

Research Article

A Study of Inverse Problems Based on Two Kinds of Special Matrix Equations in Euclidean Space

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Received 11 January 2014; Accepted 13 April 2014; Published 26 May 2014

Academic Editor: Ricardo L. Soto Montero

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Two special classes of symmetric coefficient matrices were defined based on characteristics matrix; meanwhile, the expressions of the solution to inverse problems are given and the conditions for the solvability of these problems are studied relying on researching. Finally, the optimal approximation solution of these problems is provided.

1. Introduction

In recent years, a lot of matrix problems have been used widely in the fields of structural design, automatic control, physical, electrical, nonlinear programming and numerical calculation, for example, a matrix Eigen value problem was applied for mixed convection stability analysis in the Darcy media by Serebriiskii et al. [1] and some of the problems based on the nonskew symmetric orthogonal matrices were studied by Hamed and Bennacer in 2008 [2], but some of the matrix inverse problems still need further research in order to make it easier to discuss relevant issues. Therefore, in this paper, we studied the inverse problems of two kinds of special matrix equations based on the existing research achievements, moreover, the expressions and conditions of the matrix solutions are given by related matrix-calculation methods. Some definitions and assumptions of the inverse problem for two forms of special matrices are given in Section 2. In Sections 3 and 5 we discuss the existence and expressions of general solution based on the two classes of matrices, and in Sections 4 and 6 we prove the uniqueness of matrices for researching related inverse problems.

2. Definitions and Assumptions of Inverse Problems for Two Forms of Special Matrices

In order to research some inverse problems of related matrices, we give the following definitions and assumptions.

Definition 1. When $P \in R^{n \times n}$, $P = P^T = P^{-1}$, $A = (a_{ij}) \in R^{n \times n}$, $i, j = 1, 2, 3, \dots, n$, $(PA)^T = -PA$, and $a_{ij} = -a_{ji}$, A will be called the first-class special symmetric matrix and the set of these special symmetric matrices is denoted by $A1R^{n \times n}$. The corresponding problems are as follows.

Problem 1. When $X, B \in R^{n \times m}$, $A \in A1R^{n \times n}$ can be obtained, so that $AX = B$.

Problem 2. When $\dot{A} \in R^{n \times n}$, $\hat{A} \in S_E$ can be obtained, so that $\|\dot{A} - \hat{A}\| = \min_{A \in S_E} \|\dot{A} - A\|$, where S_E is the solution set of the first problem.

Definition 2. When $A = (a_{ij}) \in R^{n \times n}$, $i, j = 1, 2, 3, \dots, n$ and $a_{ij} = -a_{n+1-i, n+1-j}$, A will be called the second-class special symmetric matrix and the set of these special symmetric matrices is denoted by $A2R^{n \times n}$. The corresponding problems are as follows.

Problem 1. When $X, B \in R^{n \times m}$, $A \in A2R^{n \times n}$ can be found, so that $AX = B$.

Problem 2. When $\dot{A} \in R^{n \times n}$, $\hat{A} \in \tilde{S}_E$ can be found, so that $\|\dot{A} - \hat{A}\| = \min_{A \in \tilde{S}_E} \|\dot{A} - A\|$, where \tilde{S}_E is the solution set of the first problem.

3. Existence and Expression of General Solutions Based on the First-Class Special Symmetric Matrix for Problem 1

To research the structure and properties of the special symmetric matrix $A \in A1R^{n \times n}$, first of all, we have the following conclusion from Definition 1.

Conclusion 1. Consider $A = (a_{ij}) \in R^{n \times n}$, and the sufficient and necessary conditions for $A \in A1R^{n \times n}$ are

$$A^T = -A, \quad A^T P + PA = 0. \quad (1)$$

Theorem 3. Consider $A = (a_{ij}) \in R^{n \times n}$, and the sufficient and necessary conditions for $A \in A1R^{n \times n}$ are

$$A = U \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} U^T, \quad (2)$$

where $A_{11} \in A1R^{k \times k}$, $A_{22} \in A1R^{(n-k) \times (n-k)}$, and U is an orthogonal matrix. Consider $U \in R^{n \times n}$ and $UU^T = I_n$.

Proof. Rely on the decomposition theorem of symmetric orthogonal matrix [3, 4]. When P is a symmetric orthogonal matrix, $P \in R^{n \times n}$, the P can be represented as the following equation by an orthogonal matrix U , $U \in R^{n \times n}$:

$$P = U \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} U^T, \quad (3)$$

where I is identity matrix [5].

When $A \in A1R^{n \times n}$, we can derive from (1) and (3) the following:

$$U^T A^T U \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} + \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} U^T A U = 0. \quad (4)$$

Based on $A^T = -A$ and $U^T A U \in A1R^{n \times n}$, there will be

$$U^T A U = \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^T & A_{22} \end{bmatrix} \quad (5)$$

and $A = U \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^T & A_{22} \end{bmatrix} U^T$ can be derived from (4) and (5).

Conversely, when

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^T & A_{22} \end{bmatrix} U^T, \quad (6)$$

$A^T = -A$, $A^T P + PA = 0$ can be obtained, and, relying on Conclusion 1, $A \in A1R^{n \times n}$ can be obtained. \square

Theorem 4. When $X, B \in R^{n \times m}$ and the X singular value decomposition is

$$X = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T, \quad (7)$$

where $U_1 = [U^{(1)} \ U^{(2)}] \in OR^{n \times n}$, $V_1 = [V^{(1)} \ V^{(2)}] \in OR^{m \times m}$, and $\Sigma_1 = \text{diag}(\delta_1, \dots, \delta_r) > 0$, $r = \text{rank}(X)$, $U^{(1)} \in R^{n \times r}$, $U^{(2)} \in R^{n \times (n-r)}$, $V^{(1)} \in R^{m \times r}$, $V^{(2)} \in R^{m \times (m-r)}$, $OR^{n \times n}$ is the set of orthogonal matrix, and X^+ is Moore-Penrose generalized inverse matrix [6–8].

The sufficient and necessary conditions for the existence of solution $AX = B$ are

$$B^T X = -X^T B, \quad BX^+ X = B. \quad (8)$$

And the general solution is as follows:

$$A = BX^+ - (BX^+)^T (I_n - XX^+) + U^{(2)} G (U^{(2)})^T, \quad (9)$$

where $G \in A1R^{(n-r) \times (n-r)}$.

Proof. Define

$$\begin{aligned} U_1^T A U_1 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ U_1^T B V_1 &= \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \end{aligned} \quad (10)$$

where $A_{11} \in R^{r \times r}$ and $B_{ij} = (U^{(1)})^T B V^{(1)}$, $i, j = 1, 2$.

The equation $AX = B$ can be represented as the following equation by (7) and (10):

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \quad (11)$$

According to (7) and (8), there will be

$$B_{11}^T \Sigma_1 = -\Sigma_1 B_{11}, \quad B_{12} = 0, \quad B_{22} = 0. \quad (12)$$

Then (11) is equivalent to the following equation:

$$\begin{aligned} A_{11} \Sigma_1 &= B_{11}, \\ A_{21} \Sigma_1 &= B_{21}, \end{aligned} \quad (13)$$

and $A_{11} = B_{11} \Sigma_1^{-1} \in A1R^{r \times r}$ and $A_{21} = B_{21} \Sigma_1^{-1}$ can be obtained from (12) and (13), so the equation $AX = B$ has solutions, $A \in A1R^{n \times n}$. And the general solution can be represented as the following equation:

$$A = U_1 \begin{bmatrix} B_{11} \Sigma_1^{-1} & -\Sigma_1^{-1} B_{21}^T \\ B_{21} \Sigma_1^{-1} & A_{22} \end{bmatrix} U_1^T \quad (14)$$

$$= BX^+ - (BX^+)^T (I_n - XX^+) + U^{(2)} G (U^{(2)})^T,$$

where $G \in A1R^{(n-r) \times (n-r)}$.

Conversely, when the equation $AX = B$ has solutions and $A \in A1R^{n \times n}$, $BX^+ X = B$ and $X^T B = X^T A X$ can be obtained relying on the Penrose theorem [9, 10].

There will be

$$B^T X = (X^T B)^T = (X^T A X)^T = -X^T A X = -X^T B. \quad (15)$$

The equations $B^T X = -X^T B$ and $BX^+ X = B$ are provided.

Finally the proof of Theorems 3 and 4 is completed. \square

From Theorems 3 and 4 we have a corollary as follows.

Corollary 5. Consider that $X, B \in R^{n \times m}$, X_i and B_i ($i = 1, 2$), $X_2, B_2 \in R^{(n-k) \times m}$, and $X_1, B_1 \in R^{k \times m}$ are given by these equations:

$$U^T X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad U^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \quad (16)$$

The X_i singular value decompositions are

$$X_1 = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T, \quad X_2 = U_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} V_2^T, \quad (17)$$

where $U_1 = [U_1^{(1)} \ U_1^{(2)}] \in OR^{k \times k}$, $V_1 = [V_1^{(1)} \ V_1^{(2)}] \in OR^{m \times m}$, $U_2 = [U_2^{(1)} \ U_2^{(2)}] \in OR^{(n-k) \times (n-k)}$, $V_2 = [V_2^{(1)} \ V_2^{(2)}] \in OR^{m \times m}$, $\Sigma_1 = \text{diag}(\delta_1^1, \dots, \delta_{r_1}^1) > 0$, $\Sigma_2 = \text{diag}(\delta_1^2, \dots, \delta_{r_2}^2) > 0$, $r_1 = \text{rank}(X_1)$, $r_2 = \text{rank}(X_2)$, $V_1^{(1)} \in R^{m \times r_1}$, $U_2^{(1)} \in R^{(n-k) \times r_2}$, $U_1^{(1)} \in R^{k \times r_1}$, $V_2^{(1)} \in R^{m \times r_2}$.

The sufficient and necessary conditions for the existence of solution $AX = B$ are

$$B_i^T X_i = -X_i^T B_i, \quad B_i X_i^+ X_i = B_i. \quad (18)$$

And the general solution is as follows:

$$A = U \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} U^T + U \begin{bmatrix} U_1^{(2)} G_1 (U_1^{(2)})^T & 0 \\ 0 & U_2^{(2)} G_2 (U_2^{(2)})^T \end{bmatrix} U^T, \quad (19)$$

$$A_{11} = B_1 X_1^+ - (B_1 X_1^+)^T (I_k - X_1 X_1^+)$$

$$A_{22} = B_2 X_2^+ - (B_2 X_2^+)^T (I_{n-k} - X_2 X_2^+),$$

where $G_1 \in A1R^{(k-r_1) \times (k-r_1)}$ and $G_2 \in A1R^{(n-k-r_2) \times (n-k-r_2)}$.

The proof is completed.

4. The Unique Solutions on the First-Class Special Symmetric Matrix for Problem 2

Consider the following theorem.

Theorem 6. When $X, B \in R^{n \times m}$, $A \in R^{n \times n}$, and X_i and B_i ($i = 1, 2$) are given by (16) and obey (18), Problem 2 has the unique solution $\hat{A} \in S_E$.

Define

$$U^T \hat{A} U = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{A}_{11} \in R^{k \times k}. \quad (20)$$

Then the unique solution \hat{A} can be represented as follows:

$$\hat{A} = U \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix} U^T, \quad (21)$$

where

$$\begin{aligned} \hat{A}_{11} &= A_{11}^0 + 0.5 \\ &\cdot (I_k - X_1 X_1^+) (\dot{A}_{11} - \dot{A}_{11}^T) (I_k - X_1 X_1^+), \end{aligned} \quad (22)$$

$$\begin{aligned} \hat{A}_{22} &= A_{22}^0 + 0.5 \\ &\cdot (I_{n-k} - X_2 X_2^+) (\dot{A}_{22} - \dot{A}_{22}^T) (I_{n-k} - X_2 X_2^+). \end{aligned}$$

A_0 is defined by the following equation:

$$A_0 = U \begin{bmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{bmatrix} U^T, \quad (23)$$

$$A_{11}^0 = B_1 X_1^+ - (B_1 X_1^+)^T (I_k - X_1 X_1^+), \quad (24)$$

$$A_{22}^0 = B_2 X_2^+ - (B_2 X_2^+)^T (I_{n-k} - X_2 X_2^+).$$

Finally S_E can represent the following set by Theorem 4:

$$S_E = A_0 + U \begin{bmatrix} U_1^{(2)} G_2 (U_1^{(2)})^T & 0 \\ 0 & U_2^{(2)} G_2 (U_2^{(2)})^T \end{bmatrix} U^T. \quad (25)$$

Consider $G_1 \in A1R^{(k-r_1) \times (k-r_1)}$ and $G_2 \in A1R^{(n-k-r_2) \times (n-k-r_2)}$.

Proof. Define these following equations:

$$\begin{aligned} U_1^T (\dot{A}_{11} - A_{11}^0) U_1 &= \begin{bmatrix} \dot{A}_{11}^1 & \dot{A}_{12}^1 \\ \dot{A}_{21}^1 & \dot{A}_{22}^1 \end{bmatrix}, \\ U_2^T (\dot{A}_{22} - A_{22}^0) U_2 &= \begin{bmatrix} \dot{A}_{11}^2 & \dot{A}_{12}^2 \\ \dot{A}_{21}^2 & \dot{A}_{22}^2 \end{bmatrix}, \end{aligned} \quad (26)$$

where $\dot{A}_{11}^1 \in R^{r_1 \times r_1}$ and $\dot{A}_{22}^1 \in R^{r_2 \times r_2}$.

Because of $A \in S_E$ we can obtain

$$\begin{aligned} A &= A_0 + U \begin{bmatrix} U_1^{(2)} G_1 (U_1^{(2)})^T & 0 \\ 0 & U_2^{(2)} G_2 (U_2^{(2)})^T \end{bmatrix} U^T \\ &= A_0 + U \begin{bmatrix} U_1 \begin{bmatrix} 0 & 0 \\ 0 & G_1 \end{bmatrix} U_1^T & 0 \\ 0 & U_2 \begin{bmatrix} 0 & 0 \\ 0 & G_2 \end{bmatrix} U_2^T \end{bmatrix} U^T, \end{aligned}$$

(27)

$$\begin{aligned} \|\dot{A} - A\|^2 &= \left\| \begin{bmatrix} U_1^{(2)} G_2 (U_1^{(2)})^T & 0 \\ 0 & U_2^{(2)} G_2 (U_2^{(2)})^T \\ -U^T (\dot{A} - A_0) U \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} U_1^{(2)} G_2 (U_1^{(2)})^T & 0 \\ 0 & U_2^{(2)} G_2 (U_2^{(2)})^T \\ -\begin{bmatrix} \dot{A}_{11} - A_{11}^0 & \dot{A}_{12} \\ \dot{A}_{21} & \dot{A}_{22} - A_{22}^0 \end{bmatrix} \end{bmatrix} \right\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|\dot{A}_{12}\|^2 + \|\dot{A}_{21}\|^2 \\
&+ \left\| \begin{bmatrix} 0 & 0 \\ 0 & G_1 \end{bmatrix} - U_1^T (\dot{A}_{11} - A_{11}^0) U_1 \right\|^2 \\
&+ \left\| \begin{bmatrix} 0 & 0 \\ 0 & G_2 \end{bmatrix} - U_2^T (\dot{A}_{22} - A_{22}^0) U_2 \right\|^2.
\end{aligned} \tag{28}$$

Finally,

$$\begin{aligned}
\|\dot{A} - A\|^2 &= \|\dot{A}_{12}\|^2 + \|\dot{A}_{21}\|^2 + \|\dot{A}_{11}^1\|^2 \\
&+ \|\dot{A}_{12}^1\|^2 + \|\dot{A}_{21}^1\|^2 + \|\dot{A}_{11}^2\|^2 + \|\dot{A}_{12}^2\|^2 \\
&+ \|\dot{A}_{21}^2\|^2 + \|G_1 - \dot{A}_{22}^1\|^2 + \|G_2 - \dot{A}_{22}^2\|^2.
\end{aligned} \tag{29}$$

When $H = -H^T$, $H \in R^{n \times m}$ and $E \in R^{n \times m}$, we can obtain:

$$\left\| E - \frac{E - E^T}{2} \right\| \leq \|E - H\|. \tag{30}$$

$$C = \begin{bmatrix} U_1^{(2)} (U_1^{(2)})^T (\dot{A}_{11} - \dot{A}_{11}^T) U_1^{(2)} (U_1^{(2)})^T & 0 \\ 0 & U_2^{(2)} (U_2^{(2)})^T (\dot{A}_{22} - \dot{A}_{22}^T) U_2^{(2)} (U_2^{(2)})^T \end{bmatrix} \tag{35}$$

and $\hat{A} = U \begin{bmatrix} \dot{A}_{11} & 0 \\ 0 & \dot{A}_{22} \end{bmatrix} U^T$ can be obtained because of $U_1^{(2)} (U_1^{(2)})^T = I_{n-k} - X_1 X_1^+$ and $U_2^{(2)} (U_2^{(2)})^T = I_k - X_2 X_2^+$.

Finally, rely on the optimal approximation [11, 12], and S_E is a closed convex set; Problem 2 has the unique solution $\hat{A} \in S_E$, $S_E \in R^{n \times n}$, so that $\|\dot{A} - \hat{A}\| = \min_{A \in \tilde{S}_E} \|\dot{A} - A\|$.

The proof is completed. \square

5. Conditions for the Existence and Expression of General Solutions Based on the Second-Class Special Symmetric Matrix for Problem 1

Consider the following theorem.

Theorem 7. Consider that $A = (a_{ij}) \in R^{n \times n}$, and the sufficient and necessary conditions for $A \in A2R^{n \times n}$ are

$$A = U \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix} U^T, \tag{36}$$

where U is an orthogonal matrix. $U \in R^{n \times n}$ and $UU^T = I_n$.

When $n = 2k + 1$, define

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ H_k & 0 & -H_k \end{bmatrix}. \tag{37}$$

When $G_1 = 0.5 \cdot (\dot{A}_{22}^1 - (\dot{A}_{22}^1)^T)$ and $G_2 = 0.5 \cdot (\dot{A}_{22}^2 - (\dot{A}_{22}^2)^T)$, so that $\|\dot{A} - A\| = \min_{A \in S_E}$,

$$\begin{aligned}
G_1 &= 0.5 \cdot (U_1^{(2)})^T (\dot{A}_{11} - \dot{A}_{11}^T - 2A_{11}^0) U_1^{(2)}, \\
G_2 &= 0.5 \cdot (U_2^{(2)})^T (\dot{A}_{22} - \dot{A}_{22}^T - 2A_{22}^0) U_2^{(2)}
\end{aligned} \tag{31}$$

can be derived from (26) and

$$(U_1^{(2)})^T A_{11}^0 U_1^{(2)} = 0, \quad (U_2^{(2)})^T A_{22}^0 U_2^{(2)} = 0 \tag{32}$$

can be obtained according to (17), (18), and (24).

Therefore,

$$\begin{aligned}
G_1 &= 0.5 \cdot (U_1^{(2)})^T (\dot{A}_{11} - \dot{A}_{11}^T) U_1^{(2)}, \\
G_2 &= 0.5 \cdot (U_2^{(2)})^T (\dot{A}_{22} - \dot{A}_{22}^T) U_2^{(2)}.
\end{aligned} \tag{33}$$

Relying on (27) the solution \hat{A} can be provided as follows:

$$\hat{A} = A_0 + \frac{1}{2} U C U^T, \tag{34}$$

where

When $n = 2k$, define

$$\begin{aligned}
U &= \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ H_k & -H_k \end{bmatrix}, \\
H_k &= \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}, \quad H_k^T H_k = I_k, \quad H_k^T = H_k,
\end{aligned} \tag{38}$$

$$F_1 \in A2R^{(n-k) \times k}, \quad F_2 \in A2R^{k \times (n-k)}.$$

Proof. First relying on the definition and property of the matrix $A \in A2R^{n \times n}$, $A2R^{n \times n}$ can represent the following set.

When $n = 2k + 1$,

$$A = \begin{bmatrix} C_1 & u & C_2 H_k \\ -v^T & 0 & v^T H_k \\ -H_k C_2 & -H_k u & -H_k C_1 H_k \end{bmatrix}, \tag{39}$$

$C_1, C_2 \in R^{k \times k}$ and $u, v \in R^k$.

When $n = 2k$,

$$A = \begin{bmatrix} C_1 & C_2 H_k \\ -H_k C_2 & -H_k C_1 H_k \end{bmatrix}, \tag{40}$$

$C_1, C_2 \in R^{k \times k}$.

We first discuss the topic $n = 2k + 1$.

From (37) and (39),

$$U^T A U = \frac{1}{2} \begin{bmatrix} I_k & 0 & H_k \\ 0 & \sqrt{2} & 0 \\ I_k & 0 & -H_k \end{bmatrix} C \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ H_k & 0 & -H_k \end{bmatrix}, \quad (41)$$

$$C = \begin{bmatrix} C_1 & u & C_2 H_k \\ -v^T & 0 & v^T H_k \\ -H_k C_2 & -H_k u & -H_k C_1 H_k \end{bmatrix},$$

$$U^T A U = \begin{bmatrix} 0 & 0 & C_1 - C_2 \\ 0 & 0 & -\sqrt{2} v^T \\ C_1 + C_2 & \sqrt{2} u & 0 \end{bmatrix}. \quad (42)$$

Define $F_1 = \begin{bmatrix} C_1 - C_2 \\ -\sqrt{2} v^T \end{bmatrix}$, $F_2 = \begin{bmatrix} C_1 + C_2 & \sqrt{2} u \end{bmatrix}$.

According to $C_1, C_2 \in R^{k \times k}$ and $u, v \in R^k$, we have $F_1 \in R^{(n-k) \times k}$ and $F_2 \in R^{k \times (n-k)}$, and

$$A = U U^T A U U^T = U \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix} U^T \quad (43)$$

can be obtained from (42).

Conversely, when $A = U \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix} U^T$, $F_1 \in R^{(n-k) \times k}$, $F_2 \in R^{k \times (n-k)}$, $A = -H_n A H_n$ can be obtained and $A \in A2R^{n \times n}$ will be provided.

When $n = 2k$, in the same way, we can prove the above theorem.

The proof is completed. \square

Theorem 8. When $X, B \in R^{n \times m}$, $A \in A2R^{n \times n}$, and X_i and B_i ($i = 1, 2$) are given by (16), the sufficient and necessary conditions for the existence of solution $AX = B$ are

$$B_i X_i^+ X_i = B_i. \quad (44)$$

And the general solution is as follows:

$$A = U \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} U^T, \quad (45)$$

$$A_{12} = B_1 X_2^+ + M_1 (I_k - X_2 X_2^+),$$

$$A_{21} = B_2 X_1^+ + M_2 (I_{n-k} - X_1 X_1^+),$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ H_k & 0 & -H_k \end{bmatrix}, \quad n = 2k + 1, \quad (46)$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ H_k & -H_k \end{bmatrix}, \quad n = 2k.$$

Meanwhile $M_1 \in R^{(n-k) \times k}$, $M_2 \in R^{k \times (n-k)}$, and according to (17) we can obtain X_i singular value decompositions:

$$X_1 = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T, \quad X_2 = U_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} V_2^T. \quad (47)$$

Proof. Relying on Theorem 7, if there is $A \in A2R^{n \times n}$, it will have

$$A = U \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix} U^T, \quad F_1 \in A2R^{(n-k) \times k}, \quad F_2 \in A2R^{k \times (n-k)}. \quad (48)$$

Because U is an orthogonal matrix and $U U^T = I_n$,

$$\|AX - B\|^2 = \|U^T A U U^T X - U^T B\|^2. \quad (49)$$

We can derive from (16), (48), and (49)

$$\begin{aligned} \|AX - B\|^2 &= \left\| \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right\|^2 \\ &= \|F_1 X_2 - B_1\|^2 + \|F_2 X_1 - B_2\|^2. \end{aligned} \quad (50)$$

It will be known that, when $X \in R^{n \times k}$, $B \in R^{n \times k}$, and $\text{rank}(X) = r$ [13], the sufficient and necessary conditions for the existence of solution $AX = B$ are $BX^+ X = B$ and the general solution is

$$A = BX^+ + M(I_m - XX^+), \quad M \in R^{n \times m}. \quad (51)$$

There will be

$$\begin{aligned} F_1 &= B_1 X_2^+ + M_1 (I_k - X_2 X_2^+), \quad M_1 \in R^{(n-k) \times k}; \\ F_2 &= B_2 X_1^+ + M_2 (I_{n-k} - X_1 X_1^+), \quad M_2 \in R^{k \times (n-k)}. \end{aligned} \quad (52)$$

The general solution can obtain

$$A = U \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} U^T, \quad (53)$$

$$A_{12} = B_1 X_2^+ + M_1 (I_k - X_2 X_2^+),$$

$$A_{21} = B_2 X_1^+ + M_2 (I_{n-k} - X_1 X_1^+).$$

The sufficient and necessary conditions for the existence of solution $AX = B$ are $B_i X_i^+ X_i = B_i$.

The proof is completed. \square

6. The Unique Solutions on the Second-Class Special Symmetric Matrix for Problem 2

Consider the following theorem.

Theorem 9. When $B \in R^{n \times m}$, $A \in R^{n \times n}$, and X_i and B_i ($i = 1, 2$) are given by (16) and (17) and obey (18), Problem 2 has the unique solution $\hat{A} \in \hat{S}_E$.

The unique solution \hat{A} can be represented as follows:

$$\hat{A} = UCU^T,$$

C

$$= \begin{bmatrix} 0 & B_1 X_2^+ + A_{12} (I_k - X_2 X_2^+) \\ B_2 X_1^+ + A_{21} (I_{n-k} - X_1 X_1^+) & 0 \end{bmatrix}, \quad (54)$$

where

$$A_0 = U \begin{bmatrix} 0 & B_1 X_2^+ \\ B_2 X_1^+ & 0 \end{bmatrix} U^T; \quad (55)$$

$$n = 2k + 1, \dot{A}_2 \in A2R^{n \times n};$$

$$A_{12} = 0.5 \cdot \begin{bmatrix} I_k & 0 & H_k \\ 0 & \sqrt{2} & 0 \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ 0 \\ -H_k \end{bmatrix}, \quad (56)$$

$$A_{21} = 0.5 \cdot \begin{bmatrix} I_k & 0 & -H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k & 0 \\ 0 & \sqrt{2} \\ H_k & 0 \end{bmatrix}; \quad (57)$$

$$n = 2k, \dot{A}_2 \in A2R^{n \times n};$$

$$\begin{aligned} A_{12} &= 0.5 \cdot \begin{bmatrix} I_k & H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ -H_k \end{bmatrix}, \\ A_{21} &= 0.5 \cdot \begin{bmatrix} I_k & -H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ H_k \end{bmatrix}. \end{aligned} \quad (58)$$

From Theorem 9 \tilde{S}_E can be represented as follows:

$$\begin{aligned} \tilde{S}_E &= A_0 + U \begin{bmatrix} 0 & M_1 (I_k - X_2 X_2^+) \\ M_2 (I_{n-k} - X_1 X_1^+) & 0 \end{bmatrix} U^T \\ M_1 &\in R^{(n-k) \times k}, \quad M_2 \in R^{k \times (n-k)}. \end{aligned} \quad (59)$$

Proof. Defining this following equation:

$$U^T (\dot{A}_2 - A_0) U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (60)$$

$$\text{where } n = 2k + 1, \dot{A}_2 \in A2R^{n \times n},$$

$$\begin{aligned} A_{11} &= 0.5 \cdot \begin{bmatrix} I_k & 0 & H_k \\ 0 & \sqrt{2} & 0 \end{bmatrix} (\dot{A}_2 - A_0), \\ A_{12} &= 0.5 \cdot \begin{bmatrix} I_k & 0 & H_k \\ 0 & \sqrt{2} & 0 \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ 0 \\ -H_k \end{bmatrix}, \\ A_{21} &= 0.5 \cdot \begin{bmatrix} I_k & 0 & -H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k & 0 \\ 0 & \sqrt{2} \\ H_k & 0 \end{bmatrix}, \\ A_{22} &= 0.5 \cdot \begin{bmatrix} I_k & 0 & -H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ 0 \\ -H_k \end{bmatrix}; \end{aligned} \quad (61)$$

$$n = 2k, \dot{A}_2 \in A2R^{n \times n};$$

$$\begin{aligned} A_{11} &= 0.5 \cdot \begin{bmatrix} I_k & H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ H_k \end{bmatrix}, \\ A_{12} &= 0.5 \cdot \begin{bmatrix} I_k & H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ -H_k \end{bmatrix}, \\ A_{21} &= 0.5 \cdot \begin{bmatrix} I_k & -H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ H_k \end{bmatrix}, \\ A_{22} &= 0.5 \cdot \begin{bmatrix} I_k & -H_k \end{bmatrix} (\dot{A}_2 - A_0) \begin{bmatrix} I_k \\ -H_k \end{bmatrix}. \end{aligned} \quad (62)$$

For $A \in \tilde{S}_E$ the following equation can be obtained:

$$\begin{aligned} A &= A_0 + UCU^T, \\ C &= \begin{bmatrix} 0 & M_1 (I_k - X_2 X_2^+) \\ M_2 (I_{n-k} - X_1 X_1^+) & 0 \end{bmatrix}. \end{aligned} \quad (63)$$

When $\dot{A} \in R^{n \times n}$ is known and relies on the definition of $A \in A2R^{n \times n}$, there will be the unique set $\dot{A}_1 = (\bar{a}_{ij}) \in R^{n \times n}$ ($\bar{a}_{ij} = \bar{a}_{(n+1-i), (n+1-j)}$) and $\dot{A}_2 \in A2R^{n \times n}$ to make the following equations true:

$$\begin{aligned} \dot{A} &= \dot{A}_1 + \dot{A}_2, \quad (\dot{A}_1, \dot{A}_2) = 0, \\ \dot{A}_2 &= 0.5 \cdot (\dot{A} + H_n \dot{A} H_n), \quad \dot{A}_1 = 0.5 \cdot (\dot{A} - H_n \dot{A} H_n), \end{aligned} \quad (64)$$

$$\|\dot{A} - A\|^2 = \|\dot{A}_1 + \dot{A}_2 - A\|^2 = \|\dot{A}_1\|^2 + \|\dot{A}_2 - A\|^2. \quad (65)$$

From (60), (63), and (65) we obtain

$$\begin{aligned} \|\dot{A} - A\|^2 &= \|\dot{A}_1\|^2 + \left\| U \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} U^T - UCU^T \right\|^2 \\ &= \|\dot{A}_1\|^2 + \left\| \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ -C \end{bmatrix} \right\|^2, \end{aligned} \quad (66)$$

$$C = \begin{bmatrix} 0 & M_1 (I_k - X_2 X_2^+) \\ M_2 (I_{n-k} - X_1 X_1^+) & 0 \end{bmatrix}.$$

Equation (67) can be known from (17) and (64) as follows:

$$\begin{aligned} U_2^{(2)} (U_2^{(2)})^T &= I_k - X_2 X_2^+, \\ U_1^{(2)} (U_1^{(2)})^T &= I_{n-k} - X_1 X_1^+, \\ (U_2^{(2)})^T U_2^{(2)} &= I_{k-r_2}, \end{aligned}$$

$$\begin{aligned}
(U_1^{(2)})^T U_1^{(2)} &= I_{n-k-r_1}, \\
(U_2^{(2)})^T U_2^{(1)} &= 0, \\
(U_1^{(2)})^T U_1^{(1)} &= 0,
\end{aligned} \tag{67}$$

$$\begin{aligned}
\|\dot{A} - A\|^2 &= \|\dot{A}_1\|^2 + \|A_{11}\|^2 + \|A_{22}\|^2 \\
&\quad + \|A_{12} - M_1 U_2^{(2)} (U_2^{(2)})^T\|^2 + \|A_{12} - M_2 U_1^{(2)} (U_1^{(2)})^T\|^2 \\
&= \|\dot{A}_1\|^2 + \|A_{11}\|^2 + \|A_{22}\|^2 + \|A_{12} U_2^{(1)}\|^2 + \|A_{21} U_1^{(1)}\|^2 \\
&\quad + \|A_{12} U_2^{(2)} - M_1 U_2^{(2)}\|^2 + \|A_{21} U_1^{(2)} - M_2 U_1^{(2)}\|^2.
\end{aligned} \tag{68}$$

Therefore, $\|\dot{A} - A\| = \min_{A \in \tilde{S}_E}$ is equivalent to

$$\begin{aligned}
\begin{cases} \|A_{12} U_2^{(2)} - M_1 U_2^{(2)}\| = \min_{M_1 \in R^{(n-k) \times k}} \\ \|A_{12} U_1^{(2)} - M_2 U_1^{(2)}\| = \min_{M_2 \in R^{k \times (n-k)}} \end{cases} \\
\Rightarrow \begin{cases} A_{12} U_2^{(2)} - M_1 U_2^{(2)} = 0 \\ A_{12} U_1^{(2)} - M_2 U_1^{(2)} = 0 \end{cases} \\
\Rightarrow \begin{cases} A_{12} U_2^{(2)} = M_1 U_2^{(2)} \\ A_{12} U_1^{(2)} = M_2 U_1^{(2)}, \end{cases}
\end{aligned} \tag{69}$$

and $A_{12}(I_k - X_2 X_2^+) = M_1(I_k - X_2 X_2^+)$, $A_{21}(I_{n-k} - X_1 X_1^+) = M_2(I_{n-k} - X_1 X_1^+)$ can be obtained.

From the above results and (63), the solution \hat{A} ($\hat{A} \in \tilde{S}_E$) can be represented as follows:

$$\hat{A} = UCU^T,$$

C

$$= \begin{bmatrix} 0 & B_1 X_2^+ + A_{12}(I_k - X_2 X_2^+) \\ B_2 X_1^+ + A_{21}(I_{n-k} - X_1 X_1^+) & 0 \end{bmatrix}. \tag{70}$$

Finally, because \tilde{S}_E is a closed convex set, Problem 2 has the unique solution $\hat{A} \in \tilde{S}_E$, $\tilde{S}_E \in R^{n \times n}$, so that $\|\dot{A} - \hat{A}\| = \min_{A \in \tilde{S}_E} \|\dot{A} - A\|$.

The proof is completed. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors are grateful to the referee for the useful comments and valuable suggestions.

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