Research Article

Boundedness of Singular Integral Operators with Variable Kernels on Weighted Weak Hardy Spaces

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Let T_Ω be the singular integral operator with variable kernel $\Omega(x, z)$. In this paper, by using the atomic decomposition theory of weighted weak Hardy spaces, we will obtain the boundedness properties of T_Ω on these spaces, under some Dini type conditions imposed on the variable kernel $\Omega(x, z)$.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, $r \geq 1$, if it satisfies the following conditions:

- (A) for all $\lambda > 0$ and $x, z \in \mathbb{R}^n$, $\Omega(x, \lambda z) = \Omega(x, z)$;
- (B) for any $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0$;
- (C) $\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} := \sup_{x \in \mathbb{R}^n} (\int_{S^{n-1}} |\Omega(x, z')|^r d\sigma(z'))^{1/r} < \infty$,

where $z' = z/|z|$ for any $z \in \mathbb{R}^n \setminus \{0\}$. Set $K(x, z) = \Omega(x, z')/|z|^n$. In this paper, we consider the singular integral operator with variable kernel which is defined by

$$T_\Omega f(x) = \text{P.V.} \int_{\mathbb{R}^n} K(x, x-y) f(y) dy. \quad (1)$$

In [1, 2], Calderón and Zygmund investigated the L^p boundedness of singular integral operators with variable kernels. They found that these operators T_Ω are closely related to the problem about second order elliptic partial differential equations with variable coefficients. We will denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$. In [3], Calderón and Zygmund proved the following theorem.

Theorem A (see [3]). *Let $1 < p, r < \infty$ satisfy*

- (i) $1/r < (1/p') + (1/p'(n-1))$ if $1 < p \leq 2$, or
- (ii) $1/r < (1/p') + (1/p(n-1))$ if $2 \leq p < \infty$.

Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Then there exists a constant $C > 0$ independent of f such that

$$\|T_\Omega(f)\|_{L^p} \leq C \|f\|_{L^p}. \quad (2)$$

In particular, T_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \geq r'$.

In 1971, Muckenhoupt and Wheeden [4] studied the weighted norm inequalities for T_Ω with power weights. In 2008, Lee et al. [5] also considered the weighted boundedness of T_Ω with more general weights and showed that if the kernel $K(x, y)$ satisfies the L^r -Hörmander condition with respect to x and y variables, respectively, then T_Ω is bounded on $L_w^p(\mathbb{R}^n)$. More precisely, they proved the following.

Theorem B (see [5]). *Let $1 < r < \infty$. Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ such that the following two inequalities*

$$\sup_{\substack{x \in \mathbb{R}^n \\ 0 < |y| < R}} \sum_{k=1}^{\infty} (2^k R)^{n/r'} \left(\int_{2^k R \leq |z| < 2^{k+1} R} |K(x, z-y) - K(x, z)|^r dz \right)^{1/r} < \infty,$$

$$\sup_{\substack{x, y \in \mathbb{R}^n \\ 0 < |x-y| < R}} \sum_{k=1}^{\infty} (2^k R)^{n/r'} \left(\int_{2^k R \leq |z| < 2^{k+1} R} |K(x, z) - K(y, z)|^r dz \right)^{1/r} < \infty \quad (3)$$

hold for all $R > 0$. If $r' \leq p < \infty$ and $w \in A_{p/r'}$, then T_Ω is bounded on $L_w^p(\mathbb{R}^n)$.

It should be pointed out that the above L^r -Hörmander conditions on the variable kernels were also considered by Rubio de Francia et al. in [6].

In [7, 8], Ding et al. introduced some definitions about the variable kernel $\Omega(x, z)$ when they studied the $H^1 - L^1$ boundedness of Marcinkiewicz integral. Replacing the condition (C) mentioned above, they strengthened it to the condition

$$(C') \sup_{\substack{x \in \mathbb{R}^n \\ \rho \geq 0}} \left(\int_{S^{n-1}} |\Omega(x + \rho z', z')|^r d\sigma(z') \right)^{1/r} < \infty.$$

For $r \geq 1$, a function $\Omega(x, z)$ is said to satisfy the L^r -Dini condition if the conditions (A), (B), and (C') hold and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty, \quad (4)$$

where $\omega_r(\delta)$ is the integral modulus of continuity of order r of Ω defined by

$$\omega_r(\delta) := \sup_{\substack{x \in \mathbb{R}^n \\ \rho \geq 0}} \left(\int_{S^{n-1}} \sup_{\substack{y' \in S^{n-1} \\ |y' - z'| \leq \delta}} |\Omega(x + \rho z', y') - \Omega(x + \rho z', z')|^r d\sigma(z') \right)^{1/r}. \quad (5)$$

In order to obtain the $H_w^p - L_w^p$ boundedness of T_Ω , Lee et al. [5] generalized the L^r -Dini condition by replacing (4) to the following stronger condition (see also [9]):

$$\int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 \leq \alpha \leq 1. \quad (6)$$

If Ω satisfies (6) for some $r \geq 1$ and $0 \leq \alpha \leq 1$, we say that it satisfies the $L^{r,\alpha}$ -Dini condition. For the special case $\alpha = 0$, it reduces to the L^r -Dini condition. For $0 \leq \beta < \alpha \leq 1$, if Ω satisfies the $L^{r,\alpha}$ -Dini condition, then it also satisfies the $L^{r,\beta}$ -Dini condition. We thus denote by $\text{Din}_\alpha^r(S^{n-1})$ the class of all functions which satisfy the $L^{r,\beta}$ -Dini condition for all $0 < \beta < \alpha$.

Theorem C (see [5]). *Let $0 < \alpha \leq 1$ and $n/(n + \alpha) < p < 1$. Suppose $\Omega \in \text{Din}_\alpha^r(S^{n-1})$ such that (3) hold for a certain large*

number $r > 1$. If $w^{r'} \in A_{(p+(p\alpha/n)-(1/r))r'}$, then there exists a constant $C > 0$ independent of f such that

$$\|T_\Omega(f)\|_{L_w^p} \leq C \|f\|_{H_w^p}. \quad (7)$$

It is easy to check that

$$\text{Din}_\alpha^r(S^{n-1}) \subset \text{Din}_\alpha^s(S^{n-1}), \quad \text{if } 1 \leq s < r < \infty. \quad (8)$$

Then for $0 < \alpha \leq 1$, we define

$$\text{Din}_\alpha^\infty(S^{n-1}) = \bigcap_{r \geq 1} \text{Din}_\alpha^r(S^{n-1}). \quad (9)$$

The main purpose of this paper is to study the corresponding estimates of T_Ω on the weighted weak Hardy spaces $WH_w^p(\mathbb{R}^n)$ (see Section 2 for its definition). We now present our main result as follows.

Theorem 1. *Let $0 < \alpha \leq 1$, $n/(n + \alpha) < p \leq 1$, and $w \in A_{p(1+(\alpha/n))}$. Suppose $\Omega \in \text{Din}_\alpha^\infty(S^{n-1})$ such that (3) hold. Then there exists a constant $C > 0$ independent of f such that*

$$\|T_\Omega(f)\|_{WL_w^p} \leq C \|f\|_{WH_w^p}. \quad (10)$$

2. Notations and Preliminaries

The definition of A_p class was first used by Muckenhoupt [10], Hunt et al. [11], and Coifman and Fefferman [12] in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions and singular integrals. Let w be a nonnegative, locally integrable function defined on $\mathbb{R}^n$; all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C, \quad (11)$$

for every cube $Q \subseteq \mathbb{R}^n$,

where C is a positive constant which is independent of the choice of Q . For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \cdot \text{ess inf}_{x \in Q} w(x), \quad (12)$$

for every cube $Q \subseteq \mathbb{R}^n$.

The smallest value of $C > 0$ such that the above inequalities hold is called the A_p characteristic constant of w and denoted by $[w]_{A_p}$. For the case $p = \infty$, $w \in A_\infty$, if it satisfies the A_p condition for some $1 < p < \infty$.

A weight function w is said to belong to the reverse Hölder class RH_s if there exist two constants $s > 1$ and $C > 0$ such that the following reverse Hölder inequality holds:

$$\left(\frac{1}{|Q|} \int_Q w(x)^s dx \right)^{1/s} \leq C \left(\frac{1}{|Q|} \int_Q w(x) dx \right), \quad (13)$$

for every cube $Q \subseteq \mathbb{R}^n$.

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w . Moreover, if $w \in A_p$ with $1 \leq p < \infty$, then there exists $s > 1$ such that $w \in RH_s$. It follows directly from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for all $1 < s < r$.

Given a cube Q and $\lambda > 0$, λQ stands for the cube with the same center as Q whose side length is λ times that of Q . $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r . For a weight function w and a measurable set E , we denote the Lebesgue measure of E by $|E|$ and set the weighted measure $w(E) = \int_E w(x)dx$.

We give the following results that will be used in the sequel.

Lemma 2 (see [13]). *Let $w \in A_q$ with $q \geq 1$. Then, for any cube Q , there exists an absolute constant $C > 0$ such that*

$$w(2Q) \leq Cw(Q). \quad (14)$$

In general, for any $\lambda > 1$, one has

$$w(\lambda Q) \leq C \cdot \lambda^{nq} w(Q), \quad (15)$$

where C does not depend on Q or λ .

Given a weight function w on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L_w^p(\mathbb{R}^n)$ the weighted space of all functions f satisfying

$$\|f\|_{L_w^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty. \quad (16)$$

When $p = \infty$, $L_w^\infty(\mathbb{R}^n)$ will be taken to mean $L^\infty(\mathbb{R}^n)$ and

$$\|f\|_{L_w^\infty} = \|f\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|. \quad (17)$$

We also denote by $WL_w^p(\mathbb{R}^n)$ the weighted weak L^p space which is formed by all measurable functions f satisfying

$$\|f\|_{WL_w^p} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty. \quad (18)$$

Let us now turn to the weighted weak Hardy spaces. The (unweighted) weak H^p spaces have first appeared in the work of Fefferman et al. [14], which are the intermediate spaces between two Hardy spaces through the real method of interpolation. The atomic decomposition characterization of weak H^1 space on $\mathbb{R}^n$ was given by Fefferman and Soria in [15]. Later, Liu [16] established the weak H^p spaces on homogeneous groups for the whole range $0 < p \leq 1$. The corresponding results related to $\mathbb{R}^n$ can be found in [17]. For the boundedness properties of some operators on weak Hardy spaces, we refer the readers to [18–24]. In 2000, Quek and Yang [25] introduced the weighted weak Hardy spaces $WH_w^p(\mathbb{R}^n)$ and established their atomic decompositions. Moreover, by using the atomic decomposition theory of $WH_w^p(\mathbb{R}^n)$, Quek and Yang [25] also obtained the boundedness of Calderón-Zygmund type operators on these weighted spaces.

We write $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz space of all rapidly decreasing infinitely differentiable functions and

$\mathcal{S}'(\mathbb{R}^n)$ to denote the space of all tempered distributions, that is, the topological dual of $\mathcal{S}(\mathbb{R}^n)$. Let $w \in A_\infty$, $0 < p \leq 1$ and $N = [n(q_w/p - 1)]$. Define

$$\mathcal{A}_{N,w} = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N+1} (1 + |x|)^{N+n+1} |D^\alpha \varphi(x)| \leq 1 \right\}, \quad (19)$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}. \quad (20)$$

For any given $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of f is defined by

$$G_w f(x) = \sup_{\varphi \in \mathcal{A}_{N,w}} \sup_{|y-x| < t} |(\varphi_t * f)(y)|. \quad (21)$$

Then we can define the weighted weak Hardy space $WH_w^p(\mathbb{R}^n)$ by $WH_w^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_w f \in WL_w^p(\mathbb{R}^n)\}$. Moreover, we set $\|f\|_{WH_w^p} = \|G_w f\|_{WL_w^p}$.

Theorem 3 (see [25]). *Let $0 < p \leq 1$ and $w \in A_\infty$. For every $f \in WH_w^p(\mathbb{R}^n)$, there exists a sequence of bounded measurable functions $\{f_k\}_{k=-\infty}^\infty$ such that*

- (i) $f = \sum_{k=-\infty}^\infty f_k$ in the sense of distributions;
- (ii) each f_k can be further decomposed into $f_k = \sum_i b_i^k$, where $\{b_i^k\}$ satisfies that

- (a) each b_i^k is supported in a cube Q_i^k with $\sum_i w(Q_i^k) \leq c2^{-kp}$ and $\sum_i \chi_{Q_i^k}(x) \leq c$. Here χ_E denotes the characteristic function of the set E and $c \sim \|f\|_{WH_w^p}^p$;
- (b) $\|b_i^k\|_{L^\infty} \leq C2^k$, where $C > 0$ is independent of i and k ;
- (c) $\int_{\mathbb{R}^n} b_i^k(x) x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq [n(q_w/p - 1)]$.

Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ has a decomposition satisfying (i) and (ii), then $f \in WH_w^p(\mathbb{R}^n)$. Moreover, one has $\|f\|_{WH_w^p}^p \sim c$.

Throughout this paper C always denotes a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

3. Proof of Theorem 1

Following the same arguments as in the proof of Lemma 5 in [26], we can also establish the following lemma on the variable kernel $\Omega(x, z)$ (see [5, 8]).

Lemma 4. *Let $r \geq 1$. Suppose that $\Omega(x, z) \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ satisfies the L^r -Dini condition in Section 1. If there*

exists a constant $0 < \gamma \leq 1/2$ such that $|y| < \gamma R$, then for any $x_0 \in \mathbb{R}^n$, one has

$$\left(\int_{R \leq |x| < 2R} |K(x + x_0, x - y) - K(x + x_0, x)|^r dx \right)^{1/r} \leq C \cdot R^{-n/r'} \left(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_r(\delta)}{\delta} d\delta \right), \quad (22)$$

where the constant $C > 0$ is independent of R and y .

We are now in a position to give the proof of Theorem 1.

Proof of Theorem 1. For any given $\lambda > 0$, we may choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \lambda < 2^{k_0+1}$. For every $f \in WH_w^p(\mathbb{R}^n)$, then by Theorem 3, we can write

$$f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2, \quad (23)$$

where $F_1 = \sum_{k=-\infty}^{k_0} f_k = \sum_{k=-\infty}^{k_0} \sum_i b_i^k$, $F_2 = \sum_{k=k_0+1}^{\infty} f_k = \sum_{k=k_0+1}^{\infty} \sum_i b_i^k$, and $\{b_i^k\}$ satisfies (a)–(c) in Theorem 3. Then we have

$$\begin{aligned} & \lambda^p \cdot w(\{x \in \mathbb{R}^n : |T_\Omega(f)(x)| > \lambda\}) \\ & \leq \lambda^p \cdot w\left(\left\{x \in \mathbb{R}^n : |T_\Omega(F_1)(x)| > \frac{\lambda}{2}\right\}\right) \\ & \quad + \lambda^p \cdot w\left(\left\{x \in \mathbb{R}^n : |T_\Omega(F_2)(x)| > \frac{\lambda}{2}\right\}\right) \\ & = I_1 + I_2. \end{aligned} \quad (24)$$

First we claim that the following inequality holds:

$$\|F_1\|_{L_w^2} \leq C \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2}. \quad (25)$$

In fact, since $\text{supp } b_i^k \subseteq Q_i^k = Q(x_i^k, r_i^k)$ and $\|b_i^k\|_{L^\infty} \leq C2^k$ according to Theorem 3, then it follows directly from Minkowski's inequality that

$$\begin{aligned} \|F_1\|_{L_w^2} & \leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L_w^2} \\ & \leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^\infty} w(Q_i^k)^{1/2}. \end{aligned} \quad (26)$$

For each $k \in \mathbb{Z}$, by using the bounded overlapping property of the cubes $\{Q_i^k\}$ and the fact that $1 - p/2 > 0$, we thus obtain

$$\begin{aligned} \|F_1\|_{L_w^2} & \leq C \sum_{k=-\infty}^{k_0} 2^k \left(\sum_i w(Q_i^k) \right)^{1/2} \\ & \leq C \sum_{k=-\infty}^{k_0} 2^{k(1-p/2)} \|f\|_{WH_w^p}^{p/2} \\ & \leq C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)(1-p/2)} \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2} \\ & \leq C \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2}. \end{aligned} \quad (27)$$

Since $w \in A_{p(1+(\alpha/n))}$ and $p(1+(\alpha/n)) \leq 1+(\alpha/n) < 2$, then we have $w \in A_2$. In this case, we know that there exists a number $s > 1$ such that $w \in RH_s$. More specifically, by using the sharp reverse Hölder's inequality for A_2 weights obtained recently in [27], we find that for $w \in A_2$,

$$w \in RH_s \quad \text{with } s = 1 + \frac{1}{2^{n+5}[w]_{A_2}}. \quad (28)$$

Observe that $\Omega \in \text{Din}_\alpha^\infty(S^{n-1})$, then we are able to find a positive number $r > 1$ large enough such that $r > \max\{s', (2n)/(n-\alpha)\}$ and $\Omega \in \text{Din}_\alpha^r(S^{n-1})$. By the choice of r , we can easily check that $2/r' > 1 + \alpha/n \geq p(1 + \alpha/n)$, which implies $w \in A_{2/r'}$. Hence, by using Theorem B, we know that T_Ω is bounded on $L_w^2(\mathbb{R}^n)$. This fact together with Chebyshev's inequality and (25) yields

$$\begin{aligned} I_1 & \leq \lambda^p \cdot \frac{4}{\lambda^2} \|T_\Omega(F_1)\|_{L_w^2}^2 \\ & \leq C \cdot \lambda^{p-2} \|F_1\|_{L_w^2}^2 \\ & \leq C \|f\|_{WH_w^p}^p. \end{aligned} \quad (29)$$

We now turn our attention to the estimate of I_2 . Setting

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q}_i^k, \quad (30)$$

where $\widetilde{Q}_i^k = Q(x_i^k, \tau^{(k-k_0)/(n+\alpha)}(2\sqrt{n})r_i^k)$ and τ is a fixed positive number such that $1 < \tau < 2$. Thus, we can further decompose I_2 as

$$\begin{aligned} I_2 & \leq \lambda^p \cdot w\left(\left\{x \in A_{k_0} : |T_\Omega(F_2)(x)| > \frac{\lambda}{2}\right\}\right) \\ & \quad + \lambda^p \cdot w\left(\left\{x \in (A_{k_0})^c : |T_\Omega(F_2)(x)| > \frac{\lambda}{2}\right\}\right) \\ & = I_2' + I_2''. \end{aligned} \quad (31)$$

Let us first deal with the term I_2' . Since $w \in A_{p(1+(\alpha/n))}$, then by Lemma 2, we can deduce that

$$\begin{aligned} I_2' & \leq \lambda^p \sum_{k=k_0+1}^{\infty} \sum_i w(\widetilde{Q}_i^k) \\ & \leq C \cdot \lambda^p \sum_{k=k_0+1}^{\infty} \tau^{(k-k_0)p} \sum_i w(Q_i^k) \\ & \leq C \|f\|_{WH_w^p}^p \sum_{k=k_0+1}^{\infty} \left(\frac{\tau}{2}\right)^{(k-k_0)p} \\ & \leq C \|f\|_{WH_w^p}^p. \end{aligned} \quad (32)$$

On the other hand, it follows immediately from Chebyshev's inequality that

$$\begin{aligned} I_2''' &\leq 2^p \int_{(A_{k_0})^c} |T_\Omega(F_2)(x)|^p w(x) dx \\ &\leq 2^p \sum_{k=k_0+1}^{\infty} \sum_i \int_{(\widetilde{Q_i^k})^c} |T_\Omega(b_i^k)(x)|^p w(x) dx \quad (33) \\ &= 2^p \sum_{k=k_0+1}^{\infty} \sum_i J_i^k. \end{aligned}$$

Now denote $\tau_{i,\ell}^k = 2^{\ell-1} \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k$, $\widetilde{Q_{i,\ell}^k} = Q(x_i^k, \tau_{i,\ell}^k)$, and

$$E_{i,\ell}^k = \{x \in \mathbb{R}^n : \tau_{i,\ell}^k \leq |x - x_i^k| < 2\tau_{i,\ell}^k\}, \quad \ell = 1, 2, \dots \quad (34)$$

An application of Hölder's inequality gives us that

$$\begin{aligned} J_i^k &\leq \sum_{\ell=1}^{\infty} \int_{E_{i,\ell}^k} |T_\Omega(b_i^k)(x)|^p w(x) dx \\ &\leq \sum_{\ell=1}^{\infty} \left(\int_{E_{i,\ell}^k} w(x) dx \right)^{1-p} \left(\int_{E_{i,\ell}^k} |T_\Omega(b_i^k)(x)|^p w(x) dx \right)^p. \quad (35) \end{aligned}$$

Let $q = p(1 + (\alpha/n))$ for simplicity. Then for any $n/(n + \alpha) < p \leq 1$ and $w \in A_q$ with $q > 1$, we can easily see that $[n(q_w/p - 1)] = 0$. Hence, by the cancellation condition of $b_i^k \in L^\infty(\mathbb{R}^n)$, we get

$$\begin{aligned} &\int_{E_{i,\ell}^k} |T_\Omega(b_i^k)(x)| w(x) dx \\ &= \int_{E_{i,\ell}^k} \left| \int_{Q_i^k} [K(x, x-y) - K(x, x-x_i^k)] b_i^k(y) dy \right| w(x) dx \\ &\leq \int_{Q_i^k} \left\{ \int_{E_{i,\ell}^k} |K(x, x-y) - K(x, x-x_i^k)| w(x) dx \right\} |b_i^k(y)| dy \\ &\leq \|b_i^k\|_{L^\infty} |Q_i^k| \left(\int_{E_{i,\ell}^k} |K(x, x-y) - K(x, x-x_i^k)| w(x) dx \right). \quad (36) \end{aligned}$$

When $y \in Q_i^k$ and $x \in (\widetilde{Q_i^k})^c$, then a trivial computation shows that

$$|x - x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k > \sqrt{n} r_i^k \geq 2|y - x_i^k|. \quad (37)$$

We also observe that $w \in RH_s$ and $r > s'$, then $w \in RH_{r'}$. Using Hölder's inequality, the estimate (37), and Lemma 4, we

can see that for any $y \in Q_i^k$, the integral of the above expression is dominated by

$$\begin{aligned} &\left(\int_{E_{i,\ell}^k} |K(x, x-y) - K(x, x-x_i^k)|^r dx \right)^{1/r} \\ &\quad \times \left(\int_{E_{i,\ell}^k} w(x)^{r'} dx \right)^{1/r'} \\ &\leq C \cdot \frac{w(\widetilde{Q_{i,\ell+1}^k})}{|Q_{i,\ell+1}^k|^{1/r}} \left(\int_{\tau_{i,\ell}^k \leq |x| < 2\tau_{i,\ell}^k} |K(x + x_i^k, x - (y - x_i^k)) \right. \\ &\quad \left. - K(x + x_i^k, x) \right|^r dx \Big)^{1/r} \\ &\leq C \cdot \frac{w(\widetilde{Q_{i,\ell+1}^k})}{|Q_{i,\ell+1}^k|^{1/r}} \cdot (\tau_{i,\ell}^k)^{-n/r'} \\ &\quad \times \left(\frac{|y - x_i^k|}{\tau_{i,\ell}^k} + \int_{|y - x_i^k|/2\tau_{i,\ell}^k}^{|y - x_i^k|/\tau_{i,\ell}^k} \frac{\omega_r(\delta)}{\delta} d\delta \right) \\ &\leq C \cdot \frac{w(\widetilde{Q_{i,\ell+1}^k})}{|Q_{i,\ell+1}^k|^{1/r}} \cdot (\tau_{i,\ell}^k)^{-n/r'} \\ &\quad \times \left(\frac{|y - x_i^k|}{\tau_{i,\ell}^k} + \frac{|y - x_i^k|^\alpha}{(\tau_{i,\ell}^k)^\alpha} \times \int_{|y - x_i^k|/2\tau_{i,\ell}^k}^{|y - x_i^k|/\tau_{i,\ell}^k} \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta \right) \\ &\leq C \cdot \frac{w(\widetilde{Q_{i,\ell+1}^k})}{|Q_{i,\ell+1}^k|} \\ &\quad \cdot \left(\frac{1}{2^\ell \tau^{(k-k_0)/(n+\alpha)}} + \left[\frac{1}{2^\ell \tau^{(k-k_0)/(n+\alpha)}} \right]^\alpha \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta \right) \\ &\leq C \cdot \left(1 + \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta \right) \cdot \frac{w(\widetilde{Q_{i,\ell+1}^k})}{|Q_{i,\ell+1}^k|} \left(\frac{1}{2^\ell \tau^{(k-k_0)/(n+\alpha)}} \right)^\alpha. \quad (38) \end{aligned}$$

Recall that $\|b_i^k\|_{L^\infty} \leq C2^k$. From the above estimate (38), it follows that

$$\begin{aligned} J_i^k &\leq C \cdot 2^{kp} \sum_{\ell=1}^{\infty} w(\widetilde{Q_{i,\ell+1}^k}) \left(\frac{|Q_i^k|}{|Q_{i,\ell+1}^k|} \right)^p \\ &\quad \times \left(\frac{1}{2^\ell \tau^{(k-k_0)/(n+\alpha)}} \right)^{\alpha p}. \quad (39) \end{aligned}$$

In addition, for $w \in A_q$ with $q > 1$, then we can take a sufficiently small number $\varepsilon > 0$ such that $w \in A_{q-\varepsilon}$. Thus, by using Lemma 2 again, we finally obtain

$$\begin{aligned} J_i^k &\leq C \cdot 2^{kp} w(Q_i^k) \\ &\quad \times \sum_{\ell=1}^{\infty} \left(2^{\ell} \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} \right)^{n(q-\varepsilon)-np} \\ &\quad \times \left(\frac{1}{2^{\ell} \tau^{(k-k_0)/(n+\alpha)}} \right)^{\alpha p} \\ &\leq C \cdot 2^{kp} w(Q_i^k) \sum_{\ell=1}^{\infty} \left(2^{\ell} \tau^{(k-k_0)/(n+\alpha)} \right)^{-n\varepsilon} \\ &\leq C \cdot 2^{kp} w(Q_i^k) \left(\tau^{(k-k_0)/(n+\alpha)} \right)^{-n\varepsilon}. \end{aligned} \quad (40)$$

Therefore

$$\begin{aligned} I_2'' &\leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} \left(\tau^{(k-k_0)/(n+\alpha)} \right)^{-n\varepsilon} w(Q_i^k) \\ &\leq C \|f\|_{WH_w^p}^p \sum_{k=k_0+1}^{\infty} \left(\tau^{(k-k_0)/(n+\alpha)} \right)^{-n\varepsilon} \\ &\leq C \|f\|_{WH_w^p}^p. \end{aligned} \quad (41)$$

Combining the above inequality (41) with (29) and (32), and then taking the supremum over all $\lambda > 0$, we conclude the proof of Theorem 1. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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