Hindawi Publishing Corporation
International Journal of Computational Mathematics
Volume 2014, Article ID 436140, 8 pages
http://dx.doi.org/10.1155/2014/436140

Research Article

Modified Eccentric Connectivity of Generalized Thorn Graphs

Nilanjan De,[1] Anita Pal,[2] and Sk. Md. Abu Nayeem[3]

[1] *Department of Basic Sciences and Humanities (Mathematics), Calcutta Institute of Engineering and Management, Kolkata 700 040, India*
[2] *Department of Mathematics, National Institute of Technology, Durgapur 713 209, India*
[3] *Department of Mathematics, Aliah University, DN 20, Sector V, Salt Lake, Kolkata 700 091, India*

Correspondence should be addressed to Sk. Md. Abu Nayeem; nayeem.math@aliah.ac.in

Received 26 June 2014; Accepted 9 December 2014; Published 22 December 2014

Academic Editor: Sheung-Hung Poon

The thorn graph G^T of a given graph G is obtained by attaching $t(>0)$ pendent vertices to each vertex of G. The pendent edges, called thorns of G^T, can be treated as P_2 or K_2, so that a thorn graph is generalized by replacing P_2 by P_m and K_2 by K_p and the respective generalizations are denoted by G_{P_m} and G_{K_p}. The modified eccentric connectivity index of a graph is defined as the sum of the products of eccentricity with the total degree of neighboring vertices, over all vertices of the graph in a hydrogen suppressed molecular structure. In this paper, we give the modified eccentric connectivity index and the concerned polynomial for the thorn graph G^T and the generalized thorn graphs G_{K_p} and G_{P_m}.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, so that $|V(G)| = k$ and $|E(G)| = e$. Let the vertices of G be labeled as $v_1, v_2, \ldots, v_k$. For any vertex $v_i \in V(G)$ the number of neighbors of v_i is defined as the degree of the vertex v_i and is denoted by $d_G(v_i)$. Let $N(v_i)$ denote the set of vertices which are the neighbors of the vertex v_i, so that $|N(v_i)| = d_G(v_i)$. Also let $\delta_G(v_i) = \sum_{v_j \in N(v_i)} d_G(v_j)$, that is, sum of degrees of the neighboring vertices of $v_i \in G$. The distance between the vertices v_i and v_j is equal to the length of the shortest path connecting v_i and v_j. Also for a given vertex $v_i \in V(G)$, the eccentricity $\varepsilon_G(v_i)$ is the largest distance from v_i to any other vertices of G and the sum of eccentricities of all the vertices of G is denoted by $\theta(G)$ [1]. The eccentric connectivity index of a graph G was proposed by Sharma et al. [2]. A lot of results related to chemical and mathematical study on eccentric connectivity index have taken place in the literature [3–5]. There are numerous modifications of eccentric connectivity index reported in the literature till date. These include edge versions of eccentric connectivity index [6], eccentric connectivity topochemical index [7], augmented eccentric connectivity index [8], superaugmented eccentric connectivity index [9], and connective eccentricity index [10]. A modified version of eccentric connectivity index was proposed by Ashrafi and Ghorbani [11].

Similar to other topological polynomials, the corresponding polynomial, that is, the modified eccentric connectivity polynomial of a graph, is defined as

$$\xi_c(G, x) = \sum_{i=1}^{k} \delta_G(v_i)\, x^{\varepsilon_G(v_i)}, \tag{1}$$

so that the modified eccentric connectivity index is the first derivative of this polynomial for $x = 1$. Several studies on this modified eccentric connectivity index are also found in the literature. In [11], the modified eccentric connectivity polynomials for three infinite classes of fullerenes were computed. In [12], a numerical method for computing modified eccentric connectivity polynomial and modified eccentric connectivity index of one-pentagonal carbon nanocones was presented. In [13], some exact formulas for the modified eccentric connectivity polynomial of Cartesian product, symmetric difference, disjunction, and join of graphs were presented. Some upper and lower bounds for this modified eccentric connectivity index are recently obtained by the present authors [14].

The first and the second Zagreb indices of G, denoted by $M_1(G)$ and $M_2(G)$, respectively, are two of the oldest topological indices introduced in [15] by Gutman and Trinajstić and were defined as

$$M_1(G) = \sum_{i=1}^{k} \{d_G(v_i)\}^2 = \sum_{(v_i,v_j)\in E(G)} [d_G(v_i) + d_G(v_j)] = \sum_{i=1}^{k} \delta_G(v_i), M_2(G) = \sum_{(v_i,v_j)\in E(G)} d_G(v_i)\, d_G(v_j). \quad (2)$$

Let $p = (p_1, p_2, \ldots, p_k)$ be a k-tuple of nonnegative integers. The thorn graph G^T is the graph obtained from G by attaching p_i pendent vertices to the vertex v_i, $i = 1, 2, \ldots, k$ of G. In this paper, we assume $p_1 = p_2 = \cdots = p_k = t$. These pendent vertices are termed as thorns. The concept of thorn graphs was first introduced by Gutman [16]. A lot of studies on thorn graphs for different topological indices are made by several researchers in the recent past [17–24]. Very recently, De [25, 26] studied two eccentricity related topological indices, such as eccentric connectivity index and augmented eccentric connectivity indices, on thorn graphs.

The thorns of the thorn graph G^T can be treated as P_2 or K_2, so that the thorn graph can be generalized by replacing P_2 by P_m and K_2 by K_p and the generalizations are, respectively, denoted by G_{P_m} and G_{K_p}. In the following section, we present the explicit expressions of the modified eccentric connectivity index of thorn graph G^T and its generalized forms G_{K_p} and G_{P_m}.

2. Evaluation of Modified Eccentric Connectivity Index

The eccentric connectivity index $\xi^c(G)$ [2] and connective eccentric index $C^\xi(G)$ [10] of a graph G are defined as

$$\xi^c(G) = \sum_{i=1}^{k} d_G(v_i)\, \varepsilon_G(v_i), \qquad C^\xi(G) = \sum_{v\in V(G)} \frac{d_G(v)}{\varepsilon_G(v)}. \quad (3)$$

The modified eccentric connectivity index $\xi_c(G)$ [11] is defined as

$$\xi_c(G) = \sum_{i=1}^{k} \delta_G(v_i)\, \varepsilon_G(v_i). \quad (4)$$

Total eccentricity index is defined as $\zeta(G) = \sum_{v\in V(G)} \epsilon_G(v)$. Total eccentricity index of the generalized hierarchical product of graphs has been studied by De et al. recently [14].

Since the modified eccentric connectivity index is likely to have an application in drug discovery process, therefore, we evaluate the index in comparison to some other well known indices in this section.

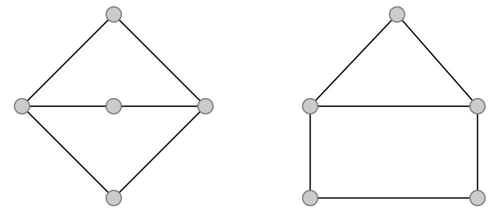

Figure 1: Two graphs with the same ξ^c and C^ξ, but with different ξ_c.

The two graphs shown in Figure 1 have the same eccentricity connectivity index and connective eccentricity index, but they have different modified eccentric connectivity indices.

To evaluate the modified eccentric connectivity index (MECI) in terms of degeneracy and intercorrelation with other well known indices we compute different topological indices such as eccentric connectivity index (ECI), total eccentricity index (TEI), connective eccentricity index, (CEI) and augmented eccentric connectivity index (AECI) for octane isomers as given in Table 1.

For modified eccentric connectivity index we observe that maximum value = 136, minimum value = 93, ratio = 1.46, and degeneracy = 4/18.

Intercorrelation of modified eccentric connectivity index with some well known vertex eccentricity based topological indices is given in Table 2.

3. Main Results

First we find the modified eccentric connectivity index of the thorn graph G^T in terms of modified eccentric connectivity index of G, total eccentricity of G, and first Zagreb index of G.

Theorem 1. *For any simple connected graph G, $\xi_c(G^T)$ and $\xi_c(G)$ are related as $\xi_c(G^T) = \xi_c(G) + 2t\xi^c(G) + t(t+1)\theta(G) + M_1(G) + 6et + kt + 2kt^2$, where $|V(G)| = k$ and G^T is the thorn graph of G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_k\}$ and $G^T = (V^T, E^T)$, where $V^T = V(G)\cup V_1 \cup V_2 \cdots \cup V_k$. Here, V_i are the set of degree one vertices attached to the vertices v_i in G^T and $V_i \cap V_j = \varphi$, $i \neq j$. Let the vertices of the set V_i be denoted by $v_{i1}, v_{i2}, \ldots, v_{ik}$ for $i = 1, 2, \ldots, k$.

Then the degree of the vertices v_i in G^T is given by $d_{G^T}(v_i) = d_G(v_i) + t$, for $i = 1, 2, \ldots, k$. Hence, $\delta_{G^T}(v_i) = \delta_G(v_i) + d_G(v_i)t + t$ for $i = 1, 2, \ldots, k$, and $\delta_{G^T}(v_{ij}) = d_G(v_i) + t$. Similarly the eccentricity of the vertices v_i, $i = 1, 2, \ldots, k$ in G^T is given by $\varepsilon_{G^T}(v_i) = \varepsilon_G(v_i) + 1$, for $i = 1, 2, \ldots, k$, and the eccentricity of the vertices v_{ij} is given by $\varepsilon_{G^T}(v_{ij}) = \varepsilon_G(v_i) + 2$, for $j = 1, 2, \ldots, t$ and $i = 1, 2, \ldots, k$.

Then the modified eccentric connectivity index of G^T is given by

$$\xi_c(G^T) = \sum_{i=1}^{k} \delta_{G^T}(v_i)\, \varepsilon_{G^T}(v_i) + \sum_{i=1}^{k}\sum_{j=1}^{t} \delta_{G^T}(v_{ij})\, \varepsilon_{G^T}(v_{ij}). \quad (5)$$

Table 1: Different topological indices of octane isomers.

Molecule	MECI	ECI	TEI	CEI	AECI
Octane	136	74	44	2.352	3.838
2-Methyl-heptane	126	65	39	3.167	6.167
3-Methyl-heptane	121	63	38	3.25	6.267
4-Methyl-heptane	118	61	37	3.383	6.55
3-Ethyl-hexane	107	54	33	3.767	7.867
2,2-Dimethyl-hexane	132	56	34	3.633	7.8
2,3-Dimethyl-hexane	117	54	33	3.767	7.6
2,4-Dimethyl-hexane	117	54	33	3.767	7.933
2,5-Dimethyl-hexane	114	56	34	3.633	7.4
3,3-Dimethyl-hexane	114	52	32	5.4	8.3
3,4-Dimethyl-hexane	112	52	32	3.9	7.8
2-Methyl-3-ethyl-pentane	93	43	27	4.833	11.5
3-Methyl-3-ethyl-pentane	95	41	26	5.083	10.833
2,2,3-Trimethyl-pentane	107	43	27	4.833	10.5
2,2,4-Trimethyl-pentane	115	45	28	4.583	11.833
2,3,3-Trimethyl-pentane	103	41	26	5.083	10.333
2,3,4-Trimethyl-pentane	101	43	43	4.833	11.5
2,2,3,3-Tetramethyl-butane	100	34	34	6	10

Table 2: Intercorrelation of indices.

	MECI	ECI	TEI	CEI	AECI
MECI	1	0.737	0.325	0.628	0.652
ECI		1	0.426	0.866	0.852
TEI			1	0.34	0.405
CEI				1	0.689
AECI					1

Now,

$$\begin{aligned}
&\sum_{i=1}^{k}\delta_{G^T}(v_i)\,\varepsilon_{G^T}(v_i)\\
&\quad=\sum_{i=1}^{k}\{\delta_G(v_i)+td_G(v_i)+t\}\{\varepsilon_G(v_i)+1\}\\
&\quad=\xi_c(G)+t\xi^c(G)+t\theta(G)+M_1(G)+2et+kt,\\
&\sum_{i=1}^{k}\sum_{j=1}^{t}\delta_{G^T}(v_{ij})\,\varepsilon_{G^T}(v_{ij})\\
&\quad=\sum_{i=1}^{k}\sum_{j=1}^{t}\{d_G(v_i)+t\}\{\varepsilon_G(v_i)+2\}\\
&\quad=\sum_{i=1}^{k}\sum_{j=1}^{t}d_G(v_i)\,\varepsilon_G(v_i)+\sum_{i=1}^{k}\sum_{j=1}^{t}t\varepsilon_G(v_i)\\
&\qquad+2\sum_{i=1}^{k}\sum_{j=1}^{t}d_G(v_i)+\sum_{i=1}^{k}\sum_{j=1}^{t}2t\\
&\quad=t\xi^c(G)+t^2\theta(G)+4et+2kt^2.
\end{aligned}\tag{6}$$

Combining the above equations, we get

$$\begin{aligned}
\xi_c\left(G^T\right)&=\xi_c(G)+2t\xi^c(G)+t(t+1)\theta(G)+M_1(G)\\
&\quad+6et+kt+2kt^2.
\end{aligned}\tag{7}$$

□

The eccentric connectivity polynomial and total eccentricity polynomial of G are defined as $\xi^c(G,x)=\sum_{i=1}^{k}d_G(v_i)x^{\varepsilon_G(v_i)}$ and $\theta(G,x)=\sum_{i=1}^{k}x^{\varepsilon_G(v_i)}$, respectively. It is easy to see that the eccentric connectivity index and the total eccentricity of a graph can be obtained from the corresponding polynomials by evaluating their first derivatives at $x=1$.

Now we express the modified eccentric polynomial of a thorn graph in terms of eccentric connectivity polynomial, modified eccentric connectivity polynomial, and total eccentric polynomial of the parent graph.

Theorem 2. *For any simple connected graph G, the polynomials $\xi_c(G^T,x)$ and $\xi_c(G,x)$ are related as $\xi_c(G^T,x)=x\xi_c(G,x)+tx(x+1)\xi^c(G,x)+tx(tx+1)\theta(G,x)$, where G^T is the thorn graph of G.*

Proof. Following the previous theorem, the modified eccentric connectivity polynomial of G^T is given by $\xi_c(G^T, x) = \sum_{i=1}^{k} \delta_{G^T}(v_i)x^{\varepsilon_{G^T}(v_i)} + \sum_{i=1}^{k}\sum_{j=1}^{t} \delta_{G^T}(v_{ij})x^{\varepsilon_{G^T}(v_{ij})}$.

Now,

$$\begin{aligned}
&\sum_{i=1}^{k}\delta_{G^T}(v_i)\,x^{\varepsilon_{G^T}(v_i)}\\
&\quad=\sum_{i=1}^{k}\{\delta_G(v_i)+td_G(v_i)+t\}\,x^{\{\varepsilon_G(v_i)+1\}}\\
&\quad= x\xi_c(G,x)+tx\xi^c(G,x)+tx\theta(G,x),\\
&\sum_{i=1}^{k}\sum_{j=1}^{t}\delta_{G^T}(v_{ij})\,x^{\varepsilon_{G^T}(v_{ij})}\\
&\quad=\sum_{i=1}^{k}\sum_{j=1}^{t}\{d_G(v_i)+t\}\,x^{\{\varepsilon_G(v_i)+2\}}\\
&\quad= x^2t\xi^c(G,x)+t^2x^2\theta(G,x).
\end{aligned}\tag{8}$$

After addition, we get

$$\begin{aligned}\xi_c\left(G^T,x\right)&=x\xi_c(G,x)+tx(x+1)\,\xi^c(G,x)\\
&\quad+tx(tx+1)\,\theta(G,x).\end{aligned}\tag{9}$$

□

It can be easily verified that expression (7) is obtained by differentiating (9) with respect to x and by putting $x = 1$.

Let G_{K_p} be the graph obtained from G by attaching t complete graphs of order p, that is, K_p, at every vertex of G. Let the vertices attached to the vertex v_i be denoted by $v_{i1}^{(r)}, v_{i2}^{(r)}, \ldots, v_{ip}^{(r)}$, $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$. Let the vertex v_i be identified with $v_{ip}^{(r)}$, $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$.

Theorem 3. *For any simple connected graph G, $\xi_c(G_{K_p})$ and $\xi_c(G)$ are related as $\xi_c(G_{K_p}) = \xi_c(G) + 2t(p-1)\xi^c(G) + t(2t+1)(p-1)^2\theta(G)+M_1(G)+6et(p-1)+kt^2(p-1)^2(2t+2p-3)$, where G_{K_p} is the graph obtained from G by attaching t complete graphs K_p at each vertex of G.*

Proof. The eccentricities of the vertices of G_{K_p} are given by $\varepsilon_{G_{K_p}}(v_i) = \varepsilon_G(v_i) + 1$, for $i = 1, 2, \ldots, k$, and $\varepsilon_{G_{K_p}}(v_{ij}^{(r)}) = \varepsilon_G(v_i) + 2$, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, p$; $r = 1, 2, \ldots, t$.

The degree of the vertices of G_{K_p} is given by $d_{G_{K_p}}(v_i) = d_G(v_i) + t$, for $i = 1, 2, \ldots, k$, and $d_{G_{K_p}}(v_{ij}^{(r)}) = (p-1)$, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, (p-1)$; $r = 1, 2, \ldots, t$.

Thus, $\delta_{G_{K_p}}(v_i) = \delta_G(v_i) + t(p-1) + t(p-1)^2$, for $i = 1, 2, \ldots, k$, and $\delta_{G_{K_p}}(v_{ij}^{(r)}) = d_G(v_i)+t(p-1)d_G(v_i)+(p-1)(p-2)$, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, (p-1)$; $r = 1, 2, \ldots, t$.

Therefore the modified eccentric connectivity index of G_{K_p} is given by $\xi^c(G_{K_p}) = \sum_{i=1}^{k} \delta_{G_{K_p}}(v_i)\varepsilon_{G_{K_p}}(v_i) + \sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t} \delta_{G_{K_p}}(v_{ij}^{(r)})\varepsilon_{G_{K_p}}(v_{ij}^{(r)})$.

Now,

$$\begin{aligned}
&\sum_{i=1}^{k}\delta_{G_{K_p}}(v_i)\,\varepsilon_{G_{K_p}}(v_i)\\
&\quad=\sum_{i=1}^{k}\left\{\delta_G(v_i)+t(p-1)\,d_G(v_i)+t(p-1)^2\right\}\\
&\qquad\times\{\varepsilon_G(v_i)+1\}\\
&\quad=\sum_{i=1}^{k}\delta_G(v_i)\,\varepsilon_G(v_i)+t(p-1)\sum_{i=1}^{k}d_G(v_i)\,\varepsilon_G(v_i)\\
&\qquad+t(p-1)^2\sum_{i=1}^{k}\varepsilon_G(v_i)\\
&\qquad+\sum_{i=1}^{k}\delta_G(v_i)+t(p-1)\sum_{i=1}^{k}d_G(v_i)\\
&\qquad+\sum_{i=1}^{k}t(p-1)^2\\
&\quad=\xi_c(G)+t(p-1)\,\xi^c(G)+t(p-1)^2\,\theta(G)\\
&\qquad+M_1(G)+2et(p-1)+kt(p-1)^2,\\
&\sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\delta_{G_{K_p}}\left(v_{ij}^{(r)}\right)\varepsilon_{G_{K_p}}\left(v_{ij}^{(r)}\right)\\
&\quad=\sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\{d_G(v_i)+t(p-1)+(p-1)(p-2)\}\\
&\qquad\times\{\varepsilon_G(v_i)+2\}\\
&\quad=\left[\xi^c(G)+t(p-1)\,\theta(G)+(p-1)(p-2)\,\theta(G)\right.\\
&\qquad\left.+4e+2kt(p-1)+2k(p-1)(p-2)\right]t(p-1)\\
&\quad=t(p-1)\,\xi^c(G)+t^2(p-1)^2\,\theta(G)+t(p-1)^2\\
&\qquad\times(p-2)\,\theta(G)+4et(p-1)\\
&\qquad+2kt^2(p-1)^2+2kt(p-1)^2(p-2).
\end{aligned}\tag{10}$$

Adding, we get

$$\begin{aligned}\xi_c\left(G_{K_p}\right)&=\xi_c(G)+2t(p-1)\,\xi^c(G)+t(2t+1)\\
&\quad\times(p-1)^2\,\theta(G)+M_1(G)+6et(p-1)\\
&\quad+kt^2(p-1)^2(2t+2p-3).\end{aligned}\tag{11}$$

□

Since for $p = 2$, the generalized thorn graph G_{K_p} reduces to the usual thorn graph G^T, Theorem 1 follows from Theorem 3 by substituting $p = 2$.

In the following, we find the modified eccentric connectivity polynomial of the graph G_{K_p} in terms of eccentric connectivity polynomial, modified eccentric connectivity polynomial, and total eccentric polynomial of the parent graph G.

Theorem 4. *For any simple connected graph G, the $\xi_c(G_{K_p}, x)$ is given by $\xi_c(G_{K_p}, x) = \xi_c(G, x) + xt(p-1)(x+1)\xi^c(G, x) + tx(p-1)^2(1+tx+x(p-2))\theta(G, x)$, where G_{K_p} is the graph obtained from G by attaching t complete graphs K_p at each vertex of G.*

Proof. From definition, the modified eccentric connectivity polynomial of G_{K_p} is given by

$$\xi^c\left(G_{K_p}, x\right) = \sum_{i=1}^{k}\delta_{G_{K_p}}(v_i)\, x^{\varepsilon_{G_{K_p}}(v_i)} + \sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\delta_{G_{K_p}}\left(v_{ij}^{(r)}\right) x^{\varepsilon_{G_{K_p}}(v_{ij}^{(r)})}. \tag{12}$$

Now,

$$\begin{aligned}
&\sum_{i=1}^{k}\delta_{G_{K_p}}(v_i)\, x^{\varepsilon_{G_{K_p}}(v_i)} \\
&= \sum_{i=1}^{k}\left\{\delta_G(v_i) + t(p-1)d_G(v_i) + t(p-1)^2\right\} x^{\{\varepsilon_G(v_i)+1\}} \\
&= x\left[\sum_{i=1}^{k}\delta_G(v_i)\, x^{\varepsilon_G(v_i)} + t(p-1)\sum_{i=1}^{k}d_G(v_i)\, x^{\varepsilon_G(v_i)} + t(p-1)^2\sum_{i=1}^{k}x^{\varepsilon_G(v_i)}\right] \\
&= x\xi_c(G,x) + tx(p-1)\xi^c(G,x) + tx(p-1)^2\theta(G,x).
\end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\delta_{G_{K_p}}\left(v_{ij}^{(r)}\right) x^{\varepsilon_{G_{K_p}}(v_{ij}^{(r)})} \\
&= \sum_{i=1}^{k}\sum_{j=1}^{p-1}\sum_{r=1}^{t}\{d_G(v_i) + t(p-1) + (p-1)(p-2)\} \times x^{\{\varepsilon_G(v_i)+2\}} \\
&= x^2\left[\xi^c(G,x) + t(p-1)\theta(G,x) + (p-1)(p-2)\theta(G,x)\right] t(p-1).
\end{aligned} \tag{14}$$

Adding the above two, we get

$$\xi_c\left(G_{K_p}, x\right) = \xi_c(G,x) + xt(p-1)(x+1)\xi^c(G,x) + tx(p-1)^2(1+tx+x(p-2))\theta(G,x). \tag{15}$$

□

It can also be verified that expression (11) is obtained by differentiating (15) with respect to x and by putting $x = 1$. Also Theorem 2 follows from Theorem 4 by substituting $p = 2$.

Let us now construct a graph G_{P_m} by attaching t paths of order m (≥ 2) at each vertex $v_i, 1 \leq i \leq k$ of G. The vertices of the rth path attached to v_i are denoted by $v_{i1}^{(r)}, v_{i2}^{(r)}, \ldots, v_{im}^{(r)}$, $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$. Let the vertex $v_{i1}^{(r)}$ be identified with the ith vertex v_i of G. Clearly the resulting graph G_{P_m} consists of $\{(m-1)t + k\}$ number of vertices.

Theorem 5. *For any simple connected graph G with k vertices, $\xi_c(G_{P_m})$ and $\xi_c(G)$ are related as $\xi_c(G_{P_m}) = \xi_c(G) + 2t\xi^c(G) + (t^2 - 7t + 4mt)\theta(G) + (m-1)M_1(G) + 2et(2m-1) + kt(6m^2 - 16m + mt + 9)$, where G_{P_m} is the graph obtained from G by attaching t paths each of length m at each vertex of G.*

Proof. The eccentricities of the vertices of G_{P_m} are given by $\varepsilon_{G_{P_m}}(v_i) = \varepsilon_G(v_i) + (m-1)$, for $i = 1, 2, \ldots, k$, and $\varepsilon_{G_{P_m}}(v_{ij}^{(r)}) = \varepsilon_G(v_i) + m + j - 2$, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, m$; $r = 1, 2, \ldots, t$.

The degrees of the vertices of G_{P_m} are given by $d_{G_{P_m}}(v_i) = d_G(v_i) + t$, for $i = 1, 2, \ldots, k$; $d_{G_{P_m}}(v_{ij}^{(r)}) = 2$, for $i = 1, 2, \ldots, k$; $j = 2, \ldots, (m-1)$; $r = 1, 2, \ldots, t$ and $d_{G_{P_m}}(v_{im}^{(r)}) = 1$, for $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$.

Thus, $\delta_{G_{P_m}}(v_i) = \delta_G(v_i) + td_G(v_i) + 2t$, for $i = 1, 2, \ldots, k$; $\delta_{G_{P_m}}(v_{ij}^{(r)}) = 4$, for $i = 1, 2, \ldots, k$; $j = 3, 4, \ldots, (m-2)$; $r = 1, 2, \ldots, t$; $\delta_{G_{P_m}}(v_{i2}^{(r)}) = 2 + d_G(v_i) + t$, for $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$; $\delta_{G_P}(v_{i(m-1)}^{(r)}) = 3$, for $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$ and $\delta_{G_P}(v_{im}^{(r)}) = 2$, for $i = 1, 2, \ldots, k$; $r = 1, 2, \ldots, t$.

Therefore, the modified eccentric connectivity index of G_{P_m} is given by

$$\xi^c\left(G_{P_m}\right) = \sum_{i=1}^{k}\delta_{G_{P_m}}(v_i)\,\varepsilon_{G_{P_m}}(v_i) + \sum_{i=1}^{k}\sum_{j=2}^{m}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{ij}^{(r)}\right)\varepsilon_{G_{P_m}}\left(v_{ij}^{(r)}\right). \tag{16}$$

Now,

$$\begin{aligned}
&\sum_{i=1}^{k}\delta_{G_{P_m}}(v_i)\,\varepsilon_{G_{P_m}}(v_i) \\
&= \sum_{i=1}^{k}\{\delta_G(v_i) + td_G(v_i) + 2t\}\{\varepsilon_G(v_i) + (m-1)\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \delta_G(v_i)\,\varepsilon_G(v_i) + t\sum_{i=1}^{k} d_G(v_i)\,\varepsilon_G(v_i) + 2t\sum_{i=1}^{k}\varepsilon_G(v_i)\\
&\quad + (m-1)\sum_{i=1}^{k}\delta_G(v_i) + t(m-1)\\
&\quad \times \sum_{i=1}^{k} d_G(v_i) + 2tk(m-1)\\
&= \xi_c(G) + t\xi^c(G) + 2t\theta(G) + (m-1)M_1(G)\\
&\quad + 2et(m-1) + 2kt(m-1).
\end{aligned}
\tag{17}
$$

Also,

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=2}^{m}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{ij}^{(r)}\right)\varepsilon_{G_{P_m}}\left(v_{ij}^{(r)}\right)\\
&\quad= \sum_{i=1}^{k}\sum_{j=3}^{m-2}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{ij}^{(r)}\right)\varepsilon_{G_{P}}\left(v_{ij}^{(r)}\right)\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{i2}^{(r)}\right)\varepsilon_{G_{P_m}}\left(v_{i2}^{(r)}\right)\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{i(m-1)}^{(r)}\right)\varepsilon_{G_{P_m}}\left(v_{i(m-1)}^{(r)}\right)\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{im}^{(r)}\right)\varepsilon_{G_{P_m}}\left(v_{im}^{(r)}\right)\\
&\quad= \sum_{i=1}^{k}\sum_{j=3}^{m-2}\sum_{r=1}^{t}4\left\{\varepsilon_G(v_i)+m+j-2\right\}\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}\left\{2+d_G(v_i)+t\right\}\left\{\varepsilon_G(v_i)+m\right\}\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}3\left\{\varepsilon_G(v_i)+2m-3\right\}\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}2\left\{\varepsilon_G(v_i)+2(m-1)\right\}\\
&\quad= t\xi^c(G)+\left(t^2+2t\right)\theta(G)+2emt+mt(t+2)k+3t\theta(G)\\
&\qquad+3kt(2m-3)+2t\theta(G)+4kt(m-1)\\
&\qquad+4t(m-4)\theta(G)+4kt(m-2)(m-4)\\
&\qquad+2kt\left\{(m-1)(m-2)-6\right\}.
\end{aligned}
\tag{18}
$$

Combining the above, we have

$$
\begin{aligned}
\xi_c\left(G_{P_m}\right) &= \xi_c(G) + 2t\xi^c(G) + \left(t^2-7t+4mt\right)\theta(G)\\
&\quad + (m-1)M_1(G) + 2et(2m-1)\\
&\quad + kt\left(6m^2-16m+mt+9\right).
\end{aligned}
\tag{19}
$$

□

Since the generalized thorn graph G_{P_m} also reduces to the usual thorn graph G^T for $m = 2$, Theorem 1 follows from Theorem 5 by substituting $m = 2$.

In the following, we find the modified eccentric connectivity polynomial of the graph G_{P_m} in terms of eccentric connectivity polynomial, modified eccentric connectivity polynomial, and total eccentric polynomial of the parent graph G.

Theorem 6. *For any simple connected graph G, the modified eccentric connectivity polynomial $\xi_c(G_{P_m}, x)$ is given by $\xi_c(G_{P_m}, x) = tx^{m-1}\xi_c(G,x) + tx^{m-1}(tx+1)\xi^c(G,x) + tx^{m-1}\{2 + x(t+2) + x^{m-2}(3+2x) + 4x^2\sum_{i=0}^{m-5}x^i\}\theta(G,x)$, where G_{P_m} is the graph obtained from G by attaching t paths each of length m at each vertex of G.*

Proof. The modified eccentric connectivity polynomial of G_{P_m} is given by

$$
\begin{aligned}
\xi^c\left(G_{P_m}, x\right) &= \sum_{i=1}^{k}\delta_{G_{P_m}}(v_i)\,x^{\varepsilon_{G_{P_m}}(v_i)}\\
&\quad + \sum_{i=1}^{k}\sum_{j=2}^{m}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{ij}^{(r)}\right)x^{\varepsilon_{G_{P_m}}(v_{ij}^{(r)})}.
\end{aligned}
\tag{20}
$$

Now, proceeding as above theorem, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}\delta_{G_{P_m}}(v_i)\,\varepsilon_{G_{P_m}}(v_i)\\
&\quad= \sum_{i=1}^{k}\left\{\delta_G(v_i)+td_G(v_i)+2t\right\}x^{\{\varepsilon_G(v_i)+(m-1)\}}\\
&\quad= x^{m-1}\Bigg[\sum_{i=1}^{k}\delta_G(v_i)\,x^{\varepsilon_G(v_i)} + t\sum_{i=1}^{k}d_G(v_i)\,x^{\varepsilon_G(v_i)}\\
&\qquad\qquad + 2t\sum_{i=1}^{k}x^{\varepsilon_G(v_i)}\Bigg]\\
&\quad= x^{m-1}\left[\xi_c(G,x)+t\xi^c(G,x)+2t\theta(G,x)\right].
\end{aligned}
\tag{21}
$$

Again,

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=2}^{m}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{ij}^{(r)}\right)x^{\varepsilon_{G_{P_m}}(v_{ij}^{(r)})}\\
&\quad= \sum_{i=1}^{k}\sum_{j=3}^{m-2}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{ij}^{(r)}\right)x^{\varepsilon_{G_P}(v_{ij}^{(r)})} + \sum_{i=1}^{k}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{i2}^{(r)}\right)x^{\varepsilon_{G_{P_m}}(v_{i2}^{(r)})}\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{i(m-1)}^{(r)}\right)x^{\varepsilon_{G_{P_m}}(v_{i(m-1)}^{(r)})}\\
&\qquad+\sum_{i=1}^{k}\sum_{r=1}^{t}\delta_{G_{P_m}}\left(v_{im}^{(r)}\right)x^{\varepsilon_{G_{P_m}}(v_{im}^{(r)})}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\sum_{j=3}^{m-2}\sum_{r=1}^{t} 4x^{\{\varepsilon_G(v_i)+m+j-2\}} \\
&\quad + \sum_{i=1}^{k}\sum_{r=1}^{t} \{2 + d_G(v_i) + t\}\, x^{\{\varepsilon_G(v_i)+m\}} \\
&\quad + \sum_{i=1}^{k}\sum_{r=1}^{t} 3x^{\{\varepsilon_G(v_i)+2m-3\}} + \sum_{i=1}^{k}\sum_{r=1}^{t} 2x^{\{\varepsilon_G(v_i)+2(m-1)\}} \\
&= 4tx^{(m+1)} \left(1 + x + x^2 + \cdots + x^{m-5}\right) \theta(G, x) \\
&\quad + tx^m (t+2)\, \theta(G, x) + x^m t \xi^c(G, x) + 3x^{2m-3} t\theta(G, x) \\
&\quad + 2tx^{2(m-1)} \theta(G, x).
\end{aligned} \tag{22}
$$

Combining the above two, we get

$$
\begin{aligned}
&\xi_c\left(G_{P_m}, x\right) \\
&\quad = tx^{m-1}\xi_c(G, x) + tx^{m-1}(tx + 1)\,\xi^c(G, x) + tx^{m-1} \\
&\qquad \times \left\{2 + x(t+2) + x^{m-2}(3 + 2x) + 4x^2 \sum_{i=0}^{m-5} x^i\right\} \theta(G, x).
\end{aligned} \tag{23}
$$

□

Here also, differentiating (23) with respect to x and putting $x = 1$, we get relation (19).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors are grateful to anonymous reviewer whose suggestions improved the presentation of the paper.

References

[1] P. Dankelmann, W. Goddard, and C. S. Swart, "The average eccentricity of a graph and its subgraphs," *Utilitas Mathematica*, vol. 65, pp. 41–51, 2004.

[2] V. Sharma, R. Goswami, and A. K. Madan, "Eccentric connectivity index: a novel highly discriminating topological descriptor for structure-property and structure-activity studies," *Journal of Chemical Information and Computer Sciences*, vol. 37, no. 2, pp. 273–282, 1997.

[3] T. Došlic, M. Saheli, and D. Vukičević, "Eccentric connectivity index: extremal graphs and values," *Iranian Journal of Mathematical Chemistry*, vol. 1, pp. 45–56, 2010.

[4] A. Ilić and I. Gutman, "Eccentric connectivity index of chemical trees," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 65, pp. 731–744, 2011.

[5] B. Zhou and Z. Du, "On eccentric connectivity index," *Match Communications in Mathematical and in Computer Chemistry*, vol. 63, no. 1, pp. 181–198, 2010.

[6] X. Xu and Y. Guo, "The edge version of eccentric connectivity index," *International Mathematical Forum*, vol. 7, no. 6, pp. 273–280, 2012.

[7] V. Kumar, S. Sardana, and A. K. Madan, "Predicting anti-HIV activity of 2,3-diaryl-1,3-thiazolidin-4-ones: computational approach using reformed eccentric connectivity index," *Journal of Molecular Modeling*, vol. 10, no. 5-6, pp. 399–407, 2004.

[8] T. Došlić and M. Saheli, "Augmented eccentric connectivity index," *Miskolc Mathematical Notes*, vol. 12, no. 2, pp. 149–157, 2011.

[9] H. Dureja and A. K. Madan, "Superaugmented eccentric connectivity indices: new-generation highly discriminating topological descriptors for QSAR/QSPR modeling," *Medicinal Chemistry Research*, vol. 16, no. 7–9, pp. 331–341, 2007.

[10] S. Gupta, M. Singh, and A. K. Madan, "Connective eccentricity index: a novel topological descriptor for predicting biological activity," *Journal of Molecular Graphics and Modelling*, vol. 18, no. 1, pp. 18–25, 2000.

[11] A. R. Ashrafi and M. Ghorbani, "A study of fullerenes by MEC polynomials," *Electronic Materials Letters*, vol. 6, no. 2, pp. 87–90, 2010.

[12] M. Alaeiyan, J. Asadpour, and R. Mojarad, "A numerical method for MEC polynomial and MEC index of one-pentagonal carbon nanocones," *Fullerenes, Nanotubes and Carbon Nanostructures*, vol. 21, no. 10, pp. 825–835, 2013.

[13] A. R. Ashrafi, M. Ghorbani, and M. A. Hossein-Zadeh, "The eccentric connectivity polynomial of some graph operations," *Serdica Journal of Computing*, vol. 5, no. 2, pp. 101–116, 2011.

[14] N. De, S. M. A. Nayeem, and A. Pal, "Total eccentricity index of the generalized hierarchical product of graphs," *International Journal of Applied and Computational Mathematics*, 2014.

[15] I. Gutman and N. Trinajstić, "Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons," *Chemical Physics Letters*, vol. 17, no. 4, pp. 535–538, 1972.

[16] I. Gutman, "Distance of thorny graphs," *Publications de l'Institut Mathématique*, vol. 63, pp. 31–36, 1998.

[17] D. Bonchev and D. J. Klein, "On the wiener number of thorn trees, stars, rings, and rods," *Croatica Chemica Acta*, vol. 75, no. 2, pp. 613–620, 2002.

[18] A. Heydari and I. Gutman, "On the terminal Wiener index of thorn graphs," *Kragujevac Journal of Science*, vol. 32, pp. 57–64, 2010.

[19] K. M. Kathiresan and C. Parameswaran, "Certain generalized thorn graphs and their Wiener indices," *Journal of Applied Mathematics and Informatics*, vol. 30, no. 5-6, pp. 793–807, 2012.

[20] S. Li, "Zagreb polynomials of thorn graphs," *Kragujevac Journal of Science*, vol. 33, pp. 33–38, 2011.

[21] D. Vukičević, B. Zhou, and N. Trinajstićc, "Altered Wiener indices of thorn trees," *Croatica Chemica Acta*, vol. 80, no. 2, pp. 283–285, 2007.

[22] H. B. Walikar, H. S. Ramane, L. Sindagi, S. S. Shirakol, and I. Gutman, "Hosoya polynomial of thorn trees, rods, rings, and stars," *Kragujevac Journal of Science*, vol. 28, pp. 47–56, 2006.

[23] B. Zhou, "On modified Wiener indices of thorn trees," *Kragujevac Journal of Mathematics*, vol. 27, pp. 5–9, 2005.

[24] B. Zhou and D. Vukičević, "On Wiener-type polynomials of thorn graphs," *Journal of Chemometrics*, vol. 23, no. 12, pp. 600–604, 2009.

[25] N. De, "On eccentric connectivity index and polynomial of thorn graph," *Applied Mathematics*, vol. 3, pp. 931–934, 2012.

[26] N. De, "Augmented eccentric connectivity index of some thorn graphs," *International Journal of Applied Mathematical Research*, vol. 1, no. 4, pp. 671–680, 2012.