Research Article

A Meshless Method for the Numerical Solution of a Two-Dimension IHCP

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This paper uses the collocation method and radial basis functions (RBFs) to analyze the solution of a two-dimension inverse heat conduction problem (IHCP). The accuracy of the method is tested in terms of Error and RMS errors. Also, the stability of the technique is investigated by perturbing the additional specification data by increasing the amounts of random noise. The results of numerical experiments are compared with the analytical solution in illustrative examples to confirm the accuracy and efficiency of the presented scheme.

1. Introduction

In many industrial applications one wishes to determine the temperature on the surface of a body, where the surface itself is inaccessible for measurements. In this case, it is necessary to determine the surface temperature from a measured temperature history at a fixed location inside the body. This is called an IHCP and has been an interesting subject recently [1]. Inverse problems have practical implications in thermal transport systems which involve conduction, convection, and radiation. In thermal radiation [2], for example, identifying the distribution of the radiation source has been stimulated by a wide range of applications, including thermal control in space technology, combustion, high-temperature forming and coating technology, solar energy utilization, high-temperature engines, and furnace technology [3]. The importance of inverse heat conduction problems and appropriate solution algorithms are established in numerous works and the books (see, e.g., [1, 4–7] and the references therein). Several techniques have been proposed for solving a one-dimensional IHCP [8–14]. Among the methods proposed for higher dimensional IHCP, boundary element [15], finite difference [16], and finite element [17] have been widely adopted for problems in two-dimension. Besides, the sequential function specification method [1, 15]

and differential method [18] have also been used in solving the IHCP. There is, however, still a need on numerical scheme for two-dimensional IHCP. The traditional mesh-dependent finite difference and finite element methods are so far the principal numerical tool of choice for the modeling and simulation of the IHCP. A major disadvantage of these methods, however, is their mesh-dependent characteristics which normally requires enormous computational effort and induces numerical instability when large number of grids or elements are required.

In this paper, a two-dimensional IHCP is solved by RBFs as a truly meshless/meshfree method. A meshfree method does not require a mesh to discretize the domain or boundary of the problem under consideration and the approximate solution is constructed entirely based on a set of scattered nodes. It is considered as the main advantage of these methods over the mesh-dependent techniques.

In the recent rapid development of meshless computational schemes, the RBF has been successfully developed as an efficient meshless scheme for solving various kinds of partial differential equations (PDEs) and ordinary differential equations (ODEs). For example, see [19–25]. Also some applications of this approach in solving inverse problems can be found in [20, 21, 26, 27].

2. Problem Formulation

In this section, we consider the following two-dimension IHCP, in the dimensionless form:

$$\begin{aligned}
 U_t &= U_{xx} + U_{yy}; \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < t < t_{\text{fin}}, \\
 U(x, y, 0) &= f(x, y); \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \\
 U(0, y, t) &= g_0(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \\
 U(1, y, t) &= g_1(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \\
 U(x, 0, t) &= h_0(x, t); \quad 0 \leq x \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \\
 U(x, 1, t) &= h_1(x, t); \quad 0 \leq x \leq 1, \quad 0 \leq t \leq t_{\text{fin}},
 \end{aligned} \tag{1}$$

and the overspecified condition:

$$U(x_1, y, t) = E(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{2}$$

where $x_1 \in (0, 1)$ is known, t_{fin} represent the final time of interest for the time evolution of the problem, and f, g_1, h_0, h_1 , and E are known functions in their domain satisfying the compatibility conditions while $U(x, y, t)$ and $g_0(y, t)$ remain to be determined from some interior temperature measurements.

The existence and uniqueness of the solution of this problem are discussed in [28]. The problem (1)–(2) may be divided into two separate problems.

The first problem is

$$U_t = U_{xx} + U_{yy}; \quad x_1 < x < 1, \quad 0 < y < 1, \quad 0 < t < t_{\text{fin}}, \tag{3}$$

$$U(x, y, 0) = f(x, y); \quad x_1 \leq x \leq 1, \quad 0 \leq y \leq 1, \tag{4}$$

$$U(x_1, y, t) = E(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{5}$$

$$U(1, y, t) = g_1(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{6}$$

$$U(x, 0, t) = h_0(x, t); \quad x_1 \leq x \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{7}$$

$$U(x, 1, t) = h_1(x, t); \quad x_1 \leq x \leq 1, \quad 0 \leq t \leq t_{\text{fin}}. \tag{8}$$

This problem may be analyzed as a direct problem, for the portion of the body from $x = x_1$ to $x = 1$ with known boundary conditions. There is a unique stable solution to the direct problem (3)–(8) and may be found in [29]. The second problem is the following IHCP:

$$U_t = U_{xx} + U_{yy}; \quad 0 < x < x_1, \quad 0 < y < 1, \quad 0 < t < t_{\text{fin}}, \tag{9}$$

$$U(x, y, 0) = f(x, y); \quad 0 \leq x \leq x_1, \quad 0 \leq y \leq 1, \tag{10}$$

$$U(0, y, t) = g_0(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{11}$$

$$U(x_1, y, t) = E(y, t); \quad 0 \leq y \leq 1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{12}$$

$$U(x, 0, t) = h_0(x, t); \quad 0 \leq x \leq x_1, \quad 0 \leq t \leq t_{\text{fin}}, \tag{13}$$

$$U(x, 1, t) = h_1(x, t); \quad 0 \leq x \leq x_1, \quad 0 \leq t \leq t_{\text{fin}}. \tag{14}$$

Therefore, for solving the inverse problem (1)–(2), we will investigate the direct problem (3)–(8) and IHCP (9)–(14). In

TABLE 1: Some well-known functions that generate RBFs.

Name of radial basis function	Definition
Multiquadric (MQ)	$\varphi(r) = \sqrt{c^2 + r^2}$
Inverse quadratic (IQ)	$\varphi(r) = \frac{1}{(c^2 + r^2)}$
Inverse multiquadric (IMQ)	$\varphi(r) = \frac{1}{\sqrt{c^2 + r^2}}$
Gaussian (GA)	$\varphi(r) = \exp(-c^2 r^2)$
Thin plate splines (TPS)	$\varphi(r) = r^2 \log(r)$

the next section, the above problems will be considered; the heat in the body and heat at the boundary $x = 0$ will be obtained by solving these problems numerically.

3. Radial Basis Functions

Radial basis functions are very efficient instruments for interpolating a scattered set of points. The use of the radial basis function for solving partial differential equations has some advantages over mesh-dependent methods, such as finite difference methods, finite element methods, spectral methods, finite volume methods, and boundary element methods. The use of radial basis functions as a meshless method for numerical solution of partial differential equations is based on the collocation method. Because of the collocation technique, this method does not need to evaluate any integral.

3.1. Definition of the Space Radial Basis Functions. Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$, $\|\cdot\|_2$ denote the Euclidean norm and let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function with $\varphi(0) \geq 0$. A radial basis function on $\mathbb{R}^d$; $d = 1, 2, 3$ is a function of the form

$$\phi_s(\mathbf{x}) = \varphi(\|\mathbf{x} - \mathbf{x}_s\|_2), \tag{15}$$

which depended only on the distance between $\mathbf{x} \in \mathbb{R}^d$ and a fixed point $\mathbf{x}_s \in \mathbb{R}^d$ [30]. So that the radial basis function ϕ_s is radially symmetric about the center $\mathbf{x}_s$. Some best-known RBFs are listed in Table 1, where $r = \|\mathbf{x} - \mathbf{x}_s\|_2$ and c is a free positive parameter, often referred to as the shape parameter, to be specified by the user. Despite many research works, which have been done to find algorithms for selecting the optimum values of c [31–33], the optimal choice of shape parameter is an open problem, which is still under intensive investigation.

The standard radial basis functions are categorized into two major classes [34, 35].

Class 1. Infinitely smooth RBFs.

These basis functions are infinitely differentiable and involve a parameter c (such as multiquadric (MQ), inverse multiquadric (IMQ), and Gaussian) which needs to be selected so that the required accuracy of the solution is attained.

Class 2. Infinitely smooth (except at centers) RBFs.

These basis functions are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in Class 1. Examples are thin plate splines.

4. Numerical Procedures

Now we use the RBFs for discretization of both time and space variables. Let $\{(x_i, y_j, t_k), 0 \leq x_i \leq 1, 0 \leq y_j \leq 1, 0 \leq t_k \leq t_{\text{fin}}, i = 1, \dots, N, j = 1, \dots, M, k = 1, \dots, S\}$ be a set of scattered nodes. Then, the function $U(x, y, t)$ of two spatial dimensions to be interpolated can be represented by RBFs as

$$\tilde{U}(x, y, t) \approx \sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x, y, t), \quad (16)$$

where $\phi_{i,j}^k(x, y, t) = \varphi(\|(x, y, t) - (x_i, y_j, t_k)\|_2)$ for a radial function φ and $\lambda_{i,j}^k; i = 1, \dots, N, j = 1, \dots, M$, and $k = 1, \dots, S$ are unknown constants that must be found.

4.1. The Application of RBFs in IHCP (9)–(14). For this problem, let $x_i = x_1((i-1)/(N-1)), i = 1, 2, \dots, N; y_j = ((j-1)/(M-1)), j = 1, 2, \dots, M$; and $t_k = ((k-1)/(S-1)), k = 1, 2, \dots, S$. Now (9)–(14) are approximated using (16). The collocation technique is used for finding unknown constants $\lambda_{i,j}^k$.

We collocate (9) in $(N-1) \times (M-2) \times (S-1)$ points $\Omega_1 = \{(x_n, y_m, t_l); n = 1, \dots, N-1, m = 2, \dots, M-1, l = 2, \dots, S\}$; we have

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \left\{ \frac{\partial}{\partial t} \phi_{i,j}^k(x_n, y_m, t_l) - \frac{\partial^2}{\partial x^2} \phi_{i,j}^k(x_n, y_m, t_l) - \frac{\partial^2}{\partial y^2} \phi_{i,j}^k(x_n, y_m, t_l) \right\} = 0; \quad (17)$$

$$(x_n, y_m, t_l) \in \Omega_1.$$

Now, collocation (10) in $N \times M$ points $\Omega_2 = \{(x_n, y_m, t_l); n = 1, \dots, N, m = 1, \dots, M, l = 1\}$ yields

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - f(x_n, y_n) = 0; \quad (18)$$

$$(x_n, y_m, t_l) \in \Omega_2.$$

By collocation (12) in $M \times (S-1)$ points $\Omega_3 = \{(x_n, y_m, t_l); n = N, m = 1, \dots, M, l = 2, \dots, S\}$, we obtain

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - E(y_m, t_l) = 0; \quad (19)$$

$$(x_n, y_m, t_l) \in \Omega_3.$$

By collocating (13) in $(N-1) \times (S-1)$ points $\Omega_4 = \{(x_n, y_m, t_l); n = 1, \dots, N-1, m = 1, l = 2, \dots, S\}$, we can write

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - h_0(x_n, t_l) = 0; \quad (20)$$

$$(x_n, y_m, t_l) \in \Omega_4,$$

and by collocation (14) in $(N-1) \times (S-1)$ points $\Omega_5 = \{(x_n, y_m, t_l); n = 1, \dots, N-1, m = M, l = 2, \dots, S\}$, we have

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - h_1(x_n, t_l) = 0; \quad (21)$$

$$(x_n, y_m, t_l) \in \Omega_5.$$

Equations (17)–(21) give a $N \times M \times S$ system of linear algebraic equations with the $N \times M \times S$ unknown coefficients $\lambda_{i,j}^k$. Solving this linear system, the approximate temperature distribution of the problem (9)–(14) is obtained. Knowing the temperature distribution, we can easily determine the function describing the boundary condition:

$$g_0(y, t) \approx \tilde{U}(0, y, t). \quad (22)$$

4.2. The Application of RBFs in Direct Problem (3)–(8). Now, let $x_i = x_1 + (1-x_1)((i-1)/(N-1)); i = 1, 2, \dots, N, y_j = (j-1)/(M-1); j = 1, 2, \dots, M$, and $t_k = (k-1)/(S-1); k = 1, 2, \dots, S$. The unknown function $U(x, y, t)$ in (3)–(8) can be approximated by using (16). The collocation technique is used for finding unknown coefficients $\lambda_{i,j}^k$.

We collocate (3) in $(N-1) \times (M-2) \times (S-1)$ points $\Gamma_1 = \{(x_n, y_m, t_l); n = 1, \dots, N-1, m = 2, \dots, M-1, l = 2, \dots, S\}$; we get

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \left\{ \frac{\partial}{\partial t} \phi_{i,j}^k(x_n, y_m, t_l) - \frac{\partial^2}{\partial x^2} \phi_{i,j}^k(x_n, y_m, t_l) - \frac{\partial^2}{\partial y^2} \phi_{i,j}^k(x_n, y_m, t_l) \right\} = 0, \quad (23)$$

$$(x_n, y_m, t_l) \in \Gamma_1.$$

Now, collocation (4) in $N \times M$ points $\Gamma_2 = \{(x_n, y_m, t_l); n = 1, \dots, N, m = 1, \dots, M, l = 1\}$, yields

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - f(x_n, y_m) = 0; \quad (24)$$

$$(x_n, y_m, t_l) \in \Gamma_2.$$

By collocation (5) in $M \times (S-1)$ points $\Gamma_3 = \{(x_n, y_m, t_l); n = 1, m = 1, \dots, M, l = 2, \dots, S\}$, we obtain

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - E(y_m, t_l) = 0; \quad (25)$$

$$(x_n, y_m, t_l) \in \Gamma_3.$$

TABLE 2: The comparison between the exact, Tikhonov, SVD, and GA-RBFs solution for $g_0(y, 0.04)$ with noiseless data for Example 1.

y	Exact	Tikhonov 0	Tikhonov 1	Tikhonov 2	SVD	GA-RBFs ($N = M = S = 5$)	GA-RBFs ($N = 11, M = S = 8$)
0.1	0.955989	0.977515	0.953963	0.953359	0.977522	0.956061	0.955988
0.2	0.941638	0.961234	0.937549	0.937005	0.961242	0.941742	0.941636
0.3	0.917877	0.936400	0.912705	0.912258	0.936409	0.918004	0.917875
0.4	0.884946	0.903014	0.879430	0.879118	0.903023	0.885090	0.884943
0.5	0.843172	0.861074	0.837723	0.837584	0.861083	0.843326	0.843169
0.6	0.792974	0.810581	0.787585	0.787658	0.810590	0.793124	0.792971
0.7	0.734852	0.751535	0.729016	0.729339	0.751544	0.734981	0.734850
0.8	0.669388	0.683935	0.662016	0.662626	0.683945	0.669479	0.669387
0.9	0.597236	0.607783	0.586884	0.587521	0.607793	0.597279	0.597935

TABLE 3: The comparison between the exact, Tikhonov, SVD, and GA-RBFs solution for $g_0(y, 0.04)$ with noisy data ($\delta = 0.01$) for Example 1.

y	Exact	Tikhonov 0	Tikhonov 1	Tikhonov 2	SVD	GA-RBFs ($N = M = S = 5$)	GA-RBFs ($N = 11, M = S = 8$)
0.1	0.955989	0.989553	0.959390	0.958599	0.989560	0.956861	0.955980
0.2	0.941638	0.971284	0.940893	0.940179	0.971291	0.941096	0.941640
0.3	0.917877	0.945030	0.914534	0.913948	0.945037	0.919012	0.917873
0.4	0.884946	0.910791	0.880313	0.879905	0.910798	0.880111	0.884940
0.5	0.843172	0.868567	0.838231	0.838050	0.868575	0.843935	0.843171
0.6	0.792974	0.818358	0.788287	0.788384	0.818366	0.793001	0.792979
0.7	0.734852	0.760165	0.730482	0.730907	0.760172	0.735098	0.734855
0.8	0.669388	0.693986	0.664816	0.665618	0.693994	0.669153	0.669390
0.9	0.597236	0.619823	0.591288	0.592518	0.619830	0.597291	0.597938

By collocation (6) in $M \times (S - 1)$ points $\Gamma_4 = \{(x_n, y_m, t_l); n = N, m = 1, \dots, M, l = 2, \dots, S\}$, we can write

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - g_1(x_n, t_l) = 0; \quad (26)$$

$$(x_n, y_m, t_l) \in \Gamma_4.$$

By collocation (7) in $(N - 2) \times (S - 1)$ points $\Gamma_5 = \{(x_n, y_m, t_l); n = 2, \dots, N - 1, m = 1, l = 2, \dots, S\}$, we have

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - h_0(x_n, t_l) = 0; \quad (27)$$

$$(x_n, y_m, t_l) \in \Gamma_5,$$

and by collocation (8) in $(N - 2) \times (S - 1)$ points $\Gamma_6 = \{(x_n, y_m, t_l); n = 2, \dots, N - 1, m = M, l = 2, \dots, S\}$, we have

$$\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S \lambda_{i,j}^k \phi_{i,j}^k(x_n, y_m, t_l) - h_1(x_n, t_l) = 0; \quad (28)$$

$$(x_n, y_m, t_l) \in \Gamma_6.$$

Equations (23)–(28) give a $N \times M \times S$ system of linear algebraic equations with the $N \times M \times S$ unknown coefficients $\lambda_{i,j}^k$. Solving this linear system, the approximate solution of the problem (3)–(8) is obtained.

5. Convergence Analysis and Error Bound

This section covers the error analysis of the proposed method. Also the sufficient conditions are presented to guarantee the convergence of RBFs, when applied to solve the differential equations.

5.1. Approximation Error. Here, we are concerned with the error of the approximation of a given three-variate function by its expansion in terms of radial basis functions.

Suppose that $X = L^2(\Omega)$, where $\Omega = [0, 1] \times [0, 1] \times [0, t_{\text{fin}}]$ (also note that the solution $u \in C^{4,4,2}(\Omega) \subset L^2(\Omega)$); the inner product in this space is defined by

$$\langle f_1(x, y, t), f_2(x, y, t) \rangle = \int_0^{t_{\text{fin}}} \iint_0^1 f_1(x, y, t) f_2(x, y, t) dx dy dt, \quad (29)$$

and the norm is as follows:

$$\|f(x, y, t)\| = \langle f(x, y, t), f(x, y, t) \rangle^{1/2} = \left(\int_0^{t_{\text{fin}}} \iint_0^1 |f(x, y, t)|^2 dx dy dt \right)^{1/2}. \quad (30)$$

TABLE 4: **Error** and RMS errors for $U(x, y, t)$ and $g_0(y, t)$ on the intervals $x \in [0, 0.1]$, $y \in [0, 1]$, and $t \in [0, 1]$ for Example 1.

Method	Error	$N = M = S = 3$	$N = 6, M = S = 4$	$N = 5, M = 4, S = 6$	$N = 7, M = 5, S = 6$
GA-RBFs ($c = 0.1$)	RMS(U)	$1.800E - 3$	$3.782E - 4$	$2.253E - 4$	$3.749E - 5$
	Error (U)	$2.894E - 3$	$6.169E - 4$	$3.748E - 4$	$6.209E - 5$
	RMS(g_0)	$2.795E - 3$	$6.242E - 4$	$3.672E - 4$	$6.271E - 5$
	Error (g_0)	$4.763E - 3$	$1.078E - 3$	$6.470E - 4$	$1.099E - 4$
IMQ-RBFs ($c = 50$)	RMS(U)	$1.506E - 3$	$9.596E - 4$	$7.356E - 4$	$8.187E - 5$
	Error (U)	$2.421E - 3$	$1.692E - 3$	$1.223E - 3$	$1.356E - 4$
	RMS(g_0)	$2.334E - 3$	$2.020E - 3$	$1.201E - 3$	$1.370E - 4$
	Error (g_0)	$3.977E - 3$	$3.111E - 3$	$2.117E - 3$	$2.402E - 4$
MQ-RBFs ($c = 50$)	RMS(U)	$1.378E - 3$	$9.301E - 3$	$1.135E - 3$	$1.638E - 4$
	Error (U)	$2.216E - 3$	$2.093E - 3$	$1.888E - 3$	$2.693E - 4$
	RMS(g_0)	$2.135E - 3$	$2.101E - 3$	$1.853E - 3$	$2.689E - 4$
	Error (g_0)	$3.639E - 3$	$3.491E - 3$	$3.266E - 3$	$4.682E - 4$
IQ-RBFs ($c = 50$)	RMS(U)	$1.563E - 3$	$9.301E - 3$	$1.172E - 3$	$6.588E - 5$
	Error (U)	$2.514E - 3$	$2.093E - 3$	$1.951E - 3$	$1.091E - 4$
	RMS(g_0)	$2.423E - 3$	$2.101E - 3$	$1.913E - 3$	$1.104E - 4$
	Error (g_0)	$4.129E - 3$	$3.491E - 3$	$3.371E - 3$	$1.936E - 4$

TABLE 5: **Error** and RMS errors for $U(x, y, t)$ on the intervals $x \in [0.1, 1]$, $y \in [0, 1]$, and $t \in [0, 1]$ for Example 1.

Method	Error	$N = M = S = 3$	$N = 6, M = S = 4$	$N = 5, M = 4, S = 6$	$N = 7, M = 5, S = 6$
GA-RBFs ($c = 0.1$)	RMS(U)	$7.237E - 4$	$3.782E - 4$	$1.038E - 4$	$3.361E - 6$
	Error (U)	$7.611E - 4$	$6.169E - 4$	$1.132E - 4$	$3.625E - 6$
IMQ-RBFs ($c = 50$)	RMS(U)	$1.342E - 4$	$6.216E - 4$	$5.188E - 4$	$2.424E - 6$
	Error (U)	$1.412E - 4$	$6.450E - 4$	$5.657E - 4$	$2.634E - 6$
MQ-RBFs ($c = 50$)	RMS(U)	$6.635E - 3$	$4.011E - 3$	$1.187E - 3$	$9.449E - 6$
	Error (U)	$6.978E - 3$	$4.721E - 3$	$2.043E - 3$	$1.026E - 5$
IQ-RBFs ($c = 50$)	RMS(U)	$1.181E - 3$	$8.112E - 4$	$4.305E - 4$	$2.424E - 6$
	Error (U)	$1.904E - 4$	$8.430E - 4$	$4.694E - 4$	$2.634E - 6$

Let $\{\phi_1(x, y, t), \phi_2(x, y, t), \dots, \phi_N(x, y, t)\} \subset X$ be the set of radial basis functions and

$$Y = \text{span}\{\phi_1(x, y, t), \phi_2(x, y, t), \dots, \phi_N(x, y, t)\}, \quad (31)$$

and let f be an arbitrary element in X . Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $f_0 \in Y$ [36], such that

$$\forall y \in Y, \quad \|f - f_0\| \leq \|f - y\|. \quad (32)$$

Moreover, since $f_0 \in Y$, there exist unique coefficients $\gamma_1, \gamma_2, \dots, \gamma_N$ such that

$$f(x, y, t) \approx f_0(x, y, t) = \sum_{i=1}^N \gamma_i \phi_i(x, y, t). \quad (33)$$

Theorem 1. Let $(X, \|\cdot\|)$ be a Hilbert space and let Y be a closed subspace of X such that $\dim Y < \infty$ and $\{y_1, y_2, \dots, y_n\}$ is any basis for Y . Let f be an arbitrary element in X and let f_0 be the unique best approximation to f out of Y . Then [36]

$$\|f - f_0\| = \frac{G(f, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)}, \quad (34)$$

where

$$G(y_1, y_2, \dots, y_n) = \begin{vmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle & \cdots & \langle y_1, y_n \rangle \\ \langle y_2, y_1 \rangle & \langle y_2, y_2 \rangle & \cdots & \langle y_2, y_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \langle y_n, y_2 \rangle & \cdots & \langle y_n, y_n \rangle \end{vmatrix}. \quad (35)$$

5.2. Convergence Analysis. We have the following theorem about the convergence of RBFs interpolation.

Theorem 2. Assume $\{x_i\}_{i=1}^N$ are N nodes in $\Omega \subseteq \mathbb{R}^d$ which is convex; let

$$h = \max_{x \in \Omega} \min_{1 \leq i \leq N} \|x - x_i\|_2, \quad (36)$$

when $\phi(\eta) < c(1 + |\eta|)^{-(2l+d)}$ for any $y(x)$ satisfies $\int (y(\eta))^2 / \phi(\eta) d\eta < \infty$; we have

$$\|y_N^{(\alpha)} - y^{(\alpha)}\|_{\infty} \leq ch^{1-\alpha}, \quad (37)$$

TABLE 6: The comparison between the exact, Tikhonov, SVD, and GA-RBFs solution for $g_0(y, 0.04)$ with noiseless data for Example 2.

y	Exact	Tikhonov 0	Tikhonov 1	Tikhonov 2	SVD	GA-RBFs ($N = M = S = 5$)	GA-RBFs ($N = 11, M = S = 8$)
0.1	0.095591	0.099151	0.091232	0.091254	0.099229	0.095594	0.095591
0.2	0.190879	0.194718	0.188768	0.188793	0.194778	0.190958	0.190879
0.3	0.283933	0.286510	0.282289	0.282316	0.286555	0.284056	0.283932
0.4	0.374149	0.374528	0.371793	0.371824	0.374561	0.374295	0.374148
0.5	0.460627	0.454773	0.457291	0.457315	0.458794	0.460774	0.460626
0.6	0.542503	0.539243	0.538753	0.538791	0.539256	0.542638	0.542502
0.7	0.618958	0.615939	0.616208	0.616251	0.615946	0.619078	0.618957
0.8	0.689228	0.688861	0.689647	0.689695	0.688864	0.689332	0.689227
0.9	0.752612	0.758009	0.759070	0.759124	0.758011	0.752690	0.752612

TABLE 7: The comparison between the exact, Tikhonov, SVD, and GA-RBFs solution for $g_0(y, 0.04)$ with noisy data ($\delta = 0.01$) for Example 2.

y	Exact	Tikhonov 0	Tikhonov 1	Tikhonov 2	SVD	GA-RBFs ($N = M = S = 5$)	GA-RBFs ($N = 11, M = S = 8$)
0.1	0.095591	0.101437	0.097458	0.097523	0.101531	0.095599	0.095591
0.2	0.190879	0.198779	0.195645	0.195653	0.198852	0.190983	0.190879
0.3	0.283933	0.291784	0.289378	0.289331	0.291838	0.284318	0.283938
0.4	0.374149	0.380451	0.378655	0.378557	0.380490	0.374808	0.374150
0.5	0.460627	0.464780	0.463477	0.463330	0.464807	0.460512	0.460621
0.6	0.542503	0.544772	0.543844	0.543652	0.544789	0.542840	0.542509
0.7	0.618958	0.620427	0.619756	0.619522	0.620437	0.619578	0.618957
0.8	0.689228	0.691744	0.691212	0.690939	0.691750	0.689809	0.689221
0.9	0.752612	0.758723	0.758213	0.757905	0.758728	0.752831	0.752612

FIGURE 1: Graph of absolute error for U and g_0 by using GA-RBF with $c = 0.1$ for Example 1.

FIGURE 2: Graph of absolute error for g_0 by using different RBFs for Example 2 with $N = 5$, $M = 6$, $S = 5$.

where $\phi(x)$ is RBFs, the constant c depends on the RBFs, d is space dimension, and l and α are nonnegative integer.

Proof. A complete proof is given by authors [37, 38]. $\square$

It can be seen that not only RBFs themselves but also any of their order derivatives has a good convergence.

6. Test Examples

In order to illustrate the performance of the RBFs method in solving IHCPs and justify the accuracy and efficiency of the method presented in this paper, we consider the following examples. For two examples, the true solutions are available. We tested the accuracy and stability of the method presented in this paper by performing the mentioned method for different values of N , M , and S . To study the convergence behavior of the RBFs method, we applied the following laws.

- (1) The error **Error** is described using

Error (U)

$$= \sqrt{\frac{\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S (U(x_i, y_j, t_k) - \tilde{U}(x_i, y_j, t_k))^2}{\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S (U(x_i, y_j, t_k))^2}}. \quad (38)$$

- (2) The root mean square (RMS) is described using

RMS (U)

$$= \sqrt{\frac{1}{NMS} \sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^S |U(x_i, y_j, t_k) - \tilde{U}(x_i, y_j, t_k)|^2}, \quad (39)$$

where (x_i, y_j, t_k) are interpolate nodes, U is the exact value, and $\tilde{U}$ is the RBFs approximation.

In order to investigate the stability of the numerical method, the additional specification data has been perturbed as

$$E(y, t)^\delta = E(y, t) [1 + \delta \times \text{rand}(x)], \quad (40)$$

where δ is the relative (percentage) noise level and $\text{rand}(x)$ is a random number between $(0, 1)$.

6.1. Example 1. We solve the problem (1)–(2) with [28]: $x_1 = 0.1$, $t_{\text{fin}} = 1$, $f(x, y) = \sin(x) + \cos(y)$, $g_1(y, t) = e^{-t}(\sin(1) + \cos(y))$, $h_0(x, t) = e^{-t}(1 + \sin(x))$, $h_1(y, t) = e^{-t}(\sin(x) + \cos(1))$, and $E(y, t) = e^{-t}(\sin(0.1) + \cos(y))$ for which the exact solution is $u(x, y, t) = e^{-t}(\sin(x) + \cos(y))$ and $g_0(y, t) = e^{-t} \cos(y)$.

TABLE 8: **Error** and RMS errors for $U(x, y, t)$ and $g_0(y, t)$ on the intervals $x \in [0, 0.1]$, $y \in [0, 1]$, and $t \in [0, 1]$ for Example 2.

Method	Error	$N = M = S = 3$	$N = 6, M = 4, S = 5$	$N = M = S = 5$	$N = 5, M = 6, S = 5$
GA-RBFs ($c = 0.1$)	RMS(U)	$1.801E - 3$	$3.298E - 5$	$2.040E - 5$	$1.092E - 5$
	Error (U)	$4.237E - 3$	$8.186E - 5$	$5.112E - 5$	$4.281E - 5$
	RMS(g_0)	$2.797E - 3$	$5.222E - 5$	$3.349E - 5$	$1.438E - 5$
	Error (g_0)	$7.066E - 3$	$1.397E - 4$	$9.055E - 5$	$7.367E - 4$
IMQ-RBFs ($c = 50$)	RMS(U)	$1.507E - 3$	$5.384E - 4$	$2.137E - 4$	$1.560E - 4$
	Error (U)	$3.546E - 3$	$1.336E - 3$	$5.354E - 4$	$3.931E - 4$
	RMS(g_0)	$2.336E - 3$	$8.666E - 4$	$3.507E - 4$	$2.562E - 4$
	Error (g_0)	$5.902E - 3$	$2.318E - 3$	$9.491E - 4$	$6.982E - 4$
MQ-RBFs ($c = 50$)	RMS(U)	$1.379E - 3$	$5.107E - 4$	$1.634E - 4$	$1.343E - 4$
	Error (U)	$3.243E - 3$	$1.267E - 3$	$4.094E - 4$	$3.382E - 4$
	RMS(g_0)	$2.136E - 3$	$8.221E - 4$	$2.682E - 4$	$2.220E - 4$
	Error (g_0)	$5.398E - 3$	$2.199E - 3$	$7.257E - 4$	$6.004E - 4$
IQ-RBFs ($c = 50$)	RMS(U)	$1.565E - 3$	$3.116E - 4$	$1.172E - 4$	$1.342E - 5$
	Error (U)	$3.683E - 3$	$7.736E - 4$	$2.951E - 4$	$2.383E - 4$
	RMS(g_0)	$2.426E - 3$	$5.013E - 4$	$2.913E - 4$	$2.205E - 4$
	Error (g_0)	$6.130E - 3$	$1.341E - 3$	$6.371E - 4$	$6.008E - 4$

TABLE 9: **Error** and RMS errors for $U(x, y, t)$ on the intervals $x \in [0.1, 1]$, $y \in [0, 1]$, and $t \in [0, 1]$ for Example 2.

Method	Error	$N = M = S = 3$	$N = 6, M = 4, S = 5$	$N = M = S = 5$	$N = 5, M = 6, S = 5$
GA-RBFs ($c = 0.1$)	RMS(U)	$5.167E - 4$	$5.678E - 5$	$7.900E - 6$	$5.923E - 6$
	Error (U)	$7.048E - 4$	$8.081E - 5$	$1.125E - 5$	$8.444E - 6$
IMQ-RBFs ($c = 50$)	RMS(U)	$9.580E - 5$	$5.974E - 5$	$5.022E - 6$	$4.144E - 6$
	Error (U)	$1.306E - 4$	$8.502E - 5$	$7.154E - 6$	$5.909E - 6$
MQ-RBFs ($c = 50$)	RMS(U)	$4.735E - 5$	$4.129E - 5$	$1.013E - 5$	$2.279E - 6$
	Error (U)	$6.458E - 5$	$7.015E - 5$	$1.444E - 5$	$3.250E - 6$
IQ-RBFs ($c = 50$)	RMS(U)	$1.291E - 4$	$5.921E - 5$	$3.043E - 6$	$2.677E - 6$
	Error (U)	$1.761E - 4$	$8.427E - 5$	$4.335E - 6$	$4.243E - 6$

Table 2 shows the comparison between the exact solution, RBFs solution and approximate solution result from method in [28] by Tikhonov regularization 0th, 1st, and 2nd, and SVD with noiseless data. Table 3 shows this comparison with noisy data. Also, Table 4 shows the **Error** and RMS error values for $U(x, y, t)$ and $g_0(y, t)$ on the intervals $x \in [0, 0.1]$, $y \in [0, 1]$, and $t \in [0, 1]$ for various values of N , M , and S . The corresponding results obtained for $U(x, y, t)$ on the intervals $x \in [0.1, 1]$, $y \in [0, 1]$, and $t \in [0, 1]$ are presented in Table 5. Figure 1 shows the plot of error for $U(x, y, 1)$ and $g_0(y, t)$ on the intervals $x \in [0, 0.1]$, $y \in [0, 1]$, and $t \in [0, 1]$. It can be obtained from Tables 4 and 5 and Figure 2 that the accuracy increases with the increase of the number of collocation points.

6.2. Example 2. As the second example, consider (1)–(2) with [28]: $x_1 = 0.1$, $t_{\text{fin}} = 1$, $f(x, y) = \sin(x) + \sin(y)g_1(y, t) = e^{-t}(\sin(1) + \sin(y))$, $h_0(x, t) = e^{-t}\sin(x)$, $h_1(y, t) = e^{-t}(\sin(x) + \sin(1))$, and $E(y, t) = e^{-t}(\sin(0.1) + \sin(y))$ for which the exact solution is $u(x, y, t) = e^{-t}(\sin(x) + \sin(y))$ and $g_0(y, t) = e^{-t}\sin(y)$.

Table 6 shows the comparison between the exact solution, RBFs solution and approximate solution result from method

in [28] by Tikhonov regularization 0th, 1st, and 2nd, and SVD with noiseless data. Table 7 shows this comparison with noisy data. Furthermore, Table 8 shows the **Error** and RMS error values for $U(x, y, t)$ and $g_0(y, t)$ on the intervals $x \in [0, 0.1]$, $y \in [0, 1]$, and $t \in [0, 1]$ for various values of N , M , and S . The corresponding results obtained for $U(x, y, t)$ on the intervals $x \in [0.1, 1]$, $y \in [0, 1]$, and $t \in [0, 1]$ are presented in Table 9. It can be obtained from Tables 8 and 9 that the accuracy increases with the increase of the number of collocation points. Also absolute error with different radial basis functions is depicted in Figure 2.

7. Conclusion

In the paper, an application of the RBFs for the solution of a two-dimension IHCP is presented. Problem consists in the calculation of temperature distribution in the domain, as well as in the reconstruction of functions describing the temperature on the boundary, when the temperature measurements in the domain are known. The present study successfully applies the numerical method to IHCPs and has been found stable with respect to small perturbation in the input data. The results of computing by radial basis function

method are compared to that by the method in [28]. The results of numerical examples demonstrate that this method is more accurate than the method in [28].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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