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Research Article

Complex Roots of Unity and Normal Numbers

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Given an arbitrary prime number q, set $\xi = e^{2\pi i/q}$. We use a clever selection of the values of ξ^α, $\alpha = 1, 2, \ldots$, in order to create normal numbers. We also use a famous result of André Weil concerning Dirichlet characters to construct a family of normal numbers.

1. Introduction and Statement of the Results

Let $\lambda(n)$ be the Liouville function (defined by $\lambda(n) := (-1)^{\Omega(n)}$, where $\Omega(n) := \sum_{p^\alpha \| n} \alpha$). It is well known that the statement "$\sum_{n\le x} \lambda(n) = o(x)$ as $x \to \infty$" is equivalent to the Prime Number Theorem. It is conjectured that if $b_1 < b_2 < \cdots < b_k$ are arbitrary positive integers, then $\sum_{n\le x} \lambda(n)\lambda(n+b_1)\cdots\lambda(n+b_k) = o(x)$ as $x \to \infty$. This conjecture seems presently out of reach since we cannot even prove that $\sum_{n\le x} \lambda(n)\lambda(n+1) = o(x)$ as $x \to \infty$.

The Liouville function belongs to a particular class of multiplicative functions, namely, the class $\mathscr{M}^*$ of completely multiplicative functions. Recently, Indlekofer et al. [1] considered a very special function $f \in \mathscr{M}^*$ constructed in the following manner. Let $\wp$ stand for the set of all primes. For each $q \in \wp$, let $C_q = \{\xi \in \mathbb{C} : \xi^q = 1\}$ be the group of complex roots of unity of order q. As p runs through the primes, let ξ_p be independent random variables distributed uniformly on C_q. Then, let $f \in \mathscr{M}^*$ be defined on $\wp$ by $f(p) = \xi_p$, so that $f(n)$ yields a random variable. In their 2011 paper, Indlekofer et al. proved that if $(\Omega, \mathscr{A}, \wp)$ stands for a probability space, where ξ_p $(p \in \wp)$ are the independent random variables, then, for almost all $\omega \in \Omega$, the sequence $\alpha = f(1)f(2)f(3)\cdots$ is a normal sequence over C_q (see Definition 1 below).

Let us now consider a somewhat different setup. Let $q \ge 2$ be a fixed prime number and set $A_q := \{0, 1, \ldots, q-1\}$. Given an integer $t \ge 1$, an expression of the form $i_1 i_2 \cdots i_t$, where each $i_j \in A_q$, is called a *word* of length t. We use the symbol Λ to denote the *empty word*. Then, A_q^t will stand for the set of words of length t over A_q, while A_q^* will stand for the set of all words over A_q regardless of their length, including the empty word Λ. Similarly, we define C_q^* to be the set of words over C_q regardless of their length.

Given a positive integer n, we write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t, \tag{1}$$

where $\varepsilon_i(n) \in A_q$ for $0 \le i \le t$ and $\varepsilon_t(n) \ne 0$. To this representation, we associate the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\cdots\varepsilon_t(n) \in A_q^{t+1}. \tag{2}$$

Definition 1. Given a sequence of integers $a(1), a(2), a(3), \ldots$, one will say that the concatenation of their q-ary digit expansions $\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\cdots$, denoted by $\mathrm{Concat}(\overline{a(n)} : n \in \mathbb{N})$, is a *normal sequence* if the number $0.\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\cdots$ is a q-normal number.

It can be proved using a theorem of Halász (see [2]) that if $f \in \mathscr{M}^*$ is defined on the primes p by $f(p) = \xi_a$ $(a \ne 0)$, then $\sum_{n\le x} f(n) = o(x)$ as $x \to \infty$.

Now, given $u_0, u_1, \ldots, u_{\ell-1} \in A_q$, let $Q(n) := \prod_{j=0}^{\ell-1}(n+j)^{u_j}$. We believe that if $\max_{j\in\{0,1,\ldots,\ell-1\}} u_j > 0$, then

$$\sum_{n\le x} f(Q(n)) = o(x) \quad \text{as } x \longrightarrow \infty. \tag{3}$$

If this was true, it would follow that

$$\text{Concat}\left(f(n) : n \in \mathbb{N}\right) \text{ is a normal sequence over } C_q. \tag{4}$$

We cannot prove (3), but we can prove the following. Let $q \in \wp$ and set $\xi := e^{2\pi i/q}$. Furthermore set $x_k = 2^k$ and $y_k = x_k^{1/\sqrt{k}}$ for $k = 1, 2, \ldots$. Then, consider the sequence of completely multiplicative functions f_k, $k = 1, 2, \ldots$, defined on the primes p by

$$f_k(p) = \begin{cases} \xi & \text{if } k \le p \le y_k, \\ 1 & \text{if } p < k \text{ or } p > y_k. \end{cases} \tag{5}$$

Then, set

$$\eta_k := f_k(x_k) f_k(x_k+1) f_k(x_k+2) \cdots f_k(x_{k+1}-1) \quad (k \in \mathbb{N}), \tag{6}$$

$$\theta := \text{Concat}\left(\eta_k : k \in \mathbb{N}\right).$$

Theorem 2. *The sequence θ is a normal sequence over C_q.*

We now use a famous result of André Weil to construct a large family of normal numbers.

Let q be a fixed prime and set $\xi := e^{2\pi i/q}$ and $\xi_a := e^{2\pi i a/q} = \xi^a$. Recall that C_q stands for the group of complex roots of unity of order q; that is,

$$C_q = \left\{\varsigma \in \mathbb{C} : \varsigma^q = 1\right\} = \left\{\xi^a : a = 0, 1, \ldots, q-1\right\}. \tag{7}$$

Let $p \in \wp$ be such that $q \mid p-1$. Moreover, let χ_p be a Dirichlet character modulo p of order q, meaning that the smallest positive integer t for which $\chi_p^t = \chi_0$ is q. (Here χ_0 stands for the principal character.)

Let $u_0, u_1, \ldots, u_{k-1} \in A_q$. Consider the polynomial

$$F(z) = F_{u_0,\ldots,u_{k-1}}(z) = \prod_{j=0}^{k-1} (z+j)^{u_j} \tag{8}$$

and assume that its degree is at least 1, that is, that there exists one $j \in \{0, \ldots, k-1\}$ for which $u_j \neq 0$. Further set

$$S_{u_0,\ldots,u_{k-1}}(\chi_p) = \sum_{n (\text{mod } p)} \chi_p\left(F_{u_0,\ldots,u_{k-1}}(n)\right). \tag{9}$$

According to a 1948 result of Weil [3],

$$\left|S_{u_0,\ldots,u_{k-1}}(\chi_p)\right| \le (k-1)\sqrt{p}. \tag{10}$$

For a proof, see Proposition 12.11 (page 331) in the book of Iwaniec and Kowalski [4].

We can prove the following.

Theorem 3. *Let $p_1 < p_2 < \cdots$ be an infinite set of primes such that $q \mid p_j - 1$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, let χ_{p_j} be a character modulo p_j of order q. Further set*

$$\Gamma_p = \chi_p(1)\chi_p(2)\cdots\chi_p(p-1) \quad (p = p_1, p_2, \ldots), \tag{11}$$

$$\eta := \Gamma_{p_1}\Gamma_{p_2}\cdots. \tag{12}$$

Then η is a normal sequence over C_q.

As an immediate consequence of this theorem, we have the following corollary.

Corollary 4. *Let $\varphi : C_q \to A_q$ be defined by $\varphi(\xi_a) = a$. Extend the function φ to $\varphi : C_q^* \to A_q^*$ by $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$. Let*

$$\varphi(\eta) = \varphi\left(\Gamma_{p_1}\right)\varphi\left(\Gamma_{p_2}\right)\cdots \tag{13}$$

and consider the q-ary expansion of the real number

$$\kappa = 0,\ \varphi\left(\Gamma_{p_1}\right)\varphi\left(\Gamma_{p_2}\right)\cdots. \tag{14}$$

Then κ is a normal number in base q.

Example 5. Choosing $q = 3$ and $\{p_1, p_2, p_3, \ldots\} = \{7, 13, 19, \ldots\}$ as the set of primes $p_j \equiv 1 \pmod 3$, then, the number η defined by (12) is normal sequence over $\{0, e^{2\pi i/3}, e^{4\pi i/3}\}$ while κ defined by (14) is a ternary normal number.

2. Proof of Theorem 2

Let ℓ be a fixed positive integer. Let $a_0, a_1, \ldots, a_{\ell-1} \in A_q$. Recall the notation $\xi = e^{2\pi i/q}$. Given a positive integer k, let x, y be such that $x_k \le x < x+y \le x_{k+1} - \ell$. We will now count the number $M([x, x+y] \mid (a_0, \ldots, a_{\ell-1}))$ of those $n \in [x, x+y]$ for which $f_k(n+j) = \xi^{a_j}$ $(j = 0, \ldots, \ell-1)$ holds.

Consider the polynomial

$$P_d(x) = \frac{x^q - 1}{x - \xi^d} = \prod_{\substack{h=0 \\ h \neq d}}^{q-1} \left(x - \xi^h\right), \tag{15}$$

so that in particular

$$\left(x - \xi^d\right) P_d(x) = x^q - 1. \tag{16}$$

Taking the derivatives on both sides of the above equation yields

$$P_d(x) + \left(x - \xi^d\right) P_d'(x) = qx^{q-1}. \tag{17}$$

Thus,

$$P_d(f_k(m)) + \left(f_k(m) - \xi^d\right) P_d'(f_k(m)) = q\overline{f_k(m)}, \tag{18}$$

where $\overline{z}$ stands for the complex conjugate of z.

We then have

$$P_d(f_k(m)) = \begin{cases} q\overline{f_k(m)} & \text{if } f_k(m) = \xi^d, \\ 0 & \text{if } f_k(m) \neq \xi^d. \end{cases} \tag{19}$$

Write the polynomial P_d as $P_d(m) = \sum_{u=0}^{q-1} e_u(d) m^u$, so that $P_d(0) = \overline{\xi}^d$; that is, $e_0(d) = \overline{\xi}^d$. We then have

$$\begin{aligned} &P_{a_0}(f_k(n)) \cdots P_{a_{\ell-1}}(f_k(n+\ell-1)) \\ &\quad = \prod_{h=0}^{\ell-1} \left\{ \sum_{u_h=0}^{q-1} e_{u_h}(a_h) f_k^{u_h}(n+h) \right\} \\ &\quad = \sum_{u_0,\ldots,u_{\ell-1} \in A_q} A(u_0, \ldots, u_{\ell-1}) f_k\left(\prod_{j=0}^{\ell-1} (n+j)^{u_j}\right), \end{aligned} \tag{20}$$

where $A(u_0,\dots,u_{\ell-1}) = e_{u_0}(a_0)\cdots e_{u_{\ell-1}}(a_{\ell-1})$, with $A(0,\dots,0) = \overline{\xi}^{\,a_0+\cdots+a_{\ell-1}}$.

With integers x, y such that $x_k \le x < x+y \le x_{k+1}-\ell$, we now sum both sides of (20) for $n = x,\dots,x+y$, and we then obtain that

$$\begin{aligned} q^{\ell}\prod_{j=0}^{\ell-1}\overline{\xi}^{\,a_j}\cdot M\left([x,x+y]\mid(a_0,\dots,a_{\ell-1})\right) &= y\prod_{j=0}^{\ell-1}\overline{\xi}^{\,a_j} + \sum_{\substack{u_0,\dots,u_{\ell-1}\in A_q\\ (u_0,\dots,u_{\ell-1})\neq(0,\dots,0)}} A\left(u_0,\dots,u_{\ell-1}\right) \\ &\quad\times\sum_{n=x}^{x+y} f_k\left(\prod_{j=0}^{\ell-1}(n+j)^{u_j}\right). \end{aligned} \tag{21}$$

Setting

$$Q(n) = \prod_{j=0}^{\ell-1}(n+j)^{u_j}, \tag{22}$$

it remains to prove that

$$\lim_{k\to\infty}\frac{1}{x_k}\max_{x_k\le x<x+y\le x_{k+1}-\ell}\left|\sum_{n=x}^{x+y} f_k\left(Q(n)\right)\right| = 0. \tag{23}$$

To prove this, we proceed using standard techniques. Let $\rho(\delta)$ stand for the number of solutions of the congruence $Q(n) \equiv 0 \pmod{\delta}$, in which case we have $\rho(p^{\alpha}) = \rho(p)$ for all primes $p > k$ and integers $\alpha \ge 1$. Now define the completely multiplicative function g_k implicitly by the relation

$$f_k(m) = \sum_{d\mid m} g_k(d), \tag{24}$$

thus implying, in light of (5), that

$$g_k(p) = f_k(p) - 1 = \begin{cases} 0 & \text{if } p<k \text{ or } p>y_k,\\ \xi-1 & \text{if } k\le p\le y_k. \end{cases} \tag{25}$$

It follows that

$$\begin{aligned} \sum_{n\in[x,x+y]} f_k\left(Q(n)\right) &= \sum_{n\in[x,x+y]}\sum_{\delta\mid Q(n)} g_k(\delta) \\ &= \sum_{\delta} g_k(\delta)\sum_{\substack{n\in[x,x+y]\\ Q(n)\equiv 0 \pmod{\delta}}} 1 \\ &= y\sum_{\delta}\frac{g_k(\delta)\,\rho(\delta)}{\delta} + o(1). \end{aligned} \tag{26}$$

Now, observe that since $g_k(p^{\alpha}) = f_k(p^{\alpha}) - f_k(p^{\alpha-1}) = \xi^{\alpha-1}(\xi-1)$, it follows that

$$\begin{aligned} \sum_{\delta}\frac{g_k(\delta)\,\rho(\delta)}{\delta} &= \prod_{p}\left(1+\frac{g_k(p)\,\rho(p)}{p}+\frac{g_k(p^2)\,\rho(p^2)}{p^2}+\cdots\right) \\ &= \prod_{k\le p\le y_k}\left(1+\frac{\rho(p)(\xi-1)}{p}\left(1+\frac{\xi}{p}+\frac{\xi^2}{p^2}+\cdots\right)\right) \\ &= \prod_{k\le p\le y_k}\left(1+\frac{\rho(p)(\xi-1)}{p}\cdot\frac{1}{1-\xi/p}\right) \\ &= \prod_{k\le p\le y_k}\left(1+\frac{\rho(p)(\xi-1)}{p-\xi}\right) \\ &= \exp\left\{\rho(p)(\xi-1)\sum_{k\le p\le y_k}\frac{1}{p}+O(1)\right\}. \end{aligned} \tag{27}$$

But, since $\Re(\xi-1) < 0$, we have

$$\exp\left\{\rho(p)(\xi-1)\sum_{k\le p\le y_k}\frac{1}{p}+O(1)\right\} \longrightarrow 0 \quad \text{as } k\longrightarrow\infty. \tag{28}$$

Hence, combining (28) with (27) and (26), we obtain (23).

We have thus established that

$$M\left([x,x+y]\mid(a_0,\dots,a_{\ell-1})\right) - \frac{y}{q^{\ell}} = o(x_k) \quad (k\longrightarrow\infty), \tag{29}$$

which completes the proof of Theorem 2.

3. Proof of Theorem 3

As we will see, the proof of Theorem 3 is essentially a consequence of Weil's result (10).

Let ℓ be a fixed positive integer. Fix a prime p and let $\beta = \xi_{e_0}\cdots\xi_{e_{\ell-1}}$ be any word belonging to C_q^{ℓ}. Consider the expression

$$f_{\beta}(n) = \prod_{j=0}^{\ell-1}\prod_{\substack{\xi\in C_q\\ \xi\neq\xi_{e_j}}}\left(\chi_p(n+j)-\xi\right). \tag{30}$$

Observe that $f_{\beta}(n) = 0$ if $\chi(n)\cdots\chi(n+\ell-1)\in C_q^{\ell}$ is different from β. But if $\chi(n)\cdots\chi(n+\ell-1) = \beta$, then

$$f_{\beta}(n) = \prod_{j=0}^{\ell-1}\prod_{\substack{\xi\in C_q\\ \xi\neq\xi_{e_j}}}\left(\xi_{e_j}-\xi\right). \tag{31}$$

Since, for each $j = 0, \ldots, \ell - 1$,

$$\frac{d}{dx}\left(x^q - 1\right)\Big|_{x=\xi_{e_j}} = q\xi_{e_j}^{q-1} = q\overline{\xi_{e_j}}, \tag{32}$$

it follows that

$$f_\beta(n) = q^\ell\left(\overline{\xi_{e_0}\cdots\xi_{e_{\ell-1}}}\right), \tag{33}$$

where again $\overline{z}$ stands for the complex conjugate of z. Hence, letting $M_p(\beta)$ stand for the number of occurrences of β as a subword in the word Γ_p, we have

$$\overline{\xi_{e_0}\cdots\xi_{e_{\ell-1}}}\, q^\ell M_p(\beta) = \sum_{n=1}^{p-\ell} f_\beta(n). \tag{34}$$

Now $f_\beta(n)$ can be written as

$$f_\beta(n) = \sum_{(u_0,\ldots,u_{\ell-1})\in A_q^\ell} A(u_0,\ldots,u_{\ell-1})\,\chi\left(F_{u_0,\ldots,u_{\ell-1}}(n)\right), \tag{35}$$

where

$$\begin{aligned} F_{u_0,\ldots,u_{\ell-1}}(n) &= \prod_{j=0}^{\ell-1}(n+j)^{u_j}, \\ A(0,\ldots,0) &= \overline{\xi_{e_0}}\cdots\overline{\xi_{e_{\ell-1}}}. \end{aligned} \tag{36}$$

Thus taking into account (8), the Weil inequality (10), and the above relations (34) and (35), we obtain that

$$\begin{aligned} &\left|\overline{\xi_{e_0}\cdots\xi_{e_{\ell-1}}}\left(q^\ell M_p(\beta) - (p-\ell)\right)\right| \\ &\quad\le \sum_{\substack{(u_0,\ldots,u_{\ell-1})\in A_q^\ell \\ (u_0,\ldots,u_{\ell-1})\neq(0,\ldots,0)}} \left|A(u_0,\ldots,u_{\ell-1})\right| \\ &\qquad\cdot\left|\sum_{n=1}^{p-\ell}\chi\left(F_{u_0,\ldots,u_{\ell-1}}(n)\right)\right| \\ &\quad\le \sum_{\substack{(u_0,\ldots,u_{\ell-1})\in A_q^\ell \\ (u_0,\ldots,u_{\ell-1})\neq(0,\ldots,0)}} \left|A(u_0,\ldots,u_{\ell-1})\right| \\ &\qquad\cdot\left((\ell-1)\sqrt{p}+\ell\right) \\ &\quad\le c_1(\ell)\sqrt{p}. \end{aligned} \tag{37}$$

We have thus shown that

$$\left|M_p(\beta) - \frac{p-\ell}{q^\ell}\right| \le c(\ell)\sqrt{p}, \tag{38}$$

thus completing the proof of Theorem 3.

Conflict of Interests

The authors of this paper certify that they have no conflict of interests.

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