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Research Article

On the Inequalities for the Generalized Trigonometric Functions

Edward Neuman

Mathematical Research Institute, 144 Hawthorn Hollow, Carbondale, IL 62903, USA

Correspondence should be addressed to Edward Neuman; edneuman76@gmail.com

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This paper deals with Huygens-type and Wilker-type inequalities for the generalized trigonometric functions of P. Lindqvist. A major mathematical tool used in this work is a generalized version of the Schwab-Borchardt mean introduced recently by the author of this work.

1. Introduction

Recently the generalized trigonometric and the generalized hyperbolic functions have attracted attention of several researches. These functions, introduced by Lindqvist in [1], depend on one parameter $p > 1$. They become classical trigonometric and hyperbolic functions when $p = 2$. It is known that they are eigenfunctions of the Dirichlet problem for the one-dimensional p-Laplacian. For more details concerning a recent progress in this rapidly growing area of functions theory the interested reader is referred to [1–11].

The goal of this paper is to establish some inequalities for families of functions mentioned earlier in this section. In Section 2 we give definitions of functions under discussions. Also, some preliminary results are included there. Some useful inequalities utilized in this note are established in Section 3. The main results, involving the Huygens-type and the Wilker-type inequalities, are derived in Section 4.

2. Definitions and Preliminaries

For the reader's convenience we recall first definition of the celebrated Gauss hypergeometric function $F(\alpha, \beta; \gamma; z)$:

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)} \frac{z^n}{n!}, \quad |z| < 1, \tag{1}$$

where $(\alpha, n) = \alpha(\alpha+1)\cdots(\alpha+n-1)$ $(n \neq 0)$ is the shifted factorial or Appell symbol, with $(\alpha, 0) = 1$ if $\alpha \neq 0$, and $\gamma \neq 0, -1, -2, \ldots$ (see, e.g., [12]).

In what follows, let the parameter p be strictly greater than 1. In some cases this assumption will be relaxed to $1 < p \leq 2$. We will adopt notation and definitions used in [5]. Let

$$\pi_p = 2\frac{\pi/p}{\sin(\pi/p)}. \tag{2}$$

Further, let

$$\begin{aligned} a_p &= \frac{\pi_p}{2}, \\ b_p &= 2^{-1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right), \\ c_p &= 2^{-1/p} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right). \end{aligned} \tag{3}$$

Also, let $I = (0, 1)$ and let $J = (1, \infty)$. The generalized trigonometric and hyperbolic functions needed in this paper are the following homeomorphisms:

$$\begin{aligned} &\sin_p : (0, a_p) \longrightarrow I, \qquad \cos_p : (0, a_p) \longrightarrow I, \\ &\tan_p : (0, b_p) \longrightarrow I, \\ &\sinh_p : (0, c_p) \longrightarrow I, \qquad \cosh_p : (0, \infty) \longrightarrow J, \\ &\tanh_p : (0, \infty) \longrightarrow J. \end{aligned} \tag{4}$$

The inverse functions $\sin_p^{-1}$ and $\sinh_p^{-1}$ are represented as follows [7]:

$$\sin_p^{-1} u = \int_0^u \left(1 - t^p\right)^{-1/p} dt = uF\left(\frac{1}{p}, \frac{1}{p};\ 1 + \frac{1}{p}; u^p\right), \quad (5)$$

$$\sinh_p^{-1} u = \int_0^u \left(1 + t^p\right)^{-1/p} dt = uF\left(\frac{1}{p}, \frac{1}{p};\ 1 + \frac{1}{p}; -u^p\right). \quad (6)$$

Inverse functions of the remaining four functions can be expressed in terms of $\sin_p^{-1}$ and $\sinh_p^{-1}$. We have

$$\cos_p^{-1} u = \sin_p^{-1}\left(\sqrt[p]{1 - u^p}\right), \quad (7)$$

$$\cosh_p^{-1} u = \sinh_p^{-1}\left(\sqrt[p]{u^p - 1}\right), \quad (8)$$

$$\tan_p^{-1} u = \sin_p^{-1}\left(\frac{u}{\sqrt[p]{1 + u^p}}\right), \quad (9)$$

$$\tanh_p^{-1} u = \sinh_p^{-1}\left(\frac{u}{\sqrt[p]{1 - u^p}}\right). \quad (10)$$

For the later use we recall now definition of a certain bivariate mean introduced recently in [13]

$$SB_p(x, y) = \begin{cases} \dfrac{\sqrt[p]{y^p - x^p}}{\cos_p^{-1}(x/y)}, & 0 \le x < y, \\ \dfrac{\sqrt[p]{x^p - y^p}}{\cosh_p^{-1}(x/y)}, & y < x, \\ x, & x = y, \end{cases} \quad (11)$$

and call $SB_p(x, y)$ the p-version of the Schwab-Borchardt mean. When $p = 2$, the latter mean becomes a classical Schwab-Borchardt mean which has been studied extensively in [14–20]. It is clear that $SB_p(x, y)$ is a nonsymmetric and homogeneous function of degree 1 of its variables.

A remarkable result states that the mean SB_p admits a representation in terms of the Gauss hypergeometric function [13]:

$$SB_p(x, y) = y\left[F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p};\ 1 - \left(\frac{x}{y}\right)^p\right)\right]^{-1} \quad (12)$$

(see [13]).

We will need the following.

Theorem A. *If $x > y$, then*

$$SB_p(x, y) < SB_p(y, x). \quad (13)$$

Let $\alpha = 1/p$ and let $\beta = 1 - 1/p$. Then the inequality

$$\left[SB_p(y, x)\right]^\alpha y^\beta < SB_p(x, y) \quad (14)$$

holds true for all positive and unequal numbers x and y (see [13]).

Another result of interest (see [21]) reads as follows.

Theorem B. *Let u, v be positive numbers. Further, let $\lambda \ge 1$ and let $\mu \ge 1$. Then the inequality*

$$2 < \left(\frac{1}{u}\right)^\lambda + \left(\frac{1}{v}\right)^\mu \quad (15)$$

holds true if

$$u < 1 < v \quad \text{or} \quad v < 1 < u \quad (16)$$

and if

$$1 < \frac{\lambda}{\lambda + \mu}\frac{1}{u} + \frac{\mu}{\lambda + \mu}\frac{1}{v}. \quad (17)$$

Also, we will utilize the following result [22].

Theorem C. *Let $u, v > 0$ and assume that $u \ne v$. If $uv > 1$, then*

$$\frac{1}{u} + \frac{1}{v} < u + v. \quad (18)$$

3. Inequalities

The goal of this section is to establish an inequality for the p-version of the Schwab-Borchardt mean S_p and other inequalities as well. Applications of those results to generalized trigonometric and generalized hyperbolic functions are presented in the next section.

We begin proving an extension of inequality (14).

Theorem 1. *Let $x, y > 0$ $(x \ne y)$ and let $p > 1$. Then*

$$x^\gamma y^\delta < \left[SB_p(y, x)\right]^\alpha y^\beta < SB_p(x, y), \quad (19)$$

where

$$\gamma = \frac{1}{p + 1}, \quad \delta = 1 - \gamma, \quad \alpha = \frac{1}{p}, \quad \beta = 1 - \alpha. \quad (20)$$

Proof. We need to prove the first inequality in (19). To this aim we will demonstrate first that

$$x^\gamma y^\delta < SB_p(x, y). \quad (21)$$

This can be proven using the following upper bound for Gauss' hypergeometric function:

$$F(a, b; c; z) < (1 - z)^{-ab/c}, \quad (22)$$

which holds true if $b > 0$, $c > a > 0$, and $|z| < 1$ (see [23, (3.4), (2.15)]). Application to the Gauss hypergeometric function on the right side of (12) yields

$$F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; 1 - \left(\frac{x}{y}\right)^p\right) < \left(\frac{x}{y}\right)^{-\gamma}. \quad (23)$$

This in conjunction with (12) gives the desired inequality (21). For the proof of the first inequality in (19) we apply (21) to the middle term of (19) to obtain

$$\left[SB_p\left(y,x\right)\right]^{\alpha} y^{\beta} > \left(y^{\gamma}x^{\delta}\right)^{\alpha} y^{\beta} = x^{\alpha\delta} y^{\alpha\gamma+\beta} = x^{\gamma} y^{\delta}, \quad (24)$$

where in the last step we have used (20). The proof is complete. □

Our next result reads as follows.

Theorem 2. *Let a and b be positive and unequal numbers. Also, let the number τ be such that*

$$a^{\tau} < b, \quad (25)$$

where $0 < \tau < 1$. Then the following inequalities

$$1 < \tau\frac{b}{a} + (1-\tau)\, b, \quad (26)$$

$$1 < \frac{1}{2}\left(b^{1/\tau-1} + \frac{b}{a}\right) \quad (27)$$

hold true.

Proof. We will prove now inequality (26). It follows from (25) followed by application of the inequality of arithmetic and geometric means, with weights τ and $1-\tau$, that

$$1 < \left(\frac{1}{a}\right)^{\tau} b = \left(\frac{b}{a}\right)^{\tau} b^{1-\tau} < \tau\frac{b}{a} + (1-\tau)\, b. \quad (28)$$

Inequality (27) can be established in a similar manner. We use (26) again followed by a little algebra to obtain

$$1 < b^{1/\tau}\frac{1}{a} = b^{1/\tau-1}\frac{b}{a}. \quad (29)$$

This yields

$$1 < \left(b^{1/\tau-1}\right)^{1/2}\left(\frac{b}{a}\right)^{1/2} < \frac{1}{2}\left(b^{1/\tau-1} + \frac{b}{a}\right), \quad (30)$$

where in the last step we have applied the Schwarz-Bunyakovsky inequality. □

4. Applications to Generalized Trigonometric and Hyperbolic Functions

In this section we present several inequalities for the generalized trigonometric and hyperbolic functions. Recently several inequalities for these families of functions have been obtained. We refer the interested reader to the following papers [3, 5, 7–9, 13] and to the references therein.

In order to facilitate presentation we recall first some known results for classical trigonometric functions. In particular, the following results

$$3 < 2\frac{\sin x}{x} + \frac{\tan x}{x}, \quad (31)$$

$$2 < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \quad (32)$$

$(0 < |x| < \pi/2)$ have attracted attention of several researchers. Inequalities (31) and (32) have been obtained, respectively, by Huygens [24] and Wilker [25]. Several proofs of these results can be found in mathematical literature (see, e.g., [21, 22, 26–32] and the references therein). In [22] the authors called inequalities (31) and (32) the first Huygens and the first Wilker inequalities, respectively, for the trigonometric functions.

The second Huygens and the second Wilker inequalities for the trigonometric functions also appear in mathematical literature. They read, respectively, as follows:

$$3 < 2\frac{x}{\sin x} + \frac{x}{\tan x}, \quad (33)$$

$$2 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \quad (34)$$

$(0 < |x| < \pi/2)$. For the proofs of the last two results the interested reader is referred to [22, 29], respectively.

It is worth mentioning that there are known counterparts of inequalities (31)–(34) for the hyperbolic functions. They have the same structure as (31)–(34) with following modifications: $\sin \to \sinh$ and $\tan \to \tanh$. The domains of their validity consist of all nonzero numbers. For more details and additional references see, for example, [22].

We are in a position to prove the following.

Theorem 3. *Let $t \in (0, a_p)$. Then*

$$\left(\cos_p t\right)^{1/(p+1)} < \left[\frac{\sin_p t}{\tanh_p^{-1}\left(\sin_p t\right)}\right]^{1/p} < \frac{\sin_p t}{t} \quad (35)$$

for all $p > 1$.

Proof. Let $x = \cos_p t$, where $t \in (0, a_p)$, and let $y = 1$. By making use of (11) and the formula

$$\sin_p^p t + \cos_p^p t = 1 \quad (36)$$

(see [7]) we obtain

$$\begin{aligned} SB_p\left(x,y\right) &= SB_p\left(\cos_p t, 1\right) \\ &= \frac{\sin_p t}{\cos_p^{-1}\left(\cos_p t\right)} \\ &= \frac{\sin_p t}{t}. \end{aligned} \quad (37)$$

Also, a use of (11) followed by application of (8) yields

$$\begin{aligned} SB_p\left(y,x\right) &= SB_p\left(1, \cos_p t\right) \\ &= \frac{\sin_p t}{\cosh_p^{-1}\left(\dfrac{1}{\cos_p t}\right)} \end{aligned}$$

$$= \frac{\sin_p t}{\sinh_p^{-1}\left(\tan_p t\right)} = \frac{\sin_p t}{\tanh_p^{-1}\left(\sin_p t\right)}. \tag{38}$$

To obtain the desired result it suffices to apply inequality (19). □

A particular case of (35), when $p = 2$, appears in [33].

It is worth mentioning that the counterpart of (35) for the generalized hyperbolic functions

$$\left(\cosh_p t\right)^{1/(p+1)} < \left[\frac{\sinh_p t}{\tan_p^{-1}\left(\sinh_p t\right)}\right]^{1/p} < \frac{\sinh_p t}{t}, \quad t > 0, \tag{39}$$

($p > 1$) can also be established in a similar manner. An identity

$$\left|\cosh_p t\right|^p - \left|\sinh_p t\right|^p = 1, \quad p > 1,\ t \in \mathbb{R}, \tag{40}$$

needed in the proof can be found in [7]. We omit further details. Inequalities which connect the first and the third members of (35) and (39) have been established in [7] in Theorems 3.6 and 3.8, respectively.

Our next goal is to provide short proofs of the first Huygens and the first Wilker inequalities for the generalized trigonometric functions.

Theorem 4. *Let $p > 1$ and let $0 < t < a_p$. Then*

$$p + 1 < p\frac{\sin_p t}{t} + \frac{\tan_p t}{t}, \tag{41}$$

$$2 < \left(\frac{\sin_p t}{t}\right)^p + \frac{\tan_p t}{t}. \tag{42}$$

Proof. We will employ Theorem 2 with $a = \cos_p t$, $b = \sin_p t/t$, and $\tau = 1/(p+1)$. This yields $b/a = \tan_p t/t$ and $1-\tau = p/(p+1)$. Inequality (41) follows from (26). Similarly, inequality (42) is an immediate consequence of (27) because $1/\tau - 1 = p$. □

Inequality (41) has been established by different means in [7, Theorem 3.16].

Our next result reads as follows.

Theorem 5. *For $1 < p \le 2$ the following inequalities*

$$p + 1 < p\frac{t}{\sin_p t} + \frac{t}{\tan_p t}, \tag{43}$$

$$2 < \left(\frac{t}{\sin_p t}\right)^p + \frac{t}{\tan_p t} \tag{44}$$

hold true for all $t \in (0, a_p)$.

Proof. Inequality (43) is established in [7, Theorem 3.22]. It is included here for the sake of completeness. We will prove now inequality (44). To this aim we let

$$u = \frac{\sin_p t}{t}, \qquad v = \frac{\tan_p t}{t}. \tag{45}$$

It follows from [7, (3.7)] and the proof of Lemma 3.32 in [7] that $u < 1 < v$ holds for all $p > 1$. To obtain the desired result we apply now Theorem B, with u and v as defined above and $\lambda = p$ and $\mu = 1$. It is easy to see that inequality (17) is the same as the second Huygens inequality (43). The assertion now follows. □

The counterparts of Theorems 4 and 5 for the generalized hyperbolic functions can be established in a similar fashion. We leave it to the interested reader.

The last result of this section gives an inequality which connects the first and the second inequalities of Wilker for the generalized trigonometric functions.

Theorem 6. *Let $p > 1$. If $t \in (0, a_p)$, then*

$$\left(\frac{t}{\sin_p t}\right)^p + \frac{t}{\tan_p t} < \left(\frac{\sin_p t}{t}\right)^p + \frac{\tan_p t}{t}. \tag{46}$$

Proof. For the sake of notation let

$$u = \frac{\sin_p t}{t}, \qquad v = \frac{\tan_p t}{t}. \tag{47}$$

Then the inequality connecting the first and the third members of (35) can be written as

$$1 < u\left(\frac{1}{\cos_p t}\right)^{1/(p+1)} = u^{p/(p+1)} v^{1/(p+1)}. \tag{48}$$

Exponentiation with the exponent of $p + 1$ allows us to write the inequality connecting the first and the last members as $1 < u^p v$. To complete the proof it suffices to utilize Theorem C with u replaced by u^p. □

Conflict of Interests

The author declares that he had no conflict of interests.

References

[1] P. Lindqvist, "Some remarkable sine and cosine functions," *Ricerche di Matematica*, vol. 44, no. 2, pp. 269–290, 1995.

[2] A. Baricz, B. A. Bhayo, and R. Klén, "Convexity properties of generalized trigonometric and hyperbolic functions," *Aequationes Mathematicae*, 2013.

[3] B. A. Bhayo, "Power mean inequality of generalized trigonometric functions," http://arxiv.org/abs/1209.0873.

[4] B. A. Bhayo and M. Vuorinen, "On generalized trigonometric functions with two parameters," *Journal of Approximation Theory*, vol. 164, no. 10, pp. 1415–1426, 2012.

[5] B. A. Bhayo and M. Vuorinen, "Inequalities for eigenfunctions of the p-Laplacian," *Issues of Analysis*, vol. 2, no. 20, pp. 13–35, 2013.

[6] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang, and F. Qi, "Convexity of the generalized sine function and the generalized hyperbolic sine function," *Journal of Approximation Theory*, vol. 174, pp. 1–9, 2013.

[7] R. Klén, M. Vuorinen, and X. Zhang, "Inequalities for the generalized trigonometric and hyperbolic functions," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 1, pp. 521–529, 2014.

[8] E. Neuman, "Inequalities involving generalized trigonometric and generalized hyperbolic functions," *Journal of Mathematical Inequalities*. In press.

[9] E. Neuman, "Wilker- and Huygens-type inequalities for the generalized and for the generalized hyperbolic functions," *Applied Mathematics and Computation*, vol. 230, pp. 211–217, 2014.

[10] S. Takeuchi, "Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 385, no. 1, pp. 24–35, 2012.

[11] S. Takeuchi, "The basis property of generalized Jacobian elliptic functions," http://arxiv.org/abs/1310.0597.

[12] B. C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, NY, USA, 1977.

[13] E. Neuman, "On the p-version of the Schwab-Borchardt mean," In press.

[14] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, NY, USA, 1987.

[15] J. L. Brenner and B. C. Carlson, "Homogeneous mean values: weights and asymptotics," *Journal of Mathematical Analysis and Applications*, vol. 123, no. 1, pp. 265–280, 1987.

[16] B. C. Carlson, "A hypergeometric mean value," *Proceedings of the American Mathematical Society*, vol. 16, pp. 759–766, 1965.

[17] B. C. Carlson, "Algorithms involving arithmetic and geometric means," *The American Mathematical Monthly*, vol. 78, pp. 496–505, 1971.

[18] E. Neuman, "Inequalities for the Schwab-Borchardt mean and their applications," *Journal of Mathematical Inequalities*, vol. 5, no. 4, pp. 601–609, 2011.

[19] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[20] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean. II," *Mathematica Pannonica*, vol. 17, no. 1, pp. 49–59, 2006.

[21] E. Neuman, "Inequalities for weighted sums of powers and their applications," *Mathematical Inequalities & Applications*, vol. 15, no. 4, pp. 995–1005, 2012.

[22] E. Neuman and J. Sándor, "On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities," *Mathematical Inequalities & Applications*, vol. 13, no. 4, pp. 715–723, 2010.

[23] B. C. Carlson, "Some inequalities for hypergeometric functions," *Proceedings of the American Mathematical Society*, vol. 17, pp. 32–39, 1966.

[24] C. Huygens, *Oeuvres Completes 1888–1940*, Sociéte Hollondaise des Science, Haga, Gothenburg.

[25] J. B. Wilker, "Problem E 3306," *The American Mathematical Monthly*, vol. 96, article 55, 1989.

[26] B.-N. Guo, B.-M. Qiao, F. Qi, and W. Li, "On new proofs of Wilker's inequalities involving trigonometric functions," *Mathematical Inequalities & Applications*, vol. 6, no. 1, pp. 19–22, 2003.

[27] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio, "Inequalities involving trigono-metric functions," *The American Mathematical Monthly*, vol. 98, pp. 264–267, 1991.

[28] S. Wu and A. Baricz, "Generalizations of Mitrinović, Adamović and Lazarević's inequalities and their applications," *Publicationes Mathematicae Debrecen*, vol. 75, no. 3-4, pp. 447–458, 2009.

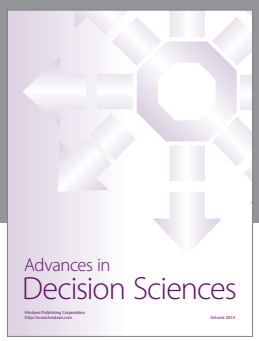
[29] S.-H. Wu and H. M. Srivastava, "A weighted and exponential generalization of Wilker's inequality and its applications," *Integral Transforms and Special Functions*, vol. 18, no. 7-8, pp. 529–535, 2007.

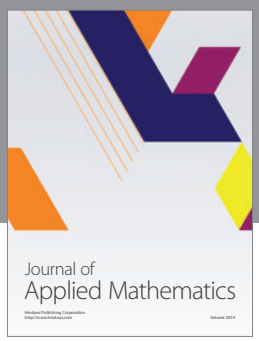
[30] L. Zhu, "A new simple proof of Wilker's inequality," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 749–750, 2005.

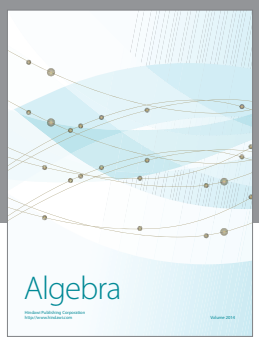
[31] L. Zhu, "On Wilker-type inequalities," *Mathematical Inequalities & Applications*, vol. 10, no. 4, pp. 727–731, 2007.

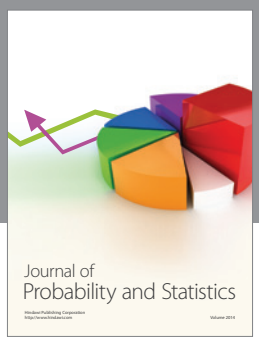
[32] L. Zhu, "Some new Wilker-type inequalities for circular and hyperbolic functions," *Abstract and Applied Analysis*, vol. 2009, Article ID 485842, 9 pages, 2009.

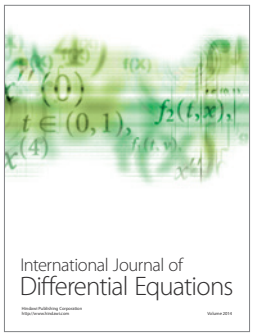
[33] E. Neuman, "Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions," *Journal of Inequalities and Applications*, vol. 1, no. 1, pp. 1–11, 2012.











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