Research Article

The Singular Temperley-Lieb Category

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We introduce and study the singular Temperley-Lieb category over $\mathbb{Z}[q, q^{-1}]$, which is a free pivotal category over two self-dual generators and is an extension of the (classical) Temperley-Lieb category. Our construction is motivated by a state model for the $sl(2)$ polynomial of an oriented link and provides a categorical perspective to this link invariant. We also construct a couple of polynomial invariants for oriented tangles from category theory point of view.

1. Introduction

The Temperley-Lieb algebra has played a central role in the discovery of the Jones polynomial [1] and in the subsequent developments relating to knot theory, topological quantum field theory, and statistical mechanics [2]. Originally presented in terms of abstract generators and relations, it was combinatorially described by Kauffman as planar diagram algebra in terms of his *bracket polynomial*. The Temperley-Lieb category provides a more structured perspective on the Temperley-Lieb algebra (for details we refer the reader to [3, 4]).

Khovanov interpreted the Jones polynomial as the Euler characteristic of a cohomology theory of a link, now so-called the Khovanov homology [5]. In [6], Khovanov extended his homology theory to tangles, by associating a bimodule to morphism in the Temperley-Lieb category and a chain complex of bimodules to a plane diagram of a tangle. The chain homotopy equivalence class of the complex is an invariant of the tangle and specializes in the Khovanov homology when the tangle is a link. We remark that Bar-Natan [7] used another approach to Khovanov homology for tangles via cobordisms modulo local relations.

The Khovanov homology satisfies the functoriality property under link cobordisms only up to a sign, but it was showed by Clark-Morrison-Walker [8], and independently by the first author [9] that there is a way to resolve the sign indeterminacy in the functoriality of the Khovanov

homology. In [9], one uses webs and seamed cobordisms to construct a homology theory for tangles that is properly functorial under tangle cobordisms, and which recovers the Khovanov homology. In particular, one starts from a state model for the $sl(2)$ link invariant, which is a state summation model for the unnormalized Jones polynomial (with $t^{1/2} = -q$) via webs, which are oriented bivalent graphs whose vertices are either “sinks” or “sources.”

This state model for the $sl(2)$ polynomial for oriented links constitutes the motivation of this paper, and our main goal is to understand this model and the corresponding geometric objects from the category theory point of view. Regarding the vertices of the webs as singularities on diagrams, we call a state associated with a planar tangle diagram a *singular flat tangle* and we regard it as morphism in a monoidal category **STL**, which we refer to as the *singular Temperley-Lieb category*. It turns out that **STL** is an autonomous category in which the objects are self-dual; that is, each object is isomorphic to its dual. The $sl(2)$ link polynomial can be interpreted via representation theory of the quantum group $U_q(sl(2))$. Specifically, an oriented edge in a singular flat tangle stands for the standard (2-dimensional) vector representation V_1 of quantum $sl(2)$. Since V_1 is self-dual (i.e., isomorphic to its dual representation V_1^*), it is not surprising that every object in the singular Temperley-Lieb category **STL** is self-dual.

We hope that our paper will be a valuable resource for young researchers interested in knot invariants, in general,

and categorical methods in representation theory and quantum invariants, in particular. Although we do not attempt it here, we remark that, as a byproduct of the construction of the singular Temperley-Lieb category, one can adapt Khovanov's work [6] to singular flat tangles and construct an $sl(2)$ tangle (co)homology, which should be isomorphic to that developed in [9].

The paper is organized as follows. We review the $sl(2)$ polynomial in Section 2. Following Turaev [10], in Section 3 we briefly review the category of oriented tangles **OTa** and present it via generators and relations. We introduce and study the singular Temperley-Lieb category **STL** in Section 4. In Section 5 we construct a monoidal functor $F : \mathbf{OTa} \rightarrow \mathbf{LSTL}$, where **LSTL** is an extension of **STL** by allowing formal linear combinations of singular flat tangles with coefficients in $\mathbb{Z}[q, q^{-1}]$. The functor F mimics the skein relations defining the $sl(2)$ polynomial, and thus it can be regarded as a categorical approach to the link invariant. Finally, in Section 6 we construct a couple of polynomial invariants for oriented tangles, while in Section 7 we take another look at the category **STL** by regarding a singular flat tangle as a directed ribbon graph.

2. A State Model for the $sl(2)$ Polynomial

The $sl(2)$ polynomial can be defined as follows. Let D be a planar projection of an oriented link L and decompose each crossing of D as explained below

$$\begin{aligned} \nearrow \searrow &= q \uparrow \uparrow - q^2 \searrow \nearrow, & \nwarrow \swarrow &= q^{-1} \uparrow \uparrow - q^{-2} \nwarrow \swarrow. \end{aligned} \quad (1)$$

We define $\langle D \rangle$, the bracket of D , as the linear combination of the brackets of all resolutions Γ of D , where Γ is evaluated according the following rules:

$$\begin{aligned} \langle \Gamma \cup \bigcirc \rangle &= \langle \Gamma \cup \bigcirc \rangle = (q + q^{-1}) \langle \Gamma \rangle, \\ \langle \nearrow \searrow \rangle &= \langle \searrow \nearrow \rangle, & \langle \nwarrow \swarrow \rangle &= \langle \swarrow \nwarrow \rangle. \end{aligned} \quad (2)$$

Specifically, $\langle D \rangle = \sum_{\Gamma} (\pm q^{\alpha(\Gamma)} \langle \Gamma \rangle)$, where the sum is over all resolutions (states) of D , and the signs $\pm$ and $\alpha(\Gamma)$ are determined by the decomposition rules given in (1).

It is easy to see that $q^2 \langle \nearrow \searrow \rangle - q^{-2} \langle \nwarrow \swarrow \rangle = (q - q^{-1}) \langle \uparrow \uparrow \rangle$. It is known that $\langle D_1 \rangle = \langle D_2 \rangle$ whenever D_1 and D_2 are ambient isotopic. This together with the above skein relations implies that $\langle D \rangle$ is the $sl(2)$ link invariant or, equivalently, the unnormalized Jones polynomial of the oriented link L , with $t^{1/2} = -q$.

A resolution Γ associated with the link diagram D is an $sl(2)$ web, which is an oriented bivalent planar graph whose vertices are either "sources" or "sinks." A resolution Γ might have a component with no vertices, that is, a closed oriented loop. For each vertex of an $sl(2)$ web there is an ordering of the two adjacent edges meeting at that vertex, in the sense that

FIGURE 1: An example of a $(4, 4)$ tangle.

$$\boxed{T} \otimes \boxed{T'} = \boxed{T \quad T'} \quad \boxed{T} \circ \boxed{T'} = \boxed{\begin{array}{c} T \\ T' \end{array}}$$

FIGURE 2: Tangle operations.

one edge is the "preferred" edge of that vertex. Throughout the paper, we represent a bivalent vertex with a red triangle pointing toward the preferred edge of that vertex.

3. The Category of Oriented Tangles

An (m, n) tangle T is an embedding of a finite collection of arcs (homeomorphic to the interval $[0, 1]$) and circles in $\mathbb{R}^2 \times [0, 1]$ such that the target n endpoints of T lie on $\mathbb{R}^2 \times \{1\}$ and the source m endpoints of T lie on $\mathbb{R}^2 \times \{0\}$. Knots and links are $(0, 0)$ tangles and braids with n strands are the most well-known class of (n, n) tangles. The number of arcs in an (m, n) tangle is $(m + n)/2$; thus, $m + n$ is required to be even.

A diagram of an (m, n) tangle is a regular projection of the tangle into $\mathbb{R} \times [0, 1]$. Two tangles are *equivalent* (or ambient isotopic) if one can be transformed into the other by an isotopy of $\mathbb{R}^2 \times [0, 1]$ which fixes the boundary.

We will work with oriented tangles, that is, tangles whose components are oriented. If T is an oriented (m, n) tangle, then the orientation of an arc in T induces orientations of its endpoints as follows: if an arc is oriented up (resp., down) near an endpoint of T , then the endpoint receives the positive orientation 1 (resp., the negative orientation -1). Thus, to each oriented (m, n) tangle T we associate two sequences of numbers, $(a_1, a_2, \dots, a_m)$ and $(b_1, b_2, \dots, b_n)$, denoted source T and target T , respectively, where $a_i = \pm 1$ and $b_i = \pm 1$. If $m = 0$ (resp., $n = 0$), then source T (resp., target T) is the empty sequence $\emptyset$. In Figure 1 we provide an example of a $(4, 4)$ tangle diagram T with source $T = (-1, 1, 1, -1)$ and target $T = (1, -1, -1, 1)$.

To avoid clutter, we will omit labeling the source and target of an oriented tangle, since these are clear from the orientations of tangle's components.

There are two operations on tangles, namely, the *tensor product* $\otimes$ and the *composition* $\circ$, which are explained in Figure 2. The composition $T \circ T'$ is defined only when source $T = \text{target } T'$.

We will denote the category of oriented tangles as **OTa**. The objects of this category are finite sequences of 1's and -1 's, together with the empty sequence $\emptyset$. A *morphism* from $A = (a_1, a_2, \dots, a_m)$ to $B = (b_1, b_2, \dots, b_n)$ is an isotopy type of

an oriented (m, n) tangle T with source $T = A$ and target $T = B$. The category $\mathbf{OTa}$ is a monoidal (or tensor) category, where the tensor product of objects is the concatenation of sequences:

$$(a_1, a_2, \dots, a_m) \otimes (b_1, b_2, \dots, b_n) = (a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n), \quad (3)$$

and the tensor product of morphisms is the tensor product of tangles. Observe that the empty sequence is the *identity object* in $\mathbf{OTa}$.

Turaev [10] showed that the morphisms in $\mathbf{OTa}$ are generated under composition and tensor product by the following morphisms:

$$\uparrow, \downarrow, \curvearrowright, \curvearrowleft, \smile, \frown, \nearrow, \nwarrow, \text{ and } \nwarrow \nearrow \quad (4)$$

subject to the relations (a)–(h) listed below

$$\begin{aligned} (a) \quad & (\curvearrowright \uparrow) \circ (\uparrow \smile) = \uparrow = (\uparrow \curvearrowleft) \circ (\frown \uparrow), \\ (b) \quad & (\curvearrowleft \downarrow) \circ (\downarrow \smile) = \downarrow = (\downarrow \curvearrowright) \circ (\smile \downarrow), \\ & (\downarrow \curvearrowright) \circ (\downarrow \uparrow \curvearrowleft \downarrow) \circ (\downarrow \nearrow \downarrow \downarrow) \\ & \quad \circ (\downarrow \smile \uparrow \downarrow \downarrow) \circ (\smile \downarrow \downarrow) \\ & = \nwarrow \swarrow = (\curvearrowleft \downarrow \downarrow) \circ (\downarrow \curvearrowleft \uparrow \downarrow \downarrow) \circ (\downarrow \nearrow \downarrow \downarrow) \\ & \quad \circ (\downarrow \uparrow \smile \downarrow) \circ (\downarrow \smile), \\ (c) \quad & \nearrow \nwarrow = \nwarrow \nearrow = \uparrow \uparrow, \\ (d) \quad & (\nearrow \uparrow) \circ (\uparrow \nwarrow) \circ (\nwarrow \uparrow) \\ & = (\uparrow \nearrow) \circ (\nwarrow \uparrow) \circ (\uparrow \nwarrow), \\ (e) \quad & (\uparrow \curvearrowright) \circ (\nwarrow \downarrow) \circ (\uparrow \frown) = \uparrow, \\ (f) \quad & Y \circ T = \downarrow \uparrow, \\ (g) \quad & T \circ Y = \uparrow \downarrow, \\ (h) \end{aligned}$$

where

$$\begin{aligned} Y &= (\downarrow \uparrow \curvearrowright) \circ (\downarrow \nearrow \downarrow) \circ (\smile \uparrow \downarrow), \\ T &= (\curvearrowleft \uparrow \downarrow) \circ (\downarrow \nwarrow \downarrow) \circ (\downarrow \uparrow \smile). \end{aligned} \quad (5)$$

These relations are illustrated in Figure 3.

FIGURE 3: Relations in the category $\mathbf{OTa}$.

4. The Singular Temperley-Lieb Category

Inspired by the state model for the $sl(2)$ polynomial described in Section 2, we define the *singular Temperley-Lieb category*, which we shall denote by $\mathbf{STL}$. The objects in $\mathbf{STL}$ are the same as those in $\mathbf{OTa}$, that is, finite sequences of 1's and -1 's, together with the empty sequence (which is the identity object in $\mathbf{STL}$). A morphism between objects $A = (a_1, a_2, \dots, a_m)$ and $B = (b_1, b_2, \dots, b_n)$ is an $sl(2)$ web with boundary, considered up to planar isotopy, and can be regarded as a cobordism with source A and target B . We call such morphism a *singular flat* (m, n) tangle, or a *singular flat tangle*. As before, $m + n$ must be even, and we use the same convention as in the case of morphisms in $\mathbf{OTa}$ for labeling the boundary points of a singular flat tangle. An example of a $(6, 4)$ -singular flat tangle with source $(-1, 1, -1, -1, 1, -1)$ and target $(1, -1, 1, 1)$ is depicted in Figure 4.

Observe that a singular flat (m, n) tangle is a resolution (state) corresponding to an oriented (m, n) tangle diagram T , obtained by applying the decomposition rules depicted in (1) to every crossing in T .

4.1. Generators and Relations. The composition and tensor product of singular flat tangles are defined in a similar manner as their analogues in the category $\mathbf{OTa}$ (see Figures 5 and 6).

It is easy to see that the singular Temperley-Lieb category $\mathbf{STL}$ is a monoidal category whose morphisms are generated under composition and tensor product by the morphisms depicted in Figure 7, subject to relations (6)–(15) given below

$$A_- = (\uparrow \curvearrowright) \circ (\uparrow A_+ \downarrow) \circ (\frown \downarrow), \quad (6)$$

FIGURE 4: A singular flat tangle.

FIGURE 5: Composition of morphisms in STL.

$$B_- = \left(\downarrow \curvearrowright \right) \circ \left(\downarrow B_+ \uparrow \right) \circ \left(\curvearrowleft \uparrow \right), \quad (7)$$

$$C_+ = \curvearrowleft \circ \left(B_+ \uparrow \right), \quad C_- = \curvearrowright \circ \left(\uparrow B_+ \right), \quad (8)$$

$$D_+ = \curvearrowright \circ \left(A_+ \downarrow \right), \quad D_- = \curvearrowleft \circ \left(\downarrow A_+ \right), \quad (9)$$

$$E_+ = \left(\uparrow A_+ \right) \circ \curvearrowleft, \quad E_- = \left(A_+ \uparrow \right) \circ \curvearrowright, \quad (10)$$

$$F_+ = \left(\downarrow B_+ \right) \circ \curvearrowright, \quad F_- = \left(B_+ \downarrow \right) \circ \curvearrowleft, \quad (11)$$

$$\left(\curvearrowright \uparrow \right) \circ \left(\uparrow \curvearrowleft \right) = \uparrow = \left(\uparrow \curvearrowright \right) \circ \left(\curvearrowleft \uparrow \right), \quad (12)$$

$$\left(\curvearrowleft \downarrow \right) \circ \left(\downarrow \curvearrowright \right) = \downarrow = \left(\downarrow \curvearrowleft \right) \circ \left(\curvearrowright \downarrow \right), \quad (13)$$

$$B_- \circ A_+ = \downarrow, \quad A_- \circ B_+ = \uparrow, \quad (14)$$

$$A_+ \circ B_- = \uparrow, \quad B_+ \circ A_- = \downarrow. \quad (15)$$

However, by the relations (6)–(11), we see that the only necessary generators are $\downarrow$, $\uparrow$, $\curvearrowright$, $\curvearrowleft$, $\curvearrowright$, $\curvearrowleft$, and, say, A_+ and B_+ .

Relations (14) and (15) are depicted in Figure 8.

We remark that the set of morphisms between objects A and B in the category **STL** is a $\mathbb{Z}[q, q^{-1}]$ module, where we mod out by the relations displayed in Figure 8, up to isotopy. We denote this space by $\text{Hom}_{\text{STL}}(A, B)$.

4.2. More on the Category STL. In this section we study in more depth the singular Temperley-Lieb category and conclude that it is a pivotal category freely generated by two self-dual generators, namely, (1) and (-1) . For that, we need to recall some concepts from category theory.

Let $\mathcal{C}$ be a monoidal category with identity object I . An object A in $\mathcal{C}$ has a *dual*, denoted by A^* , if there exists a pair of morphisms $\eta_A : I \rightarrow A^* \otimes A$ and $\epsilon_A : A \otimes A^* \rightarrow I$, such that

$$\begin{aligned} (\epsilon_A \otimes \text{id}_A) \circ (\text{id}_A \otimes \eta_A) &= \text{id}_A, \\ (\text{id}_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes \text{id}_{A^*}) &= \text{id}_{A^*}. \end{aligned} \quad (16)$$

If every object in $\mathcal{C}$ has a dual, then $\mathcal{C}$ is called an *autonomous* (or *compact*) category. The following holds in an autonomous category, for any two objects A and B :

$$(A \otimes B)^* = B^* \otimes A^*. \quad (17)$$

The *dual of morphism* $\alpha : A \rightarrow B$ is a morphism $\alpha^* : B^* \rightarrow A^*$ that satisfies the following two equations:

$$\begin{aligned} \alpha^* &= (\text{id}_{A^*} \otimes \epsilon_B) \circ (\text{id}_{A^*} \otimes \alpha \otimes \text{id}_{B^*}) \circ (\eta_A \otimes \text{id}_{B^*}), \\ \alpha &= (\epsilon_{B^*} \otimes \text{id}_{A^*}) \circ (\text{id}_{B^*} \otimes \alpha \otimes \text{id}_{A^*}) \circ (\text{id}_{B^*} \otimes \eta_{A^*}). \end{aligned} \quad (18)$$

It is well known that the category **OTa** of oriented tangles is an autonomous category, where $(1)^* = (-1)$ and $(-1)^* = (1)$. To see this, let $\eta_1 = \curvearrowleft$ and $\epsilon_1 = \curvearrowright$, and observe that

$$\curvearrowleft = \uparrow, \quad \curvearrowright = \downarrow \quad (19)$$

are equivalent to (16) for the object $A = (1)$ in **OTa**. Similarly, let $\eta_{-1} = \curvearrowright$ and $\epsilon_{-1} = \curvearrowleft$, and recall that the following holds in **OTa**

$$\curvearrowright = \downarrow, \quad \curvearrowleft = \uparrow \quad (20)$$

which are pictorial representations for (16) corresponding to the object $A = (-1)$.

Since the above relations hold in the category **STL** as well, the following statement holds.

Proposition 1. *The singular Temperley-Lieb category STL is an autonomous category.*

Relations (a)–(b) in **OTa** and equivalently relations (12)–(13) in **STL** imply that $\uparrow$ and $\downarrow$ are dual morphisms (of each other) in both categories, **OTa** and **STL**. Moreover, in **STL** the dual of $\curvearrowleft$ is $\curvearrowright$, and the dual of $\curvearrowright$ is $\curvearrowleft$. Therefore, the dual operation for morphisms in **STL** switches the preferred side of a vertex.

A morphism $\alpha : A \rightarrow B$ in a category $\mathcal{C}$ is said to be *invertible* if there exists a morphism $\beta : B \rightarrow A$ in $\mathcal{C}$ such that $\alpha \circ \beta = \text{id}_B$ and $\beta \circ \alpha = \text{id}_A$. In this case, β is said to be the *inverse* of α and is denoted by α^{-1} . If a morphism $\alpha :$

FIGURE 6: Tensor product of morphisms in **STL**.

FIGURE 7: The generators for STL.

FIGURE 8: The singularity canceling relations.

FIGURE 9: Involution W .

$A \rightarrow B$ is invertible, then A is said to be *isomorphic* to B and denoted by $A \cong B$. An invertible morphism is also called an *isomorphism*.

Observe that relations (14)-(15) in **STL** imply that the morphisms $\begin{array}{c} \uparrow \\ \textcolor{red}{\downarrow} \end{array}$, $\begin{array}{c} \textcolor{red}{\downarrow} \\ \downarrow \end{array}$, $\begin{array}{c} \textcolor{red}{\downarrow} \\ \textcolor{red}{\downarrow} \end{array}$, and $\begin{array}{c} \textcolor{red}{\downarrow} \\ \textcolor{red}{\downarrow} \end{array}$ are invertible, where

$$\left(\begin{array}{c} \uparrow \\ \text{red triangle} \\ \downarrow \end{array} \right)^{-1} = \begin{array}{c} \downarrow \\ \text{red triangle} \\ \uparrow \end{array}, \quad \left(\begin{array}{c} \downarrow \\ \text{red triangle} \\ \uparrow \end{array} \right)^{-1} = \begin{array}{c} \uparrow \\ \text{red triangle} \\ \downarrow \end{array}. \quad (21)$$

Therefore, $(1) \cong (-1)$ in the category **STL**, which implies that the objects in **STL** are *self-dual*; that is, every object is isomorphic to its dual.

A *pivotal category* is an autonomous category $\mathcal{C}$ where $(A^*)^* \cong A$, for every object A in $\mathcal{C}$. Therefore, categories **OTa** and **STL** are examples of pivotal categories. In fact, $(A^*)^* = A$, for every object A in **OTa**.

4.3. The Linear Singular Temperley-Lieb Category. We define the *linear singular Temperley-Lieb category* and denote it by **LSTL**, as the free $\mathbb{Z}[q, q^{-1}]$ -linear category generated by **STL**. The objects of **LSTL** are the same as those of **STL**, and the morphisms are $\mathbb{Z}[q, q^{-1}]$ -linear combinations of morphisms in **STL**. The composition of morphisms in **LSTL** is defined by bilinear extension of the composition of morphisms in **STL**.

This new category is subject to the same relations as in STL, together with relations (22) depicted below

$$\text{clockwise loop} = \text{anticlockwise loop} = \text{two clockwise arcs} = \text{two anticlockwise arcs} = q + q^{-1}. \quad (22)$$

Relations (22) “remove” all circles and piecewise oriented circles with two bivalent vertices (whose red triangles point to each other), and replace them with $q + q^{-1}$.

Observe that the set of morphisms between objects A and B in **LSTL** is a $\mathbb{Z}[q, q^{-1}]$ module, which we denote by $\text{Hom}_{\text{LSTL}}(A, B)$, where we mod out by the relations (22) and those displayed in Figure 8, up to isotopy.

5. The $sl(2)$ Polynomial Revisited

We define a monoidal functor $F : \mathbf{OTa} \rightarrow \mathbf{LSTL}$ from the category of oriented tangles to the linear singular Temperley-Lieb category. The functor F is the identity on objects. Moreover, the functor F is the identity on all of the generating morphisms for $\mathbf{OTa}$, except for $\nearrow$ and $\nwarrow$, for which we have

$$\begin{aligned} F\left(\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}\right) &= q \uparrow \uparrow - q^2 \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}, \\ F\left(\begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array}\right) &= q^{-1} \uparrow \uparrow - q^{-2} \begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array}. \end{aligned} \quad (23)$$

The functor F is well defined, in the sense that all of the defining relations in **OTa** hold for their images under T in **LSTL**. The planar relations (a) and (b) follow since these relations hold in both categories, and since F is the identity on the morphisms that are used to form these relations. The image under F of the first equality (c) equals

$$\begin{aligned} F\left(\text{Diagram 1}\right) &= q \text{Diagram 2} - q^2 \text{Diagram 3} \\ &= q \text{Diagram 4} - q^2 \text{Diagram 5} = F\left(\text{Diagram 6}\right). \end{aligned} \quad (24)$$

The second equality (c) is verified similarly. For relation (d), observe the following:

$$\begin{aligned} F\left(\text{Diagram 7} \circ \text{Diagram 8}\right) &= F\left(\text{Diagram 9}\right) \circ F\left(\text{Diagram 10}\right) \\ &= \left(q \text{Diagram 11} - q^{-2} \text{Diagram 12}\right) \\ &\quad \circ \left(q^{-1} \text{Diagram 13} - q^{-2} \text{Diagram 14}\right) \\ &= \left(\text{Diagram 15}\right) \circ \left(\text{Diagram 16}\right) - q^{-1} \left(\text{Diagram 17}\right) \\ &\quad \circ \left(\text{Diagram 18}\right) - q \left(\text{Diagram 19}\right) \circ \left(\text{Diagram 20}\right) \\ &\quad + \left(\text{Diagram 21}\right) \circ \left(\text{Diagram 22}\right). \end{aligned} \quad (25)$$

Therefore, we have

$$\begin{aligned} F\left(\text{Diagram 23}\right) &= \text{Diagram 24} - q^{-1} \text{Diagram 25} - q \text{Diagram 26} + \text{Diagram 27} \\ &= \text{Diagram 28} - q^{-1} \text{Diagram 29} - q \text{Diagram 30} + (q + q^{-1}) \text{Diagram 31} \\ &= \text{Diagram 32} = F\left(\text{Diagram 33}\right), \end{aligned} \quad (26)$$

which verifies that the first part of relation (d) in **OTa** holds for its image in **LSTL**. The second relation (d) is verified in a similar manner.

Now we take a look at relation (e). The image of the left hand side of the equality equals

$$\begin{aligned} F\left(\text{Diagram 34}\right) &= q \left[q \left(q \text{Diagram 35} - q^2 \text{Diagram 36} \right) \right. \\ &\quad \left. - q^2 \left(q \text{Diagram 37} - q^2 \text{Diagram 38} \right) \right] \\ &\quad - q^2 \left[q \left(q \text{Diagram 39} - q^2 \text{Diagram 40} \right) \right. \\ &\quad \left. - q^2 \left(q \text{Diagram 41} - q^2 \text{Diagram 42} \right) \right] \\ &= q^3 \text{Diagram 43} - q^4 \text{Diagram 44} - q^4 \text{Diagram 45} + q^5 \text{Diagram 46} \\ &\quad - q^4 \text{Diagram 47} + q^5 \text{Diagram 48} + q^5 \text{Diagram 49} - q^6 \text{Diagram 50} \\ &= q^3 \text{Diagram 51} - 2q^4 \text{Diagram 52} - q^4 \text{Diagram 53} + q^5 \text{Diagram 54} \\ &\quad + q^5 \left(q \text{Diagram 55} + q^{-1} \text{Diagram 56} \right) \\ &\quad + q^5 \text{Diagram 57} - q^6 \text{Diagram 58} \\ &= q^3 \text{Diagram 59} - q^4 \text{Diagram 60} - q^4 \text{Diagram 61} \\ &\quad + q^5 \text{Diagram 62} + q^5 \text{Diagram 63}. \end{aligned} \quad (27)$$

Applying similar computations to the right hand side of relation (e), we obtain

$$\begin{aligned} F\left(\text{Diagram 64}\right) &= q^3 \text{Diagram 65} - q^4 \text{Diagram 66} - q^4 \text{Diagram 67} \\ &\quad + q^5 \text{Diagram 68} + q^5 \text{Diagram 69}, \end{aligned} \quad (28)$$

And, therefore, $F\left(\text{Diagram 34}\right) = F\left(\text{Diagram 64}\right)$.

For relation (f), we have

$$\begin{aligned} F\left(\text{Diagram 70}\right) &= q \text{Diagram 71} - q^2 \text{Diagram 72} = q(q + q^{-1}) \text{Diagram 73} - q^2 \text{Diagram 74} \\ &= (q^2 + 1 - q^2) \text{Diagram 75} = \text{Diagram 76} = F\left(\text{Diagram 77}\right), \end{aligned} \quad (29)$$

as desired. Finally, we verify relation (g):

$$\begin{aligned}
 F\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array}\right) &= q\left(q^{-1}\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - q^{-2}\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array}\right) \\
 &\quad - q^2\left(q^{-1}\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} - q^{-2}\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array}\right) \\
 &= (q + q^{-1})\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} - q^{-1}\begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} - q\begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} + \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \\
 &= \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} = F\left(\begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array}\right).
 \end{aligned} \tag{30}$$

Relation (h) is verified similarly, and we leave it for the reader. Therefore, the functor F is well defined.

Our construction implies that if T is an oriented (m, n) tangle, then $F(T)$ is an ambient isotopy invariant for T . Moreover, an oriented link L is a $(0, 0)$ tangle, and $F(L)$ is an ambient isotopy invariant of L which equals the $sl(2)$ polynomial of L . Thus, our functor F provides another way of defining the $sl(2)$ link invariant.

6. Polynomial Invariants for Tangles

The scope of this section is to show how to use the geometric categories **STL** and **LSTL** to construct invariants for tangles.

6.1. The Case of (n, n) Tangle. In this section we restrict our attention to endomorphisms in **OTa**, **STL**, and **LSTL**. Recall that an *endomorphism* in a category $\mathcal{C}$ is morphism $\alpha : A \rightarrow A$ whose source and target are the same object in $\mathcal{C}$.

The *trace* of an endomorphism $\alpha : A \rightarrow A$ in a pivotal category with identity object I is the endomorphism of I given by the formula

$$\text{tr}(\alpha) = \epsilon_A \circ (\alpha \otimes \text{id}_{A^*}) \circ \eta_{A^*}. \tag{31}$$

For any endomorphism α in **STL** obtained by applying the functor F to a tangle in **OTa**, its trace, $\text{tr}(\alpha)$, is a polynomial in $\mathbb{Z}[q, q^{-1}]$. We regard tr as a map on the space of endomorphisms $\text{End}_{\text{STL}}(A)$, and extend it by bilinearity to $\text{End}_{\text{LSTL}}(A)$.

Example 2. Consider the following:

$$\begin{aligned}
 \text{tr}\left(\begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array}\right) &= \text{tr}\left(\begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array}\right) + q \text{tr}\left(\begin{array}{c} \text{Diagram 27} \\ \text{Diagram 28} \end{array}\right) \\
 &= \begin{array}{c} \text{Diagram 29} \\ \text{Diagram 30} \end{array} + q \begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \end{array} \\
 &= (q + q^{-1})^3 + q(q + q^{-1})^2 \\
 &= (q + q^{-1})^2(2q + q^{-1}).
 \end{aligned} \tag{32}$$

Definition 3. Given an (n, n) tangle T , define a polynomial $P(T) \in \mathbb{Z}[q, q^{-1}]$ by

$$P(T) = \text{tr}(F(T)). \tag{33}$$

Proposition 4. If D is a diagram of an oriented (n, n) tangle T , then $P(D)$ is an ambient isotopy invariant for T .

Proof. Since the functor $F : \mathbf{OTa} \rightarrow \mathbf{LSTL}$ is well defined, $F(D) = F(D')$ whenever D and D' differ by a Reidemeister move. In this case, $\text{tr}(F(D)) = \text{tr}(F(D'))$, and therefore, $F(T)$ is independent of the diagram D . $\square$

Example 5. Consider the oriented tangle diagram $T = \begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array}$. Applying the functor F , we have the following linear combination of singular flat tangles:

$$F\left(\begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array}\right) = q^3 \begin{array}{c} \text{Diagram 35} \\ \text{Diagram 36} \end{array} - q^4 \begin{array}{c} \text{Diagram 37} \\ \text{Diagram 38} \end{array} + q^5 \begin{array}{c} \text{Diagram 39} \\ \text{Diagram 40} \end{array} + q^4 \begin{array}{c} \text{Diagram 41} \\ \text{Diagram 42} \end{array}. \tag{34}$$

Applying the trace function, we obtain

$$\begin{aligned}
 \text{tr}\left(F\left(\begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array}\right)\right) &= q^3 \begin{array}{c} \text{Diagram 43} \\ \text{Diagram 44} \end{array} - q^4 \begin{array}{c} \text{Diagram 45} \\ \text{Diagram 46} \end{array} \\
 &\quad + q^5 \begin{array}{c} \text{Diagram 47} \\ \text{Diagram 48} \end{array} + q^4 \begin{array}{c} \text{Diagram 49} \\ \text{Diagram 50} \end{array} \\
 &= q^3(q + q^{-1})^3 - q^4(q + q^{-1})^2 \\
 &\quad + q^5(q + q^{-1}) + q^4(q + q^{-1})^2 \\
 &= 2q^6 + 4q^4 + 3q^2 + 1.
 \end{aligned} \tag{35}$$

We have obtained that $P\left(\begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array}\right) = 2q^6 + 4q^4 + 3q^2 + 1$.

6.2. The Case of Arbitrary Tangles. We consider now arbitrary (m, n) tangles and construct a polynomial invariant for them.

Given a singular flat tangle α , where $\alpha : A \rightarrow B$, we define the *involution* of α as the singular flat tangle $W(\alpha) : B \rightarrow A$ obtained by reflecting α about the line $\mathbb{R} \times \{1/2\}$, followed by reversing the orientations of the strands. The preferred side of a vertex is carried forward by the reflection operation. An example is depicted in Figure 9.

We regard W as a map on the space $\text{Hom}_{\text{STL}}(A, B)$

$$W : \text{Hom}_{\text{STL}}(A, B) \longrightarrow \text{Hom}_{\text{STL}}(B, A), \tag{36}$$

and extend it by linearity to $\text{Hom}_{\text{LSTL}}(A, B)$.

Definition 6. Given a singular flat tangle α , define $K(\alpha) = W(\alpha) \circ \alpha$.

Observe that $K : \text{Hom}_{\text{STL}}(A, B) \rightarrow \text{End}_{\text{STL}}(A)$ and that it can be extended to $\text{Hom}_{\text{LSTL}}(A, B)$ by bilinearity. Moreover,

$K(\alpha)$ is an endomorphism of A in **STL**, and thus we can take the trace of it, $\text{tr}(K(\alpha))$.

Definition 7. Given an oriented (m, n) tangle T , define the Laurent polynomial $I(T) \in \mathbb{Z}[q, q^{-1}]$ by

$$I(T) = \text{tr}(K(F(T))). \quad (37)$$

The following statement follows at once from our construction.

Proposition 8. $I(T)$ is an ambient isotopy invariant for oriented tangles.

Example 9. Find the polynomial $I(T)$ for

$$T = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \quad (38)$$

Applying the functor F to T , we have

$$F\left(\begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}\right) = q^{-1} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} - q^{-2} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \quad (39)$$

The involution tangles $W(\alpha)$ of the singular flat tangles α found in the previous step are

$$W\left(\begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}\right) = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}, \quad W\left(\begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}\right) = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \quad (40)$$

Composing all singular flat tangles α obtained in the first step with their corresponding involution tangle $W(\alpha)$, we obtain

$$q^{-1} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} - q^{-2} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \quad (41)$$

Finally, applying the trace function and the relations (22), we have

$$q^{-1} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} - q^{-2} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} = q^{-1}(q + q^{-1})^3 - q^{-2}(q + q^{-1})^3. \quad (42)$$

Therefore, $I(T) = (q + q^{-1})^3(q^{-1} - q^{-2})$.

7. Ribbon Graphs and Singular Flat Tangles

A *band* is the image of the square $[0, 1] \times [0, 1]$ under its embedding in $\mathbb{R}^3$, and an *annulus* is the image of the cylinder $S^1 \times [0, 1]$ under its embedding in $\mathbb{R}^3$. The core of the band (equivalently the core of the annulus) is the image of the segment $\{1/2\} \times [0, 1]$ (equivalently $S^1 \times \{1/2\}$) under this embedding.

Let m and n be nonnegative integers with $m + n$ even. A *ribbon (m, n) graph* is an oriented surface S embedded

in $\mathbb{R}^2 \times [0, 1]$ and decomposed into $(m + n)/2$ bands and a finite collection of annuli. The choice of orientation for S is equivalent to a choice of one side of S . Rotating a band (or an annulus) by 180° around its core results in the same band (or annulus) with the opposite orientation.

A ribbon graph is called *directed* if the cores of its bands and annuli are equipped with directions. Note that the cores of the bands and annuli of the surface S are provided with opposite directions on the two sides of S .

With each directed ribbon (m, n) graph we associate two sequences $(a_1, a_2, \dots, a_m)$ and $(b_1, b_2, \dots, b_n)$ of ± 1 in a similar fashion as we did for oriented tangles and singular flat tangles.

We can regard a singular flat (m, n) tangle α as a directed ribbon (m, n) graph S . Since a bivalent vertex in α is a singularity where orientations disagree, it corresponds to a half-twist in a band or an annulus of the ribbon graph S . If the red triangle corresponding to a bivalent vertex in α is oriented up or to the right, then the half-twist of the corresponding band/annulus is a right twist in the neighborhood of the vertex. Otherwise, it is a left twist:

$$\begin{array}{cc} \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \end{array} \quad (43)$$

As explained in Section 4.2, the above generators for **STL** are isomorphisms, and thus $(1) \cong (-1)$. It implies that the category **STL** is self-dual, where there exists an isomorphism $h_A : A \rightarrow A^*$ for every object A (represented as a half-twist).

For $A = (1)$ let $h_{(1)} = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}$, and for $A = (-1)$ let $h_{(-1)} = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}$.

Observe that if $h_{(-1)} = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}$, then its inverse is $h_{(-1)}^{-1} = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array}$, and their composition is

$$h_{(-1)}^{-1} \circ h_{(-1)} \rightsquigarrow \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \curvearrowright \text{---} \\ \text{---} \end{array} \rightsquigarrow \text{id}_{(-1)}. \quad (44)$$

We remark that the self-duality imposes a *braiding* on **STL** via the isomorphisms $c_{A,B} : A \otimes B \rightarrow B \otimes A$ given by

$$c_{A,B} := A \otimes B \xrightarrow{h_{A \otimes B}} (A \otimes B)^* \xrightarrow{\cong} B^* \otimes A^* \xrightarrow{h_B^{-1} \otimes h_A^{-1}} B \otimes A, \quad (45)$$

for any pair of objects A and B .

Moreover, for every object A in **STL**, there is an isomorphism $\theta_{A^*} : A^* \rightarrow A^*$ given by $\theta_{A^*} := h_{A^*}^* \circ h_{A^*}$ and

represented as a full twist on A^* . For example, if $A = (1)$, then $A^* = (-1)$ and θ_{A^*} is represented as

$$\theta_{(-1)} = \begin{array}{c} \text{Y} \\ \text{red triangle} \\ \text{X} \end{array} \circ \begin{array}{c} \text{Y} \\ \text{red triangle} \\ \text{X} \end{array} \rightsquigarrow \begin{array}{c} \text{Y} \\ \text{red triangle} \\ \text{X} \end{array} \quad (46)$$

Note that $\theta_{A^*} = (\theta_A)^*$ and $c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B} = \theta_{A \otimes B}$ for any objects A, B in **STL**, which implies that the category **STL** is *tortile* (see [11]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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