Research Article

A New Optimal Eighth-Order Ostrowski-Type Family of Iterative Methods for Solving Nonlinear Equations

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Received 6 September 2013; Accepted 2 October 2013; Published 13 March 2014

Academic Editors: Q. Guo and Z.-Y. Li

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Based on Ostrowski's method, a new family of eighth-order iterative methods for solving nonlinear equations by using weight function methods is presented. Per iteration the new methods require three evaluations of the function and one evaluation of its first derivative. Therefore, this family of methods has the efficiency index which equals 1.682. Kung and Traub conjectured that a multipoint iteration without memory based on n evaluations could achieve optimal convergence order 2^{n-1} . Thus, we provide a new class which agrees with the conjecture of Kung-Traub for $n = 4$. Numerical comparisons are made to show the performance of the presented methods.

1. Introduction

In this paper, we consider iterative methods to find a simple root α of a nonlinear equation $f(x) = 0$, where $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function on an open interval D . This problem is a prototype for many nonlinear numerical problems. Newton's method is the most widely used algorithm for dealing with such problems, and it is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (1)$$

which converges quadratically in some neighborhood of α (see [1, 2]).

To improve the local order of convergence, many modified methods have been proposed in the open literature; see [3–17] and references therein. King [3] developed a one-parameter family of fourth-order methods, which is written as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \end{aligned} \quad (2)$$

where $\beta \in \mathbb{R}$ is a constant. In particular, the famous Ostrowski's method [2] is a member of this family for the case $\beta = 0$, and it can be written as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}. \end{aligned} \quad (3)$$

Kung and Traub [15] who conjectured that an iteration method without memory based on n evaluations of f or its derivatives could achieve optimal convergence order 2^{n-1} . Thus, the optimal order for a method with 3 functional evaluations per step would be 4. King's method [3], Ostrowski's method, and Jarrat's method [16] are some of the optimal fourth-order methods, because they only perform three functional evaluations per step. Recently, based on Ostrowski's or King's methods, some higher-order multipoint methods have been proposed for solving nonlinear equations. Bi et al. developed a scheme of optimal order of convergence eight [17], estimating the first derivative of the function in

the second and third steps and constructing a weight function as well in the following form:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)} \\ &\quad \times \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n](z_n - y_n)}, \end{aligned} \quad (4)$$

where $\gamma \in \mathbb{R}$ is constant. Liu and Wang in [18] presented the following family of optimal order eight:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \left[\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \right)^2 + \frac{f(z_n)}{f(y_n) - \mu f(z_n)} \right. \\ &\quad \left. + \frac{4f(z_n)}{f(x_n) + \lambda f(z_n)} \right] \frac{f(z_n)}{f'(x_n)}, \end{aligned} \quad (5)$$

where μ and λ are in $\mathbb{R}$. J. R. Sharma and R. Sharma in [12] produce optimal eighth-order method in the following form:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)} \right)^2 \right] \\ &\quad \times \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]}. \end{aligned} \quad (6)$$

In this paper, based on Ostrowski's method, we present a new family of optimal eighth-order methods by using the method of weight functions and we apply a few weight functions to construct families of iterative methods with high convergence and high efficiency index. The convergence analysis is provided to establish their eighth order of convergence. In terms of computational cost, they require the evaluations of only three functions and one first-order derivative per iteration. This gives 1.682 as efficiency index of the presented methods. The new methods are comparable with Bi et al.'s method, Liu and Wang's method, and Sharma's method. The efficacy of the methods is tested on a number of numerical examples.

We use the symbols $\rightarrow$, O , and $\sim$ according to the following conventions [1]. If $\lim_{x_n \rightarrow \infty} g(x_n) = C$, we write $g(x_n) \rightarrow C$ or $g \rightarrow C$. If $\lim_{x \rightarrow a} g(x) = C$, we write $g(x) \rightarrow C$ or $g \rightarrow C$. If $f/g \rightarrow C$, where C is a nonzero constant, we write $f = O(g)$ or $f \sim Cg$. Let $f(x)$ be a function defined on an interval I , where I is the smallest interval containing $k + 1$ distinct nodes $x_1, x_2, \dots, x_k$. The divided difference $f[x_0, x_1, \dots, x_k]$ with the k th order is defined as follows: $f[x_0] = f(x_0)$

$$\begin{aligned} f[x_0] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \dots, f[x_0, x_1, \dots, x_k] \\ &= \frac{f[x_1, x_2, \dots, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_0}. \end{aligned} \quad (7)$$

Moreover, we recall the definition of efficiency index (EI) as $E = p^{1/n}$, where p is the order of convergence and n is the total number of function evaluations per iteration.

2. The Methods and Analysis of Convergence

In order to construct new methods, we consider an iteration scheme of the form

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned} \quad (8)$$

This scheme includes three evaluations of the function and two evaluations of its first derivative. Therefore, this scheme has efficiency index equal to 1.516. To improve the efficiency index, we approximate $f'(z_n)$ by the divided difference [12]

$$f'(z_n) \approx \frac{f[x_n, z_n] f[y_n, z_n]}{f[x_n, y_n]}. \quad (9)$$

Now, we present a new family of optimal eighth-order Ostrowski-type iterative methods by using the method of weight functions as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \{K(t_1) \times L(t_2) \times P(t_3)\} \\ &\quad \times \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]}, \end{aligned} \quad (10)$$

where $K(t_1)$, $L(t_2)$, and $P(t_3)$ are three real-valued weight functions when

$$t_1 = \frac{f(z)}{f(x)}, \quad t_2 = \frac{f(y)}{f(x)}, \quad t_3 = \frac{f(z)}{f(y)}, \quad (11)$$

without the index n , should be chosen such that the order of convergence arrives at the optimal level eight. If x_0 is an approximation to the zero α of f , then the corresponding iterative method is defined by

$$x_{k+1} = g(x_k), \quad k = 0, 1, \dots \quad (12)$$

Regarding (10), let us define the errors

$$\begin{aligned} e &= x - \alpha, & e_y &= y - \alpha, \\ e_z &= z - \alpha, & e_1 &= g(x) - \alpha. \end{aligned} \quad (13)$$

We will use Taylor's expansion about the zero α to express $f(x)$, $f(y)$, and $f(z)$ as series in e , e_y , and e_z , respectively. Then, according to (11), we represent t_1 , t_2 , and t_3 as Taylor's polynomials in e .

Assume that x is sufficiently close to the zero α of f then t_1 , t_2 and t_3 are close enough to 0. Let us represent real functions K , L , and P appearing in (10) by Taylor's series about 0,

$$\begin{aligned} K(t_1) &= K(0) + K'(0)t_1 + \dots, \\ L(t_2) &= L(0) + L'(0)t_2 + \frac{L''(0)}{2}t_2^2 \\ &\quad + \frac{L'''(0)}{6}t_2^3 + \frac{L^{(4)}(0)}{24}t_2^4 + \dots, \\ P(t_3) &= P(0) + P'(0)t_3 + \frac{P''(0)}{2}t_3^2 + \dots. \end{aligned} \quad (14)$$

Symbolic computations reported here to find candidates for K , L , and P have been carried out in a *Mathematica* 8.0 environment. We will find the coefficients $K(0)$, $K'(0)$, $\dots$, $P''(0)$ of the developments (14) using a simple program in *Mathematica* 8.0 and an interactive approach explained by the comments C1–C5. First, let us introduce the following abbreviations used in this program (see Algorithm 1):

$$\begin{aligned} c_k &= \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, & e &= x - \alpha, & e_1 &= g(x) - \alpha. \\ f[e] &= f(x), & f[e_y] &= f(y), & f[e_z] &= f(z), \\ f'[e] &= f'(x), & f_1a &= f'(\alpha), \\ K0 &= K(0), & K1 &= K'(0), & K2 &= K''(0), \\ K3 &= K'''(0), & K4 &= K^{(4)}(0), \\ L0 &= L(0), & L1 &= L'(0), & L2 &= L''(0), \\ L3 &= L'''(0), & L4 &= L^{(4)}(0), \\ P0 &= P(0), & P1 &= P'(0), & P2 &= P''(0), \\ P3 &= P'''(0), & P4 &= P^{(4)}(0). \end{aligned} \quad (15)$$

Comment C1: from the expression of the error $e_1 = g(x) - \alpha$, we observe that e_1 is of the form

$$\begin{aligned} e_1 &= g(x) - \alpha \\ &= a_4e^4 + a_5e^5 + a_6e^6 + a_7e^7 + a_8e^8 + O(e^9). \end{aligned} \quad (16)$$

The iterative method $x_{k+1} = g(x_k)$ will have the order of convergence equal to eight if we determine the coefficients of the developments appearing in (14) in such a way that a_4 , a_5 , a_6 , and a_7 (in (16)) vanish. We find these coefficients equalling shaded expressions in boxed formulas to 0. First, from Out [a4], we have

$$-1 + K(0)L(0)P(0) = 0. \quad (17)$$

Without loss of generality, we can take $K(0) = L(0) = P(0) = 1$ and hence, $a_4 = 0$. In what follows, a_5 , a_6 , a_7 , and a_8 are calculated using the already found coefficients.

Comment C2: from Out [a5], we see that the choice $L'(0) = 0$ gives $a_5 = 0$.

Comment C3: we obtain $a_6 = 0$ choosing $L''(0) = P'(0) = 0$.

Comment C4: a_7 vanishes if we choose simultaneously $K'(0) = 1$, $L'''(0) = 0$.

Comment C5: substituting the quantities $K(0)$, $K'(0)$, $\dots$, $P''(0)$ in the expression of e_1 , found in the described interactive procedure, we obtain

$$\begin{aligned} e_1 &= g(x) - \alpha \\ &= -\frac{1}{24} \left(c_2(c_2^2 - c_3) \right. \\ &\quad \times \left((L^{(4)}(0) + 12(-6 + P''(0)))c_2^4 \right. \\ &\quad \left. - 24(-4 + P''(0))c_2^2c_3 \right. \\ &\quad \left. + 12P''(0)c_3^2 - 24c_2c_4 \right) \\ &\quad \times e_n^8 + O(e_n^9), \end{aligned} \quad (18)$$

where $|L'''(0)| < \infty$, $|P''(0)| < \infty$.

According to the above analysis, we have proved the following theorem.

Theorem 1. Assume that $f \in C^5(D)$. Suppose $\alpha \in D$, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. If the initial point x_0 is sufficiently close to α , then the sequence x_n generated by any method of the family (10) has eighth order of convergence to α if K , L , and P are any functions with

$$\begin{aligned} K(0) &= K'(0) = 1, \\ L(0) &= 1, \quad L'(0) = L''(0) = L'''(0) = 0, \\ |L^{(4)}(0)| &< \infty, \\ P(0) &= 1, \quad P'(0) = 0, \quad |P''(0)| < \infty. \end{aligned} \quad (19)$$

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f[e_] = f1a (e^1 + c2 e^2 + c3 e^3 + c4 e^4 + c5 e^5 + c6 e^6 + c7 e^7 + c8 e^8 + c9 e^9);
ey = e - Series [f[e], {e, 0, 8}];
ez = ey - Series [f[e] / (f[e] - 2f[ey]) * f[ey] / f'[e], {e, 0, 8}];
f[x_, y_] := (f[x] - f[y]) / (x - y);
t1 := f[ez] / f[e]; t2 := f[ey] / f[e]; t3 := f[ez] / f[ey];
K = K0 + K1 * t1 + 1/2 * K2 * t1^2 + 1/6 * K3 * t1^3 + 1/24 * K4 * t1^4;
L = L0 + L1 * t2 + 1/2 * L2 * t2^2 + 1/6 * L3 * t2^3 + 1/24 * L4 * t2^4;
P = P0 + P1 * t3 + 1/2 * P2 * t3^2 + 1/6 * P3 * t3^3 + 1/24 * P4 * t3^4;
e1 = ez - Series [K * L * P * (f[ez] f[ey] / (f[e, ez] f[ey, ez])), {e, 0, 8}] // FullSimplify
a4 = Coefficient[e1, e^4] // FullSimplify
C1: Out[a4] = -(-1 + K0L0P0) c2 (c2^2 - c3)
K0 = 1; L0 = 1; P0 = 1; (* Vanish coefficient of e^4 *)
a5 = Coefficient[e1, e^5] // FullSimplify
C2: Out[a5] = (-4 + K0 (4L0 - L1) P0) c2^4 + (8 + K0 (-8L0 + L1) P0) c2^2 c3
      + 2 (-1 + K0L0P0) c3^2 + 2 (-1 + K0L0P0) c2 c4
L1 = 0; (* Vanish coefficient of e^5 *)
a6 = Coefficient[e1, e^6] // FullSimplify
C3: Out[a6] = -1/2 (-20 + K0 (20L0 - 14L1 + L2) P0 + 2K0L0P1) c2^5
      + 1/2 (-60 + K0 (60L0 - 26L1 + L2) P0 + 4K0L0P1) c2^3 c3
      + (18 + K0 (4L1P0 - L0 (18P0 + P1))) c2 c3^2
      + 2 (6 + K0 (-6L0 + L1) P0) c2^2 c4 + 7 (-1 + K0L0P0) c3 c4
L2 = 0; P1 = 0; (* Vanish coefficient of e^6 *)
a7 = Coefficient[e1, e^7] // FullSimplify
C4: Out[a7] = (-20 - K1L0P0 + K0 (21L0P0 - 30L1P0 + 5L2P0
      - L3P0/6 + 6L0P1 - L1P1)) c2^6 + 1/6 (480 + 12K1L0P0
      + K0 ((-492L0 + 480L1 - 54L2 + L3) P0
      + 12 (-9L0 + L1) P1)) c2^4 c3 + (-80 - K1L0P0 + K0 (81L0P0
      - 50L1P0 + 3L2P0 + 14L0P1 - L1P1)) c2^2 c3^2 + 2 (6
      + K0 (2L1P0 - L0 (6P0 + P1))) c3^3 + (-40
      + K0 (40L0 - 21L1 + L2) P0 + 4K0L0P1) c2^3 c4
      + 2 (26 + K0 (7L1P0 - 2L0 (13P0 + P1))) c2 c3 c4
      + 6 (-1 + K0L0P0) c4^2
K1 = 1; L3 = 0; (* Vanish coefficient of e^7 *)
e1 // FullSimplify
C5: Out[e1] = -1/24 (c2 (c2^2 - c3) ((L4 + 12 (-6 + P2)) c2^4 - 24 (-4 + P2) c2^2 c3
      + 12P2 c3^2 - 24c2 c4)) e^8 + 0[e]^9
|K1| < infinity; |L3| < infinity;

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Remark 2. Any method of the family (10) uses four evaluations per iteration and has eighth-order convergence and satisfies the conditions (19), which accords with the conjecture of Kung-Traub [15] that a multipoint iteration without memory based on n evaluations achieves optimal convergence order 2^{n-1} for the case $n = 4$.

3. The Concrete Iterative Methods

In what follows, we give some concrete iterative forms of scheme (10).

Method 1. The functions $K(t)$, $L(t)$, and $P(t)$ defined by

$$\begin{aligned} K(t) &= \sin(t) + \cos(t), \\ L(t) &= t^4 e^t + 1, \quad P(t) = e^{t^2} \end{aligned} \quad (20)$$

satisfy the conditions of Theorem 1. A new family of two-parameter eighth-order methods is obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \left(\sin\left(\frac{f(z_n)}{f(x_n)}\right) + \cos\left(\frac{f(z_n)}{f(x_n)}\right) \right) \\ &\quad \times \left(1 + \frac{f^4(y_n)}{f^4(x_n)} e^{(f(y_n)/f(x_n))} \right) e^{(f^2(z_n)/f^2(y_n))} \\ &\quad \times \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]}, \end{aligned} \quad (21)$$

and the error equation becomes

$$e_{n+1} = c_2 (c_2^2 - c_3) (c_2^4 - 2c_2^2 c_3 - c_3^2 + c_2 c_4) e_n^8 + O(e_n^9). \quad (22)$$

Method 2. The functions $K(t)$, $L(t)$, and $P(t)$ defined by

$$\begin{aligned} K(t) &= e^t - 1 + \cos(t), \\ L(t) &= e^{t^4}, \quad P(t) = 1 - t + \sin(t) \end{aligned} \quad (23)$$

satisfy the conditions of Theorem 1. A new family of two-parameter eighth-order methods is obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \end{aligned}$$

TABLE 1: The examples considered in this study.

Test functions	Zeros
$f_1(x) = x^2 - e^x - 3x + 2$	$\alpha \approx 0.25753028543986079$
$f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	$\alpha \approx -1.207647827130919$
$f_3(x) = \sin(x)e^x + \ln(x^2 + 1)$	$\alpha \approx 0$
$f_4(x) = x^5 + x^4 + 4x^2 - 15$	$\alpha \approx 1.3474280989683053$
$f_5(x) = 10xe^{-x^2} - 1$	$\alpha \approx 1.6796306104284499$
$f_6(x) = \cos(x) - x$	$\alpha \approx 0.73908513321516064$
$f_7(x) = e^{-x^2+x+2} - 1$	$\alpha \approx -1.0000000000000000$
$f_8(x) = \ln(x^2 + x + 2) - x + 1$	$\alpha \approx 4.1525907367571583$

$$\begin{aligned} x_{n+1} &= z_n - \left(e^{(f(z_n)/f(x_n))} - 1 + \cos\left(\frac{f(z_n)}{f(x_n)}\right) \right) \\ &\quad \times \left(1 - \frac{f(z_n)}{f(y_n)} + \sin\left(\frac{f(z_n)}{f(y_n)}\right) \right) e^{f^4(y_n)/f^4(x_n)} \\ &\quad \times \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]}, \end{aligned} \quad (24)$$

and the error equation becomes

$$e_{n+1} = c_2^2 (c_2^2 - c_3) (2c_2^3 - 4c_2 c_3 + c_4) e_n^8 + O(e_n^9). \quad (25)$$

Method 3. The functions $K(t)$, $L(t)$, and $P(t)$ defined by

$$K(t) = 1 + \sin(t), \quad L(t) = 1 + t^4 \cos(t), \quad P(t) = \cos(t) \quad (26)$$

satisfy the conditions of Theorem 1. A new family of two-parameter eighth-order methods is obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \left(1 + \sin\left(\frac{f(z_n)}{f(x_n)}\right) \right) \\ &\quad \times \left(1 + \frac{f^4(y_n)}{f^4(x_n)} \cos\left(\frac{f(y_n)}{f(x_n)}\right) \right) \\ &\quad \times \cos\left(\frac{f(z_n)}{f(y_n)}\right) \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]}, \end{aligned} \quad (27)$$

and the error equation becomes

$$e_{n+1} = \frac{1}{2} c_2 (c_2^2 - c_3) (5c_2^4 - 10c_2^2 c_3 + c_3^2 + 2c_2 c_4) e_n^8 + O(e_n^9). \quad (28)$$

The families (21), (24), and (27) achieve eighth-order convergence. Per iteration the presented methods require three

TABLE 2: Comparison of various iterative methods under the same total number of function evaluations (TNFE = 12).

	(21)	(24)	(27)	(BM)	(LWM)	(ShM)
$f_1, x_0 = 0$						
$ x_n - \alpha $	$0.23e - 643$	$0.18e - 654$	$0.42e - 662$	$0.46e - 650$	$0.24e - 613$	$0.90e - 656$
$ f(x_n) $	$0.85e - 643$	$0.68e - 654$	$0.16e - 661$	$0.17e - 649$	$0.91e - 613$	$0.34e - 655$
COC	8.000000	8.000000	8.000000	8.000000	8.000000	8.000000
$f_2, x_0 = -1$						
$ x_n - \alpha $	$0.24e - 307$	$0.39e - 293$	$0.26e - 297$	$0.48e - 217$	$0.14e - 340$	$0.65e - 430$
$ f(x_n) $	$0.49e - 306$	$0.79e - 292$	$0.53e - 296$	$0.97e - 216$	$0.28e - 339$	$0.13e - 428$
COC	8.000001	8.000002	8.000002	7.999999	8.000000	8.000000
$f_3, x_0 = 2$						
$ x_n - \alpha $	$0.32e - 83$	$0.610e - 92$	$0.74e - 77$	$0.10e - 107$	$0.79e - 52$	$0.14e - 96$
$ f(x_n) $	$0.32e - 83$	$0.610e - 92$	$0.74e - 77$	$0.10e - 107$	$0.79e - 52$	$0.14e - 96$
COC	7.986895	7.993900	7.984913	7.965999	7.899280	7.991216
$f_4, x_0 = 1.6$						
$ x_n - \alpha $	$0.12e - 368$	$0.11e - 359$	$0.21e - 361$	$0.66e - 303$	$0.97e - 307$	$0.51e - 341$
$ f(x_n) $	$0.45e - 367$	$0.41e - 358$	$0.78e - 360$	$0.24e - 301$	$0.36e - 305$	$0.19e - 339$
COC	7.999999	7.999997	8.000018	7.999999	7.999999	8.000000
$f_5, x_0 = 1.8$						
$ x_n - \alpha $	$0.15e - 472$	$0.13e - 471$	$0.10e - 485$	$0.26e - 446$	$0.47e - 399$	$0.17e - 444$
$ f(x_n) $	$0.42e - 472$	$0.36e - 471$	$0.28e - 485$	$0.72e - 446$	$0.13e - 398$	$0.46e - 444$
COC	8.000000	8.000000	8.000000	8.000000	8.000000	8.000000
$f_6, x_0 = 1$						
$ x_n - \alpha $	$0.11e - 632$	$0.33e - 572$	$0.99e - 558$	$0.98e - 664$	$0.49e - 581$	$0.71e - 570$
$ f(x_n) $	$0.18e - 632$	$0.56e - 572$	$0.17e - 557$	$0.16e - 663$	$0.81e - 581$	$0.12e - 569$
COC	8.000000	8.000000	8.000000	8.000000	8.000000	8.000000
$f_7, x_0 = -0.5$						
$ x_n - \alpha $	$0.27e - 235$	$0.17e - 198$	$0.84e - 192$	$0.31e - 221$	$0.13e - 221$	$0.35e - 193$
$ f(x_n) $	$0.80e - 235$	$0.52e - 198$	$0.25e - 191$	$0.94e - 221$	$0.39e - 221$	$0.11e - 192$
COC	8.015338	8.000253	8.000189	8.000058	8.000030	7.999924
$f_8, x_0 = 3.2$						
$ x_n - \alpha $	$0.31e - 518$	$0.10e - 485$	$0.12e - 476$	$0.19e - 618$	$0.58e - 487$	$0.26e - 479$
$ f(x_n) $	$0.19e - 518$	$0.62e - 486$	$0.69e - 477$	$0.11e - 618$	$0.35e - 487$	$0.16e - 479$
COC	8.000000	8.000000	8.000000	8.000000	8.000000	8.000000

evaluations of the function and one evaluation of its first derivative. If we assume that all the evaluations have the same cost as function one, we have that the family of methods (10) has the efficiency index of $8^{1/4} = 1.682$ which is better than $2^{1/2} = 1.414$ of Newton's method, $4^{1/3} = 1.587$ of King's methods, $6^{1/4} = 1.565$ of some methods with sixth-order convergence, and $7^{1/4} = 1.627$ of some variants of Ostrowski's method with seventh-order convergence [14].

4. Numerical Results

Now, Method 1 (21), Method 2 (24), and Method 3 (27) are employed to solve some nonlinear equations and compared with Bi et al.'s method (BM), (4), (with $\gamma = 1$), Liu and Wang's method (LWM), (5), (with $\mu = 5$ and $\lambda = -7$), and Sharma's method (ShM), (6). The test functions of $f(x)$ are listed in Table 1.

Numerical computations reported here have been carried out in a *Mathematica* 8.0 environment. Table 2 shows the difference of the root α and the approximation x_n to α , where α is the exact root computed with 800 significant digits ($Digits := 800$) and x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. The absolute values of the function ($|f(x_n)|$) and the computational order of convergence (COC) are also shown in Table 2. Here, COC is defined by [19]

$$\rho \approx \frac{\ln |(x_{n+1} - \alpha) / (x_n - \alpha)|}{\ln |(x_n - \alpha) / (x_{n-1} - \alpha)|}. \quad (29)$$

5. Conclusions

We have obtained a new family of variants of Ostrowski's method. The convergence order of these methods is eight, which consist of three evaluations of the function and one evaluation of the first derivative per iteration, so they have

an efficiency index equal to $8^{1/4} = 1.682$. Therefore, the family of methods agrees with the conjecture of Kung-Traub for $n = 4$. Numerical examples also show that the numerical results of our new methods, in equal iterations, improve the results of other existing three-step methods with eighth order convergence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the referees for their comments and suggestions that helped to improve the paper. This research was supported by Islamic Azad University, Hamedan Branch.

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