y(s_0) + x_\varepsilon(s_0)$ for s_0 too. Since $a \in x(s_0)$, $a \in y(s_0) + x_\varepsilon(s_0)$. So we have

$$\begin{aligned} 0 &= |a - (b + a_\varepsilon)| \geq ||a - b| - |a_\varepsilon|| \\ &\geq |d(a, y(s_0)) - |a_\varepsilon|| \\ &\geq \left|d(a, y(s_0)) - \frac{d(a, y(s_0))}{2}\right| \\ &= \left|\frac{d(a, y(s_0))}{2}\right| = \varepsilon^{1/2} \end{aligned} \tag{42}$$

for $b \in y(s_0)$ and $a_\varepsilon \in x_\varepsilon(s_0)$. This is a contradiction. Eventually $x(s_0) \subseteq y(s_0)$. Hence $x \leq y$.

It can be easily seen that Schwartz inequality

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \tag{43}$$

and triangle inequality

$$\|x + y\| \leq \|x\| + \|y\| \tag{44}$$

hold in a quasilinear inner product space.

Remark 15. The proof of the following result is similar to its classical linear counterpart. But we note that the convergence of a sequence in a quasilinear space is different from the convergence of a sequence in a linear space. Here, convergence is according to the Hausdorff metric of quasilinear space. Because of this relation there are slight differences in details of proof here.

Proposition 16. *If in a quasilinear inner product space $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Proof. Since $x_n \to x$ and $y_n \to y$, for any $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that the condition

$$x_n \leq x + a_{1,n}^\epsilon, \quad x \leq x_n + a_{2,n}^\epsilon, \quad \left\|a_{i,n}^\epsilon\right\| \leq \epsilon \tag{45}$$

holds for every $n \geq n_0$ and for any $\epsilon > 0$ there exists a $n_0' \in \mathbb{N}$ such that the condition

$$y_n \leq y + b_{1,n}^\epsilon, \quad y \leq y_n + b_{2,n}^\epsilon, \quad \left\|b_{i,n}^\epsilon\right\| \leq \epsilon \tag{46}$$

holds for every $n > n_0'$. From the condition (17), we get that

$$\begin{aligned} \langle x_n, y_n\rangle &\leq \left\langle x + a_{1,n}^\epsilon, y + b_{1,n}^\epsilon\right\rangle \\ &= \langle x, y\rangle + \left\langle x, b_{1,n}^\epsilon\right\rangle + \left\langle a_{1,n}^\epsilon, y\right\rangle + \left\langle a_{1,n}^\epsilon, b_{1,n}^\epsilon\right\rangle. \end{aligned} \tag{47}$$

Hence,

$$\begin{aligned} &|\langle x_n, y_n\rangle - \langle x, y\rangle| \leq \left|\left\langle x, b_{1,n}^\epsilon\right\rangle\right| + \left|\left\langle a_{1,n}^\epsilon, y\right\rangle\right| + \left|\left\langle a_{1,n}^\epsilon, b_{1,n}^\epsilon\right\rangle\right| \\ &\leq \|x\|\left\|b_{1,n}^\epsilon\right\| + \left\|a_{1,n}^\epsilon\right\|\|y\| + \left\|a_{1,n}^\epsilon\right\|\left\|b_{1,n}^\epsilon\right\| \\ &\longrightarrow 0 \end{aligned} \tag{48}$$

for all $n > N$ and $N = \max\{n_0, n_0'\}$. □

Lemma 17. *Let X be a quasilinear space. For every $x, y \in X$, $x + y \in X_r$ implies $x \in X_r$ and $y \in X_r$.*

Proof. Let $x + y \in X_r$. Then there exists $z \in X_r$ such that

$$(x + y) + z = 0. \tag{49}$$

Suppose that $x \notin X_r$. Then there does not exist $x' \in X_r$, such that $x + x' = 0$.

From (49), $y + z$ is inverse of x, such that

$$x + (y + z) = 0. \tag{50}$$

This contradiction shows that $x \in X_r$. Analogously we obtain $y \in X_r$. □

Lemma 18. *Let X be a quasilinear inner product space. The following inequalities are satisfied for any $x, y \in X$:*

(i) *for every $z \in X$, $x \leq y$ implies $\langle x, z\rangle \leq \langle y, z\rangle$;*

(ii) *for every $x, y, z \in X$, $\langle x, z\rangle \leq \langle y, z\rangle$ implies $x, y \in X_r$. Moreover, $x = y$.*

Proof. The proof of part (i) is obvious. If $\langle x, z\rangle \leq \langle y, z\rangle$ for every $x, y, z \in X$ then

$$\langle x, z\rangle - \langle y, z\rangle \leq 0, \qquad \langle x - y, z\rangle \leq 0. \tag{51}$$

Hence

$$\begin{aligned} \langle x - y, x - y\rangle &= \|x - y\|^2 \leq 0 \\ &\Longleftrightarrow \|x - y\| = 0 \\ &\Longleftrightarrow x - y = 0 \end{aligned} \tag{52}$$

for $z = x - y$ too. Thus by Lemma 17, $x, y \in X_r$. Further, under this condition we get $x = y$. □

Definition 19. A quasilinear inner product space X is called an inner product Ω-space, if there exists an element $B_X \neq \theta$ such that

$$\text{if } \|x\| \leq \|B_X\| \quad \text{then } x \leq B_X. \tag{53}$$

This definition is an obvious result of the Definition 7.

We recall that any normed linear space cannot be an Ω-space. Indeed, if $\|x\| \leq \|B_X\|$, then $x = B_X$. Also, $\|x/2\| \leq \|B_X\|$ implies $x/2 = B_X$. This is not true. So, the concept of Ω-space is meaningless in the normed linear spaces although it is significant for (nonlinear) the quasilinear space.

Example 20. Let E be an inner product space. Then $K(E)$ and $K_c(E)$ are the inner product Ω-spaces with inner product defined by

$$\langle A, B\rangle = \sup\left\{\langle a, b\rangle_E : a \in A, b \in B\right\}. \tag{54}$$

Indeed, let $A \subseteq B$, $C \subseteq D$. For the elements $A, B, C, D \in K(E)$ and every $a \in A$, $b \in B$, $c \in C$, $d \in D$, we have $\|a\| \leq M, \|b\| \leq M', \|c\| \leq N, \|d\| \leq N'$ for some integers $M, N, M', N' \geq 0$. Since $A \subseteq B$ and $C \subseteq D$, $M' \geq M$ and $N' \geq N$. Because of Schwarz's inequality, we have

$$\begin{aligned} |\langle a, c\rangle| &\leq \|a\| \, \|c\| \leq M \cdot N, \\ |\langle b, d\rangle| &\leq \|b\| \, \|d\| \leq M' \cdot N'. \end{aligned} \tag{55}$$

Then, $-M \cdot N \leq \langle a, c\rangle \leq M \cdot N$ and $-M' \cdot N' \leq \langle b, d\rangle \leq M' \cdot N$. So we have

$$\langle a, c\rangle \leq M \cdot N \leq M' \cdot N' \tag{56}$$

for $M' \geq M$ and $N' \geq N$. Hence,

$$\begin{gathered} \sup\left\{\langle a, c\rangle_E : a \in A, c \in C\right\} \leq \sup\left\{\langle b, d\rangle_E : b \in B, d \in D\right\}, \\ \langle A, C\rangle \leq \langle B, D\rangle. \end{gathered} \tag{57}$$

The proof of (18) is analogous to Example 11. Thus $K(E)$ is an inner product space.

Now, we show that $K(E)$ is an Ω-space. Let B_E be a unit sphere of E. Then $B_E \in K(E)$ and $\|B_E\| = 1$. We will show that if $\|A\| \leq 1$ then $A \subseteq B_E$. Let x be an arbitrary element of A. Since

$$\|A\| = \sup_{x \in A} \|x\| \leq 1 \tag{58}$$

we have $\|x\| \leq 1$ for every $x \in A$. Thus $x \in B_E$.

3.2. Hilbert Quasilinear Spaces and Hilbert Ω-Spaces. If $(X, \|\cdot\|)$ is a normed quasilinear space then we know that the relation h given by

$$h(x, y) = \inf\left\{r \geq 0 : x \leq y + a_1^r, y \leq x + a_2^r, \|a_i^r\|_X \leq r\right\} \tag{59}$$

defines a metric on X [1]. Note that, here, $h_X(x, y) = \|x - y\|_X$ may not be satisfied for every $x, y \in X$. But, the inequality

$$h_X(x, y) \leq \|x - y\|_X \tag{60}$$

is always true. This metric is called Hausdorff metric. Because of this inequality, instead of analyzing topological properties of normed quasilinear spaces, analyzing according to the metric derived from this norm is more convenient. Because

$$d(x, y) = \|x - y\| \tag{61}$$

does not define a metric. Therefore, the metric of this norm is not given with the equality of (61). Instead of that, the inequality (60) is the norm metric. If X is a normed linear space, then we know that $h(x, y) = d(x, y)$. So, if a normed quasilinear space is complete according to the norm metric $h(x, y)$ then normed quasilinear space is called complete normed space.

Definition 21. A quasilinear inner product space is called Hilbert quasilinear space, if it is complete according to the Hausdorff metric.

Definition 22. A quasilinear inner product space X is called Hilbert Ω-space, if X is a Hilbert quasilinear space and Ω-space.

Example 23. Let E be an inner product space. Then we know that $K(E)$ is a quasilinear inner product space and this space is complete with respect to Hausdorff metric; further it is Ω-space [1]. So, $K(E)$ is a Hilbert Ω-space.

Definition 24 (orthogonality). An element x of a quasilinear inner product space X is said to be orthogonal to an element $y \in X$ if

$$\langle x, y\rangle = 0. \tag{62}$$

One also says that x and y are orthogonal and one writes $x \perp y$. Similarly, for subsets $\alpha, \beta \subseteq X$ one writes $x \perp \alpha$ if $x \perp z$ for all $z \in \alpha$ and $\alpha \perp \beta$ if $a \perp b$ for all $a \in \alpha$ and $b \in \beta$.

Definition 25. An *orthonormal* set $M \subset X$ is an orthogonal set in X whose elements have norm 1; that is, for all $x, y \in M$

$$\langle x, y\rangle = \begin{cases} 0, & x \neq y, \\ 1, & x = y. \end{cases} \tag{63}$$

Example 26. (a) $X = [0, 1]$ and $Y = [-1, 0]$ are elements of $K(\mathbb{R})$. These elements are orthogonal, since

$$\langle X, Y\rangle = \sup\{a \cdot b : a \in [0, 1], b \in [-1, 0]\} = 0. \tag{64}$$

At the same time, these sets are orthonormal since X and Y have norm 1.

(b) Let

$$\begin{aligned} A_1 &= \{(0, t) : 0 \leq t \leq 1\}, \qquad A_2 = \{(t, 0) : 0 \leq t \leq 1\}, \\ A_3 &= \{(0, -t) : 0 \leq t \leq 1\}, \qquad A_4 = \{(-t, 0) : 0 \leq t \leq 1\} \end{aligned} \tag{65}$$

be orthogonal subsets of $K(\mathbb{R}^2)$ from Definition 24. Further, the set $\{A_1, A_2, A_3, A_4\}$ is orthonormal.

Corollary 27. *For orthogonal elements x, y one has $\langle x, y\rangle = 0$, so that one easily obtains the Pythagorean relation*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \tag{66}$$

and more generally, if $\{x_1, \ldots, x_n\}$ is an orthogonal set, then

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2. \tag{67}$$

Definition 28. Let A be a nonempty subset of an quasilinear inner product space X. An element $x \in X$ is said to be orthogonal to A, denoted by $x \perp A$, if $\langle x, y\rangle = 0$ for every $y \in A$. The set of all elements of X orthogonal to A, denoted by $A^{\perp}$, is called the *orthogonal complement* of A and is indicated by

$$A^{\perp} = \{x \in X : \langle x, y\rangle = 0, y \in A\}. \tag{68}$$

Theorem 29. *For any subset A of a quasilinear inner product space X, $A^{\perp}$ is a closed subspace of X.*

Proof. If $\alpha, \beta \in \mathbb{C}$ and $x, y \in A^{\perp}$, then

$$\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle = 0 \tag{69}$$

for every $z \in A$. Thus, $\alpha x + \beta y \in A^{\perp}$. So, $A^{\perp}$ is a subspace of X. We now prove that $A^{\perp}$ is closed. Let $(x_n) \in A^{\perp}$ and $x_n \to a$ for some $a \in X$. Then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that the following conditions hold for $n \geq N$:

$$x_n \leq a + b^{\epsilon}_{1,n}, \quad a \leq x_n + b^{\epsilon}_{2,n}, \quad \left\|b^{\epsilon}_{i,n}\right\| \leq \epsilon^{1/2}. \tag{70}$$

From the definition of the quasilinear inner product space, we have

$$x_n \leq a, \qquad a \leq x_n \tag{71}$$

for $\langle b^{\epsilon}_{i,n}, b^{\epsilon}_{i,n}\rangle \leq \epsilon$. This shows that $0 \leq \langle a, z\rangle$, if $x_n \leq a$ and $z \leq z$ for every $z \in A$. At the same time, we obtain $\langle a, z\rangle \leq 0$, if $a \leq x_n$ and $z \leq z$ for every $z \in A$. Consequently, $\langle a, z\rangle = 0$. Hence, $a \in A^{\perp}$. □

This theorem implies that $A^{\perp}$ is a Hilbert quasilinear space for any subset A of a Hilbert quasilinear space X.

Remark 30. Suppose that X is a classical inner product space; then we immediately show that $A^{\perp}$ is closed from the inequality $|\langle x_n - a, z\rangle| \leq \|x_n - a\|\|z\|$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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