Research Article

Common Fixed Point of (ψ, β) -Generalized Contractive Mappings in Partially Ordered Metric Spaces

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Gordji et al. (2012) gave a generalization of Geraghty's theorem. The aim of this paper is to study the necessary conditions for the existence of coincidence and common fixed point of four mappings satisfying (ψ, β) -generalized contractive condition in the setup of partial ordered metric spaces. Some examples are given to validate the definitions and results presented herein.

1. Introduction and Preliminaries

Abbas et al. [1] proved that a weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhoades [2] established the same result considering the domain of mapping a complete metric space instead of Hilbert space. The study of common fixed points of mappings satisfying certain contractive conditions can be employed to establish existence of solutions of many types of operator equations such as differential and integral equations. Beg and Abbas [3] obtained common fixed points extending a weak contractive condition to two maps. In 2009, Kadelburg et al. [4] proved common fixed point theorems for generalized (ψ, ϕ) -weakly contractive mappings. Doric [5] obtained a common fixed point theorem for four maps. For more work in this direction, we refer to [2, 3, 5–13] and references mentioned therein.

In complete metric spaces equipped with a partial ordering, $\leq$ was first investigated in 2004 by Ran and Reurings [14], and then by Nieto and López [15] who subsequently extended the result of Ran and Reurings [14] for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions (see also [16]).

Abbas et al. [8] initiated the study of common fixed points for four mappings satisfying generalized weak contractive condition in complete partially ordered metric space.

Recently, Esmaily et al. [17] coincidence point result for four mappings in partially ordered metric space and employed their result to find the common solution of two integral equations.

The aim of this paper is to obtain coincidence and common fixed points for four mappings under generalized (ψ, β) contractive condition in complete partially ordered metric space. Our results extend, unify, and generalize the comparable results in [8, 17–19]. For some details also see [20, 21].

In the sequel, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers, and the set of positive integers, respectively. The usual order on $\mathbb{R}$ (resp., on $\mathbb{R}^+$) will be indistinctly denoted by $\leq$ or by $\geq$.

The following definitions and results will be needed in the sequel.

Let P be the class of all mappings $\beta : [0, \infty) \rightarrow [0, 1)$ satisfying the condition $\beta(t_n) \rightarrow 1$ whenever $t_n \rightarrow 0$. Note that $P \neq \emptyset$ as if we take $\beta x = 1/(1 + x^2)$, $x \in [0, \infty)$, then $\beta \in P$.

Define $\Psi = \{\psi : [0, \infty) \rightarrow [0, \infty) : \psi \text{ is continuous, nondecreasing and } \psi(t) = 0 \text{ if and only if } t = 0\}$. If we define $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi x = \ln(1 + x)$, then it is easy to check that $\psi \in \Psi$.

Let f be self-mapping on a set X . If $x = fx$, for some x in X , then x is called a fixed point of f . The set of all fixed points of f is denoted by $F(f)$. If $F(f) = \{z\}$ and, for each x_0 in a complete metric space X , the sequence $x_{n+1} = fx_n =$

$f^n x_0, n = 0, 1, 2, \dots$, converges to z , then f is called a Picard operator.

In the following [22], Geraghty obtained the generalization of Banach's contraction principle.

Theorem 1. Let (X, d) be a complete metric space and f a self-map on X . If there exists $\beta \in P$ such that $d(fx, fy) \leq \beta(d(x, y))d(x, y)$ for all $x, y \in X$, then f is a Picard operator.

A nonempty set X equipped with a partial order $\leq$ is called partially ordered metric space if there exists a metric d on X . We will denote it by $(X, d, \leq)$.

Definition 2 (see [23]). Let $(X, d, \leq)$ be an ordered metric space. We say that X is regular if $\{x_n\}$ is a nondecreasing sequence in X with respect to $\leq$ such that $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Harandi and Emami [24] proved Theorem 1 in partially ordered metric spaces.

Theorem 3. Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a complete metric d on X . Let $f : X \rightarrow X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq fx_0$. If there exists $\alpha \in P$ such that $d(fx, fy) \leq \alpha(d(x, y))d(x, y)$ for each $x, y \in X$ with $x \geq y$, then f has a fixed point provided that either f is continuous or X is regular. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y , then f has a unique fixed point.

Recently, [18] Gordji et al. proved the following result.

Theorem 4. Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a complete metric d on X . Let $f : X \rightarrow X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq fx_0$. Suppose that there exist $\alpha \in P$ and $\psi \in \Psi$ such that

$$\psi(d(fx, fy)) \leq \alpha(\psi(d(x, y)))\psi(d(x, y)), \quad (1)$$

for all $x, y \in X$ with $x \leq y$. Assume that either f is continuous or X is regular. Then f has a fixed point.

It is worth to noticing that Condition (b) of subadditivity in [18] for the function ψ is superfluous. Namely, all results in ([18], Theorems 2.2, 2.3, 3.3) are true only if $\psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$. However, it is easy to see that with Assumption (b) for the function ψ [18] is not a generalization of [24]. For details see [25].

Let f and g be two self-mappings on a nonempty set X . If $x = fx = gx$, for some x in X , then x is called a common fixed point of f and g .

Let $f, g : X \rightarrow X$ be given self-mappings on a metric space X . The pair (f, g) is said to be compatible if $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that, $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$, for some $t \in X$.

Let $(X, d, \leq)$ be an ordered metric space and $f, g, S, T : X \rightarrow X$. If there exist $\beta \in P$ and $\psi \in \Psi$ such that, for every

two comparable elements $x, y \in X$, we have

$$\begin{aligned} & \psi(d(fx, gy)) \\ & \leq \beta(M(x, y))\psi(\max\{d(Sx, Ty), \\ & \quad d(Sx, fx), d(Ty, gy)\}), \end{aligned} \quad (2)$$

where

$$\begin{aligned} M(x, y) = & \psi\left(\max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \right. \right. \\ & \left. \left. \frac{1}{2}[d(Sx, gy) + d(Ty, fx)]\right\}\right), \end{aligned} \quad (3)$$

then (f, g) is said to be (ψ, β) -order contractive pair with respect to S and T .

Example 5. Let $X = \{1, 2, 3, 4, 6\}$ be a partially ordered set and d a usual metric on X . Define four self-maps f, g, S , and T on X as follows:

$$\begin{aligned} f &= \begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, & g &= \begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\ S &= \begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 1 & 1 & 3 & 1 & 4 \end{pmatrix}, & T &= \begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 1 & 1 & 3 & 2 & 6 \end{pmatrix}. \end{aligned} \quad (4)$$

Take $\psi(x) = \ln(x + 1)$ and $\beta(x) = \psi(x)/x$. Then (f, g) is (ψ, β) -order contractive pair with respect to S and T .

Let $(X, \leq)$ be a partially ordered set. Two mappings $f, g : X \rightarrow X$ are said to be weakly increasing if $fx \leq gfx$ and $gx \leq fgx$ hold for all $x \in X$ [26].

Example 6. Let $X = \mathbb{R}^+$ be endowed with usual order and usual topology. Let $f, g : X \rightarrow X$ be defined by

$$\begin{aligned} fx &= \begin{cases} x^{1/4}, & \text{if } x \in [0, 1] \\ x^4, & \text{if } x \in 1, [\infty), \end{cases} \\ gx &= \begin{cases} x, & \text{if } x \in [0, 1] \\ 3x, & \text{if } x \in 1, [\infty). \end{cases} \end{aligned} \quad (5)$$

Then, the pair (f, g) is weakly increasing where g is a discontinuous mapping on $\mathbb{R}^+$.

Let $(X, \leq)$ be a partially ordered set. Two mappings $f, g : X \rightarrow X$ are said to be partially weakly increasing if $fx \leq gfx$ for all $x \in X$ [8].

Definition 7 (see [23]). Let $(X, \leq)$ be a partially ordered set and $f, g, h : X \rightarrow X$ given mappings such that $fx \subseteq hX$ and $gx \subseteq hX$. We say that f and g are weakly increasing with respect to h if and only if, for all $x \in X$, we have: $fx \leq gy$, for all $y \in h^{-1}(fx)$, and $gx \leq fy$, for all $y \in h^{-1}(gx)$, where $h^{-1}(x) = \{u \in X \mid hu = x\}$, for x in X .

Definition 8 (see [17]). Let $(X, \leq)$ be a partially ordered set and $f, g, h : X \rightarrow X$ given mappings such that $fx \subseteq hX$. We say that f and g are partially weakly increasing with respect to h if and only if, for all $x \in X$, we have $fx \leq gy$, for all $y \in h^{-1}(fx)$.

If $f = g$, we say that f is weakly increasing with respect to h .

Note that if $h = I_X$ (identity map on X), then pair (f, g) is weakly increasing.

Let $(X, \preceq)$ be a partially ordered set. A self-mapping f on X is called (a) dominating if $x \preceq fx$ for each x in X [8] and (b) dominated if $fx \preceq x$ for each x in X .

Example 9. Let $X = [0, 1]$ be endowed with usual ordering and let $f : X \rightarrow X$ be defined by $fx = x^{1/4}$. Since $x \leq x^{1/4} = fx$ for all $x \in X$, Therefore f is a dominating map.

Example 10. Let $X = [1, \infty]$ be endowed with usual ordering and let $f : X \rightarrow X$ be defined by $fx = 1/(x + 1)$. Since $fx = 1/(x + 1) \leq x$ for all $x \in X$, therefore f is a dominated map.

Assertion similar to the following lemma was used (and proved) in the course of proofs of several fixed point results in various papers [18, 27].

Lemma 11. Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_{n+1}, y_n)$ is nonincreasing and that

$$\lim_{n \rightarrow \infty} d(y_{n+1}, y_n) = 0. \quad (6)$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \rightarrow \infty$:

$$\begin{aligned} d(y_{2m_k}, y_{2n_k}), & \quad d(y_{2m_k}, y_{2n_k+1}), \\ d(y_{2m_k-1}, y_{2n_k}), & \quad d(y_{2m_k-1}, y_{2n_k+1}). \end{aligned} \quad (7)$$

2. Common Fixed Point Result

Now we start with the following result.

Theorem 12. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X . Suppose that f, g, S , and T are self-mappings on X with $fX \subseteq TX$ and $gX \subseteq SX$ such that a pair (f, g) of dominating maps is (ψ, β) -order contractive with respect to dominated maps S and T . If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \rightarrow u$ implies that $x_n \preceq u$ and either

- (a) $\{f, S\}$ are compatible, f or S is continuous, and $\{g, T\}$ are weakly compatible or
- (b) $\{g, T\}$ are compatible, g or T is continuous, and $\{f, S\}$ are weakly compatible,

then f, g, S , and T have a common fixed point. Moreover, the set of common fixed points of f, g, S , and T is well ordered if and only if f, g, S , and T have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X . Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n-1} = Tx_{2n-1} = fx_{2n-2}$ and $y_{2n} = Sx_{2n} = gx_{2n-1}$. This can be done as $fX \subseteq TX$ and $gX \subseteq SX$. By given assumptions, $x_{2n-2} \preceq fx_{2n-2} = Tx_{2n-1} \preceq x_{2n-1}$ and $x_{2n-1} \preceq gx_{2n-1} = Sx_{2n} \preceq x_{2n}$. Thus, for all $n \geq 1$, we have

$x_n \preceq x_{n+1}$. We claim that $d(y_{2n}, y_{2n+1}) > 0$, for every n . If not, then $y_{2n} = y_{2n+1}$, for some n . From inequality (2), we obtain

$$\begin{aligned} & \psi(d(y_{2n+1}, y_{2n+2})) \\ &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \beta(M(x_{2n}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), \\ &\quad \quad d(Tx_{2n+1}, gx_{2n+1})\}) \\ &= \beta(M(x_{2n}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &\quad \quad d(y_{2n+1}, y_{2n+2})\}) \\ &= \beta(M(x_{2n}, x_{2n+1})) \psi(d(y_{2n+1}, y_{2n+2})) \\ &< \psi(d(y_{2n+1}, y_{2n+2})), \end{aligned} \quad (8)$$

a contradiction. Hence, $y_{2n+2} = y_{2n+1}$. Following the similar arguments, we obtain $y_{2n+2} = y_{2n+3}$ and so on. Thus $\{y_n\}$ becomes a constant sequence, and y_{2n} is the common fixed point of f, g, S , and T . Take $d(y_{2n}, y_{2n+1}) > 0$ for each n . As x_{2n} and x_{2n+1} are comparable, so by inequality (2) we have

$$\begin{aligned} & \psi(d(y_{2n+1}, y_{2n+2})) \\ &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \beta(M(x_{2n}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), \\ &\quad \quad d(Tx_{2n+1}, gx_{2n+1})\}) \\ &= \beta(M(x_{2n}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &\quad \quad d(y_{2n+1}, y_{2n+2})\}) \\ &< \psi(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}). \end{aligned} \quad (9)$$

Now, if $\max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2})\} = d(y_{2n+1}, y_{2n+2})$, then (9) gives a contradiction. Hence,

$$\psi(d(y_{2n+1}, y_{2n+2})) < \psi(d(y_{2n}, y_{2n+1})). \quad (10)$$

Similarly, we obtain

$$\psi(d(y_{2n+2}, y_{2n+3})) < \psi(d(y_{2n+1}, y_{2n+2})). \quad (11)$$

Thus the sequence $\{d(y_{2n}, y_{2n+1})\}$ is a nonincreasing sequence and bounded below. So $\lim_{n \rightarrow \infty} \psi(d(y_{2n}, y_{2n+1}))$ exists. We

claim that $\lim_{n \rightarrow \infty} \psi(d(y_{2n}, y_{2n+1})) = 0$. If not, assume that $\lim_{n \rightarrow \infty} \psi(d(y_{2n}, y_{2n+1})) = r > 0$; then we have

$$\begin{aligned}
 & \psi(d(y_{2n+1}, y_{2n+2})) \\
 &= \psi(d(fx_{2n}, gx_{2n+1})) \\
 &\leq \beta(M(x_{2n+1}, x_{2n})) \\
 &\quad \times \psi(\max\{d(Sx_{2n+1}, Tx_{2n}), d(Sx_{2n+1}, fx_{2n+1}), \\
 &\quad d(Tx_{2n}, gx_{2n})\}) \\
 &= \beta(M(x_{2n+1}, x_{2n})) \\
 &\quad \times \psi(\max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), \\
 &\quad d(y_{2n}, y_{2n+1})\}) \\
 &= \beta(M(x_{2n+1}, x_{2n})) \psi(d(y_{2n+1}, y_{2n})), \tag{12}
 \end{aligned}$$

and so

$$\frac{\psi(d(y_{2n+1}, y_{2n+2}))}{\psi(d(y_{2n+1}, y_{2n}))} \leq \beta(M(x_{2n+1}, x_{2n})) < 1, \tag{13}$$

which, on taking limit as $n \rightarrow \infty$, implies that $\lim_{n \rightarrow \infty} \beta(M(x_{2n+1}, x_{2n})) = 1$. By the property of β , we have $\lim_{n \rightarrow \infty} M(x_{2n+1}, x_{2n}) = 0$ and so $\lim_{n \rightarrow \infty} \psi(d(y_{2n}, y_{2n+1})) = 0$, a contradiction. Therefore $r = 0$. Now, we show that $\{y_n\}$ is a Cauchy sequence. It is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence in X . Suppose that this is not the case. Applying Lemma 11 to the sequence $\{y_n\}$, we obtain that there exist $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that the sequences

$$\begin{aligned}
 & d(y_{2m_k}, y_{2n_k}), \quad d(y_{2m_k}, y_{2n_k+1}), \\
 & d(y_{2m_k-1}, y_{2n_k}), \quad d(y_{2m_k-1}, y_{2n_k+1}) \tag{14}
 \end{aligned}$$

all tend to ε when $k \rightarrow \infty$. Putting $x = x_{2n_k}$, $y = x_{2m_k-1}$ in (2) we have

$$\begin{aligned}
 & \frac{\psi(d(y_{2n_k+1}, y_{2m_k}))}{\psi(\max\{d(y_{2m_k}, y_{2n_k}), d(y_{2n_k}, y_{2n_k-1}), d(y_{2m_k-1}, y_{2m_k})\})} \\
 & \leq \beta(M(x_{2n_k}, x_{2m_k-1})) < 1, \tag{15}
 \end{aligned}$$

where

$$\begin{aligned}
 & M(x_{2n_k}, x_{2m_k-1}) \\
 &= \max \left\{ d(y_{2n_k}, y_{2m_k-1}), d(y_{2n_k}, y_{2n_k-1}), d(y_{2m_k-1}, y_{2m_k}), \right. \\
 & \quad \left. \frac{d(y_{2m_k}, y_{2n_k}) + d(y_{2m_k-1}, y_{2n_k+1})}{2} \right\}. \tag{16}
 \end{aligned}$$

Since

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \psi(d(y_{2n_k+1}, y_{2m_k})) \\
 &= \lim_{k \rightarrow \infty} \psi(\max\{d(y_{2m_k}, y_{2n_k}), d(y_{2n_k}, y_{2n_k-1}), \\
 & \quad d(y_{2m_k-1}, y_{2m_k})\}) = \psi(\varepsilon), \tag{17}
 \end{aligned}$$

we obtain that $\lim_{k \rightarrow \infty} \beta(M(x_{2n_k}, x_{2m_k-1})) = 1$. By the property of β , we have $\lim_{k \rightarrow \infty} M(x_{2n_k}, x_{2m_k-1}) = 0$ which is a contradiction with $\lim_{k \rightarrow \infty} M(x_{2n_k}, x_{2m_k-1}) = \varepsilon$. Therefore $\{y_{2n}\}$ is a Cauchy sequence in X , and hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X , such that $\{y_{2n}\}$ converges to z . Therefore,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} y_{2n+1} &= \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} fx_{2n} = z, \\
 \lim_{n \rightarrow \infty} y_{2n+2} &= \lim_{n \rightarrow \infty} Sx_{2n+2} = \lim_{n \rightarrow \infty} gx_{2n+1} = z. \tag{18}
 \end{aligned}$$

Assume that S is continuous. Since $\{f, S\}$ are compatible, we have $\lim_{n \rightarrow \infty} fSx_{2n+2} = \lim_{n \rightarrow \infty} Sf x_{2n+2} = Sz$. Now we show that $z = Sz$. If not, that is $d(Sz, z) > 0$. As $x_{2n+1} \leq gx_{2n+1} = Sx_{2n+2}$, so from inequality (2), we have

$$\begin{aligned}
 & \psi(d(fSx_{2n+2}, gx_{2n+1})) \\
 & \leq \beta(M(Sx_{2n+2}, x_{2n+1})) \\
 & \quad \times \psi(\max\{d(SSx_{2n+2}, Tx_{2n+1}), \\
 & \quad d(SSx_{2n+2}, fSx_{2n+2}), d(Tx_{2n+1}, gx_{2n+1})\}), \tag{19}
 \end{aligned}$$

where

$$\begin{aligned}
 & M(Sx_{2n+2}, x_{2n+1}) \\
 &= \psi \left(\max \left\{ d(SSx_{2n+2}, Tx_{2n+1}), \right. \right. \\
 & \quad d(SSx_{2n+2}, fSx_{2n+2}), \\
 & \quad d(Tx_{2n+1}, gx_{2n+1}), \\
 & \quad \left. \left. \frac{1}{2} [d(SSx_{2n+2}, gx_{2n+1}) \right. \right. \\
 & \quad \left. \left. + d(Tx_{2n+1}, fSx_{2n+2})] \right\} \right). \tag{20}
 \end{aligned}$$

On taking limit as $n \rightarrow \infty$, we obtain

$$\psi(d(Sz, z)) \leq \beta(\psi(d(Sz, z))) \psi(d(Sz, z)) < \psi(d(Sz, z)), \tag{21}$$

a contradiction. Hence, $Sz = z$. Now, since $x_{2n+1} \leq gx_{2n+1}$ and $gx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$, so we have $x_{2n+1} \leq z$. We are to show that $z = fz$. If not, then from inequality (2), we obtain

$$\begin{aligned}
 & \psi(d(fz, gx_{2n+1})) \\
 & \leq \beta(M(z, x_{2n+1})) \\
 & \quad \times \psi(\max\{d(Sz, Tx_{2n+1}), d(Sz, fz), \\
 & \quad d(Tx_{2n+1}, gx_{2n+1})\}), \tag{22}
 \end{aligned}$$

which on taking limit as $n \rightarrow \infty$, implies that $\psi(d(fz, z)) \leq \lim_{n \rightarrow \infty} \beta(M(z, x_{2n+1}))\psi(d(fz, z))$. Thus $\lim_{n \rightarrow \infty} \beta(M(z, x_{2n+1})) = 1$ and so $\lim_{n \rightarrow \infty} M(z, x_{2n+1}) = 0$, a contradiction. Thus $fz = z$.

Since $f(X) \subseteq T(X)$, there exists a point $w \in X$ such that $z = fz = Tw$. Now we show that $Tw = gw$. If not, that is $d(Tw, gw) > 0$. Since $z \leq fz = Tw \leq w$ implies $z \leq w$, therefore from inequality (2), we obtain

$$\begin{aligned} \psi(d(Tw, gw)) &= \psi(d(fz, gw)) \\ &\leq \beta(M(z, w)) \\ &\quad \times \psi(\max\{d(Sz, Tw), d(Sz, fz), \\ &\quad d(Tw, gw)\}) \\ &< \psi(\max\{d(Sz, Tw), d(Sz, fz), \\ &\quad d(Tw, gw)\}) \\ &= \psi(d(Tw, gw)), \end{aligned} \quad (23)$$

a contradiction. Hence $Tw = gw$. Since g and T are weakly compatible, $gz = gfz = gTw = Tgw = Tfz = Tz$. Thus z is a coincidence point of g and T .

Next we show that $z = gz$. If not, then $d(z, gz) > 0$. As $x_{2n} \leq fx_{2n}$ and $fx_{2n} \rightarrow z$ as $n \rightarrow \infty$ implies that $x_{2n} \leq z$, so from (2), we have

$$\begin{aligned} \psi(d(fx_{2n}, gz)) &\leq \beta(M(x_{2n}, z)) \\ &\quad \times \psi(\max\{d(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n}), \\ &\quad d(Tz, gz)\}) \\ &< \psi(\max\{d(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n}), \\ &\quad d(Tz, gz)\}), \end{aligned} \quad (24)$$

which, on taking limit as $n \rightarrow \infty$, gives $\psi(d(z, gz)) < \psi(d(z, gz))$, a contradiction, thus $z = gz$. Therefore $fz = gz = Sz = Tz = z$. The proof is similar when f is continuous. Similarly, the result follows when (b) holds. $\square$

Now suppose that the set of common fixed points of f, g, S , and T is well ordered. We claim that common fixed point of f, g, S , and T is unique. Assume on contrary that $fu = gu = Su = Tu = u$ and $fv = gv = Sv = Tv = v$ but $d(u, v) > 0$. By given assumption, we can replace x by u and y by v in (2) to obtain

$$\begin{aligned} \psi(d(u, v)) &= \psi(d(fu, gv)) \\ &\leq \beta(M(u, v)) \\ &\quad \times \psi(\max\{d(Su, Tv), d(Su, fu), \\ &\quad d(Tv, gv)\}) \\ &< \psi(d(u, v)), \end{aligned} \quad (25)$$

a contradiction. Hence $u = v$. Conversely, if f, g, S , and T have only one common fixed point then the set of common fixed points of f, g, S , and T being singleton is well ordered.

Example 13. Let $X = \{1, 2, 3, 4\}$ be a partially ordered set defined as $x \leq y$ if and only if $x \geq y$ and d a usual metric on X . Define self-maps f, g, S , and T as

$$\begin{aligned} f &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, & g &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \\ S &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 4 \end{pmatrix}, & T &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \end{pmatrix}. \end{aligned} \quad (26)$$

It is easy to verify that mappings f and g are dominated and S and T are dominating. Take $\psi(x) = \ln(x+1)$, $\beta(x) = \psi(x)/x$, and $\phi(x) = \sqrt{x}$ (see Table 1).

The mappings f, g, S , and T satisfy all the conditions given in Theorem 12. Moreover, 1 is a unique common fixed point of f, g, S , and T .

Remark 14. If in above example, we take $(x, y) = (4, 4)$, then

$$d(fx, gy) \not\leq kd(Sx, Ty), \quad (27)$$

for any value of k . Also,

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(M(x, y)) \quad (28)$$

does not hold for any altering distance functions $\psi, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. So Theorem 2.1 in [8] is not applicable.

Now we prove existence of coincidence point of two pairs of compatible mappings on partially ordered metric spaces.

Theorem 15. Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X and $f, g, S, T : X \rightarrow X$ given mappings. Suppose that pairs (f, S) and (g, T) are compatible, f, g, S , and T are continuous, and (f, g) and (g, f) are partially weakly increasing with respect to T and S , respectively. If there exist $\beta \in P$ and $\psi \in \Psi$ such that

$$\begin{aligned} \psi(d(fx, gy)) &\leq \beta(M(x, y)) \psi(\max\{d(Sx, Ty), d(Sx, fx), \\ &\quad d(Ty, gy)\}) \end{aligned} \quad (29)$$

holds for every $x, y \in X$ for which Sx and Ty are comparable, where

$$\begin{aligned} M(x, y) &= \psi\left(\max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(Sx, fx) + d(Ty, gy)]\right\}\right), \end{aligned} \quad (30)$$

then the pairs (f, S) and (g, T) have a coincidence point $z \in X$. Moreover, if Sz and Tz are comparable, then $z \in X$ is a coincidence point of f, g, S , and T .

TABLE 1

(x, y)	$\psi(d(fx, fy))$	$\beta(M(x, y))\psi(\max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy)\})$
(1, 1)	0	0
(1, 2)	0	0.741276
(1, 3)	0	0.741276
(1, 4)	0	0.869742
(2, 2)	0	0.741276
(2, 3)	0	0.741276
(2, 4)	0	0.869742
(3, 3)	0	0.869742
(3, 4)	0	0.869742
(4, 4)	0.693147	0.869742

Proof. Suppose x_0 be an arbitrary point in X . Following similar arguments to those given in Theorem 12, sequences $\{x_n\}$ and $\{y_n\}$ in X are given by

$$\begin{aligned} y_{2n} &= fx_{2n} = Tx_{2n+1}, \\ y_{2n+1} &= gx_{2n+1} = Sx_{2n+2}, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (31)$$

By construction, we have $x_{2n+1} \in T^{-1}(fx_{2n})$. Using the fact that (f, g) is partially weakly increasing with respect to T , we obtain

$$\begin{aligned} Tx_{2n+1} &= fx_{2n} \leq gfx_{2n} = gx_{2n+1} = Sx_{2n+2}, \\ &\forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (32)$$

Also, $x_{2n+2} \in S^{-1}(gx_{2n+1})$. As (g, f) is partially weakly increasing with respect to S , so

$$\begin{aligned} Sx_{2n+2} &= gx_{2n+1} \leq fgx_{2n+1} = fx_{2n+2} = Tx_{2n+3}, \\ &\forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (33)$$

Hence

$$Tx_1 \leq Sx_2 \leq Tx_3 \leq \cdots \leq Tx_{2n+1} \leq Sx_{2n+2} \leq Tx_{2n+3} \leq \cdots. \quad (34)$$

That is,

$$y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_{2n} \leq y_{2n+1} \leq y_{2n+2} \leq \cdots. \quad (35)$$

We will prove the result in four steps.

Step 1. First we show that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (36)$$

First Case. There exists an $n \in \mathbb{N} \cup \{0\}$ such that $y_n = y_{n+2}$. If there exists an $n \in \mathbb{N}$ such that $y_{2n+1} = y_{2n-1}$, then by (35) we have $y_{2n-1} = y_{2n} = y_{2n+1}$. Now we claim that $y_{2n+2} = y_{2n+1}$. If not, then $d(y_{2n+1}, y_{2n+2}) > 0$. As Sx_{2n+2} and Tx_{2n+1} are comparable so from inequality (29), we have

$$\begin{aligned} &\psi(d(y_{2n+2}, y_{2n+1})) \\ &= \psi(d(fx_{2n+2}, gx_{2n+1})) \end{aligned}$$

$$\begin{aligned} &\leq \beta(M(x_{2n+2}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(Sx_{2n+2}, Tx_{2n+1}), \\ &\quad d(Sx_{2n+2}, fx_{2n+2}), \\ &\quad d(Tx_{2n+1}, gx_{2n+1})\}) \\ &= \beta(M(x_{2n+1}, x_{2n+2})) \\ &\quad \times \psi(\max\{d(y_{2n+1}, y_{2n}), \\ &\quad d(y_{2n+1}, y_{2n+2}), \\ &\quad d(y_{2n}, y_{2n+1})\}) \\ &= \beta(M(x_{2n+1}, x_{2n+2})) \\ &\quad \times \psi(\max\{0, d(y_{2n+1}, y_{2n+2})\}) \\ &< \psi(d(y_{2n+1}, y_{2n+2})), \end{aligned} \quad (37)$$

a contradiction. Thus for every $k \geq 2n$, we have $y_k = y_{2n-1}$. This implies that $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0$. Similarly, (36) remains valid if $y_{2n} = y_{2n+2}$.

Second Case. For every $n \in \mathbb{N}$, $y_n \neq y_{n+2}$. Since Sx_{2n} and Tx_{2n+1} are comparable, so by (29) we obtain

$$\begin{aligned} &\psi(d(y_{2n}, y_{2n+1})) \\ &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \beta(M(x_{2n}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(Sx_{2n}, Tx_{2n+1}), \\ &\quad d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1})\}) \\ &= \beta(M(x_{2n}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), \\ &\quad d(y_{2n}, y_{2n+1})\}) \\ &< \psi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}), \end{aligned} \quad (38)$$

which further implies that

$$\psi(d(y_{2n}, y_{2n+1})) < \psi(d(y_{2n-1}, y_{2n})). \quad (39)$$

Similarly, as Sx_{2n+2} and Tx_{2n+1} are comparable, so by inequality (29), we have

$$\begin{aligned} & \psi(d(y_{2n+2}, y_{2n+1})) \\ &= \psi(d(fx_{2n+2}, gx_{2n+1})) \\ &\leq \beta(M(x_{2n+2}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(Sx_{2n+2}, Tx_{2n+1}), \\ &\quad d(Sx_{2n+2}, fx_{2n+2}), \\ &\quad d(Tx_{2n+1}, gx_{2n+1})\}) \\ &= \beta(M(x_{2n+2}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(y_{2n+1}, y_{2n}), \\ &\quad d(y_{2n+1}, y_{2n+2}), \\ &\quad d(y_{2n}, y_{2n+1})\}) \\ &< \psi(\max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2})\}), \end{aligned} \quad (40)$$

which implies that

$$\psi(d(y_{2n+2}, y_{2n+1})) < \psi(d(y_{2n+1}, y_{2n})). \quad (41)$$

Combining (39) and (41), we have $\psi(d(y_{n+1}, y_{n+2})) < \psi(d(y_n, y_{n+1}))$ for any $n \in \mathbb{N} \cup \{0\}$. Hence $\{\psi(d(y_n, y_{n+1}))\}$ is a nonincreasing sequence and bounded below. So $\lim_{n \rightarrow \infty} \psi(d(y_n, y_{n+1}))$ exists. We claim that $\lim_{n \rightarrow \infty} \psi(d(y_n, y_{n+1})) = 0$. If not, assume that $\lim_{n \rightarrow \infty} \psi(d(y_n, y_{n+1})) = r > 0$. Note that

$$\begin{aligned} & \psi(d(y_{2n+1}, y_{2n+2})) \\ &\leq \beta(M(x_{2n+2}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(Sx_{2n+2}, Tx_{2n+1}), d(Sx_{2n+2}, fx_{2n+2}), \\ &\quad d(Tx_{2n+1}, gx_{2n+1})\}) \end{aligned} \quad (42)$$

gives

$$\begin{aligned} & \psi(d(y_{2n+1}, y_{2n+2})) \\ &\leq \beta(M(x_{2n+2}, x_{2n+1})) \\ &\quad \times \psi(\max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2})\}), \quad (43) \\ & \psi(d(y_{2n+1}, y_{2n+2})) \\ &\leq \beta(M(x_{2n+2}, x_{2n+1})) \psi(d(y_{2n+1}, y_{2n})). \end{aligned}$$

Hence,

$$\frac{\psi(d(y_{2n+1}, y_{2n+2}))}{\psi(d(y_{2n+1}, y_{2n}))} \leq \beta(M(x_{2n+2}, x_{2n+1})) < 1, \quad (44)$$

which, on taking limit as $n \rightarrow \infty$, implies that $\lim_{n \rightarrow \infty} \beta(M(x_{2n+2}, x_{2n+1})) = 1$, so $\lim_{n \rightarrow \infty} \psi(d(Sx_{2n+2}, Tx_{2n+1})) = 0$, which further implies that $\lim_{n \rightarrow \infty} \psi(d(y_{2n}, y_{2n+1})) = 0$, a contradiction. So $r = 0$. Thus $\lim_{n \rightarrow \infty} \psi(d(y_n, y_{n+1})) = 0$.

Step 2. Now, we show that $\{y_n\}$ is a Cauchy sequence. It is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence in X . Suppose that this is not the case. Applying Lemma 11 to the sequence $\{y_n\}$ we obtain that there exist $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that the sequences

$$\begin{aligned} & d(y_{2m_k}, y_{2n_k}), \quad d(y_{2m_k}, y_{2n_k+1}), \\ & d(y_{2m_k-1}, y_{2n_k}), \quad d(y_{2m_k-1}, y_{2n_k+1}) \end{aligned} \quad (45)$$

all tend to ε when $k \rightarrow \infty$. Putting $x = x_{2n_k}$, $y = x_{2m_k-1}$ in (29), we obtain

$$\begin{aligned} & \frac{\psi(d(y_{2n_k+1}, y_{2m_k}))}{\psi(\max\{d(y_{2m_k}, y_{2n_k}), d(y_{2n_k}, y_{2n_k-1}), d(y_{2m_k-1}, y_{2m_k})\})} \\ &\leq \beta(M(x_{2n_k}, x_{2m_k-1})) < 1, \end{aligned} \quad (46)$$

where

$$\begin{aligned} & M(x_{2n_k}, x_{2m_k-1}) \\ &= \max \left\{ d(y_{2n_k}, y_{2m_k-1}), \right. \\ & \quad d(y_{2n_k}, y_{2n_k-1}), d(y_{2m_k-1}, y_{2m_k}), \\ & \quad \left. \frac{d(y_{2m_k}, y_{2n_k}) + d(y_{2m_k-1}, y_{2n_k+1})}{2} \right\}. \end{aligned} \quad (47)$$

Since

$$\begin{aligned} & \lim_{k \rightarrow \infty} \psi(d(y_{2n_k+1}, y_{2m_k})) \\ &= \lim_{k \rightarrow \infty} \psi(\max\{d(y_{2m_k}, y_{2n_k}), d(y_{2n_k}, y_{2n_k-1}), \\ & \quad d(y_{2m_k-1}, y_{2m_k})\}) = \psi(\varepsilon), \end{aligned} \quad (48)$$

we now have that $\lim_{k \rightarrow \infty} \beta(M(x_{2n_k}, x_{2m_k-1})) = 1$. By the property of the function β follows that $\lim_{k \rightarrow \infty} M(x_{2n_k}, x_{2m_k-1}) = 0$ a contradiction with $\lim_{k \rightarrow \infty} M(x_{2n_k}, x_{2m_k-1}) = \varepsilon$. Hence, $\{y_{2n}\}$ is a Cauchy sequence in X , and hence $\{y_n\}$ is a Cauchy sequence.

Step 3. Now we show the existence of a coincidence point for (f, S) and (g, T) . From the completeness of (X, d) , there is a $z \in X$ such that $\lim_{n \rightarrow \infty} y_n = z$. From (31), we obtain that $\lim_{n \rightarrow \infty} d(Sx_{2n}, z) = \lim_{n \rightarrow \infty} d(fx_{2n}, z) = \lim_{n \rightarrow \infty} d(Sx_{2n+2}, z) = \lim_{n \rightarrow \infty} d(gx_{2n+1}, z) = \lim_{n \rightarrow \infty} d(Tx_{2n+1}, z) =$

0. Since the pairs (f, S) and (g, T) are compatible, so $\lim_{n \rightarrow \infty} d(S(fx_{2n}), f(Sx_{2n})) = d(T(gx_{2n+1}), g(Tx_{2n+1})) = 0$. Now using the continuity of f, g, S, T , we have $\lim_{n \rightarrow \infty} d(f(Sx_{2n}), fz) = \lim_{n \rightarrow \infty} d(g(Tx_{2n+1}), gz) = \lim_{n \rightarrow \infty} d(S(Tx_{2n+1}), Sz) = \lim_{n \rightarrow \infty} d(T(Sx_{2n+2}), Tz) = 0$. The triangular inequality and (31) yield

$$\begin{aligned} d(Sz, fz) &\leq d(Sz, S(Tx_{2n+1})) \\ &\quad + d(S(fx_{2n}), f(Sx_{2n})) + d(f(Sx_{2n}), fz), \\ d(Tz, gz) &\leq d(Tz, T(Sx_{2n+1})) \\ &\quad + d(T(gx_{2n+1}), g(Tx_{2n+1})) \\ &\quad + d(g(Tx_{2n+1}), gz). \end{aligned} \quad (49)$$

On taking limit as $n \rightarrow \infty$, we obtain $d(Sz, fz) \leq 0$, and $d(Tz, gz) \leq 0$. Hence $Sz = fz$ and $Tz = gz$.

Step 4. Existence of a coincidence point for f, g, S , and T .

Since Sz and Tz are comparable, now are to show that $Tz = Sz$. If not, that is $d(Sz, Tz) > 0$. So from inequality (29), we have

$$\begin{aligned} \psi(d(Sz, Tz)) &= \psi(d(fz, gz)) \\ &\leq \beta(M(z, z)) \\ &\quad \times \psi(\max\{d(Sz, Tz), d(Sz, fz), d(Tz, gz)\}) \\ &< \psi(\max\{d(Sz, Tz), d(Sz, fz), d(Tz, gz)\}) \\ &= \psi(d(Sz, Tz)), \end{aligned} \quad (50)$$

a contradiction. Hence $Tz = Sz$, which means that z is a coincidence point of f, g, S , and T . $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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