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*Research Article*

# Radio Numbers of Certain $m$-Distant Trees

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*Radio coloring* of a graph $G$ with diameter $d$ is an assignment $f$ of positive integers to the vertices of $G$ such that $|f(u) - f(v)| \geq 1 + d - d(u, v)$, where $u$ and $v$ are any two distinct vertices of $G$ and $d(u, v)$ is the distance between $u$ and $v$. The number max $\{f(u) : u \in V(G)\}$ is called the *span* of $f$. The minimum of spans over all radio colorings of $G$ is called *radio number* of $G$, denoted by $rn(G)$. An *$m$-distant tree* $T$ is a tree in which there is a path $P$ of maximum length such that every vertex in $V(T) \setminus V(P)$ is at the most distance $m$ from $P$. This path $P$ is called a *central path*. For every tree $T$, there is an integer $m$ such that $T$ is a $m$-distant tree. In this paper, we determine the radio number of some $m$-distant trees for any positive integer $m \geq 2$, and as a consequence of it, we find the radio number of a class of 1-distant trees (or *caterpillars*).

## 1. Introduction

The *channel assignment problem* is the problem of assigning frequencies to transmitters in some optimal manner and with no interferences; see Hale [1]. Chartrand et al. [2] introduced *radio $k$-colorings* of graphs which is a variation of Hale's channel assignment problem, in which one seeks to assign positive integers to the vertices of a graph $G$ subject to certain constraints involving the distance between the vertices. For any simple connected graph $G$ with diameter $d$ and a positive integer $k$, $1 \leq k \leq d$, specifically, a radio $k$-coloring of $G$ is an assignment $f$ of positive integers to the vertices of $G$ such that $|f(u) - f(v)| \geq 1 + k - d(u, v)$, where $u$ and $v$ are any two distinct vertices of $G$ and $d(u, v)$ is the distance between $u$ and $v$. The maximum color (positive integer) assigned by $f$ to some vertex of $G$ is called the *span* of $f$, denoted by $rc_k(f)$. The minimum of spans of all possible radio $k$-colorings of $G$ is called the radio $k$-chromatic number of $G$, denoted by $rc_k(G)$. A radio $k$-coloring with span $rc_k(G)$ is called minimal radio $k$-coloring of $G$. Radio $k$-colorings have been studied by many authors; see [3–9].

Although the positive integer $k$ can have value in-between 1 and $d$, the case $k = d$ has become a special interest for many authors. Radio $d$-coloring is simply called radio coloring and radio $d$-chromatic number is radio number. Here we concentrate on radio number of trees. Kchikech et al. [4] have found the exact value of the radio $k$-chromatic number of stars $K_{1,n}$ as $n(k-1) + 2$ and have also given an upper bound for radio $k$-chromatic number, $rc_k(T)$, $k \geq 2$, of an arbitrary non-star-tree $T$ on $n$ vertices as $(n-1)(k-1)$. Liu [5] has given a lower bound for the radio number $rn(T)$ of an $n$-vertex tree with diameter $d$ as $(n-1)(d+1) + 1 - 2w(T)$, where $w(T)$ is the weight of $T$ defined as $w(T) = \min_{u \in V(T)}\{\sum_{v \in V(T)} d(u, v)\}$. She also has characterized the trees achieving this bound. In the same paper, Liu considered spiders denoted by $S_{l_1, l_2, l_3, \ldots, l_m}$, which are trees having a vertex $v$ of degree $m \geq 3$, and $m$ number of paths of length $l_1, l_2, \ldots, l_m$ whose one end vertex is $v$ and other ends are pendant vertices. She has given a lower bound for the radio number of $S_{l_1, l_2, l_3, \ldots, l_m}$ as $\sum_{i=1}^{m} l_i(l_1 + l_2 - l_i) + \lceil (l_1 - l_2)/2 \rceil \lfloor (l_1 - l_2)/2 \rfloor + 1$, where $l_1 \geq l_2 \geq \cdots \geq l_m$, and has also characterized the spiders achieving this bound. Li et al. [10] have determined the radio number of complete $m$-ary trees ($m \geq 3$) with height $k$ ($\geq 2$), denoted by $T_{k,m}$, as $(m^{k+2} + m^{k+1} - 2km^2 + (2k-3)m + 1)/(m-1)^2$.

In this paper, we determine the radio number of some $m$-distant trees for any positive integer $m \geq 2$, and as a consequence of it, we find the radio number of a class of 1-distant trees (or *caterpillars*).

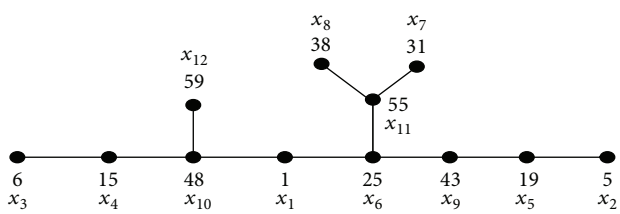


Figure 1: A radio coloring of a tree of order 12 and diameter 7.

## 2. Radio Numbers of Some $m$-Distant Trees

Recall that an $m$-distant tree $T$ is a tree in which there is a path $P$ of maximum length such that every vertex in $V(T)\setminus V(P)$ is at the most distance $m$ from $P$. This path $P$ is called a *central path*. Since we consider the path $P$ as a path of maximum length, the end vertices of $P$ are of degree one vertices in the $m$-distant tree; that is, if $P$: $a_1a_2\cdots a_k$ is a central path of $T$, then $deg_T(a_1) = deg_T(a_k) = 1$. For every tree $T$, there is an integer $m$ such that $T$ is a $m$-distant tree. Usually 1-distant trees are known as caterpillars.

Before we present the main result of the paper, we give a definition and a lemma below which will be used in the sequel. From the definition of a radio coloring $f$, one observes that for any two vertices $u$ and $v$, the quantity $|f(u)-f(v)|-\{1+d-d(u,v)\}$ is an excess for $f$ to achieve the minimum span. In the definition, we give notation for these excesses corresponding to pair of vertices. In the lemma, we show that to get an optimal radio $k$-coloring, one has to minimize this sum of excesses.

*Definition 1.* For any radio coloring $f$ of a simple connected graph $G$ on $n$ vertices and an ordering $x_1, x_2, \ldots, x_n$ of vertices of $G$ with $f(x_i) \le f(x_{i+1})$, $1 \le i \le n-1$, we define $\epsilon_i$ (or $\epsilon_i^f$ to specify the coloring $f$) $= \{f(x_i)-f(x_{i-1})\}-\{1+d-d(x_i,x_{i-1})\}$, $2 \le i \le n$. It is clear from the definition of radio coloring that $\epsilon_i \ge 0$, for all $i$. With respect to the ordering of vertices of $G$ induced by $f$, we denote $d(f) = \sum_{i=2}^{n} d(x_i, x_{i-1})$. In other words, every radio coloring $f$ is associated with a unique positive integer $d(f)$ called *distance sum* of $f$.

*Example 2.* In this example, we explain Definition 1.

In Figure 1, a radio coloring $f$ of a tree $T$ is given. The labels $x_1, x_2, x_3, \ldots, x_{12}$ are an ordering of vertices of $T$ with $f(x_i) \le f(x_{i+1})$, $1 \le i \le 11$. Here $\epsilon_2 = \{f(x_2)-f(x_1)\}-\{1+7-d(x_1,x_2)\} = (5-1)-(1+7-4) = 0$, $\epsilon_3 = 0$, $\epsilon_4 = 2$, $\epsilon_5 = 1$, $\epsilon_6 = 0$, $\epsilon_7 = 0$, $\epsilon_8 = 1$, $\epsilon_9 = 0$, $\epsilon_{10} = 0$, $\epsilon_{11} = 2$, $\epsilon_{12} = 0$, and $d(f) = \sum_{i=2}^{12} d(x_i, x_{i-1}) = d(x_2,x_1) + d(x_3,x_2) + d(x_4,x_3) + d(x_5,x_4) + d(x_6,x_5) + d(x_7,x_6) + d(x_8,x_7) + d(x_9,x_8) + d(x_{10},x_9) + d(x_{11},x_{10}) + d(x_{12},x_{11}) = 4+7+1+5+2+2+2+3+3+3+4 = 36$.

The following lemma gives the span of a radio coloring of a graph of order $n$ in terms of $n$, $d$, distance sum, and $\epsilon$'s sum.

**Lemma 3.** *For any radio coloring $f$ of $G$, the span of $f$ $rc_d(f) = f(x_n) = (n-1)(1+d) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i^f + 1$, where $x_i$'s are arranged as in Definition 1.*

*Proof.* Consider $f(x_n) - f(x_1) = \sum_{i=2}^{n}[f(x_i)-f(x_{i-1})] = \sum_{i=2}^{n}[1+d-d(x_i,x_{i-1})+\epsilon_i^f] = (n-1)(1+d) - \sum_{i=2}^{n} d(x_i,x_{i-1}) + \sum_{i=2}^{n}\epsilon_i^f$.

Since $f(x_1) = 1$, we get $f(x_n) = (n-1)(1+d) - \sum_{i=2}^{n} d(x_i,x_{i-1}) + \sum_{i=2}^{n}\epsilon_i^f + 1$. □

Lemma 3 says that to obtain a minimal radio coloring of a graph, one should maximize $d(f)$ and minimize $\sum_{i=2}^{n}\epsilon_i^f$ over all possible radio colorings of the graph. Since this fact is the basic concept to construct a minimal radio coloring, we express it as the theorem below.

**Theorem 4.** *If $g$ is a radio coloring of $G$ such that $\sum_{i=2}^{n}\epsilon_i^g = 0$ and $d(g) = \max\{d(f)$: $f$ is a radio coloring of $G\}$, then $g$ is a minimal radio coloring of $G$.*

*Proof.* For any radio coloring $f$ of $G$, Lemma 3 gives that $rc_d(f) = (n-1)(1+d) - d(f) + \sum_{i=2}^{n}\epsilon_i^f$. Then

$$\begin{aligned}
&rc_d(G)\\
&\quad = \min\{rc_d(f) : f \text{ is a radio coloring of } G\}\\
&\quad = \min\left\{(n-1)(1+d) - d(f) + \sum_{i=2}^{n}\epsilon_i^f : f \text{ is a radio coloring of } G\right\}\\
&\quad = (n-1)(1+d) - \max_f d(f) + \min_f \sum_{i=2}^{n}\epsilon_i^f\\
&\quad = (n-1)(1+d) - d(g)\\
&\qquad \left(\text{since } \epsilon_i,\ i = 2,3,\ldots,n \text{ is nonnegative, } \sum_{i=2}^{n}\epsilon_i^g = 0\right)\\
&\quad = rc_d(g).
\end{aligned} \tag{1}$$

□

Now, we determine the radio number of an $m$-distant tree $T$ with $diam(T) = 2p-1$, $p \ge 2$, $m \le p - \sqrt{p-1}$, and the degrees of the vertices on the central path satisfy certain conditions (given in the theorem below).

**Theorem 5.** *Let $T$ be an $m$-distant tree of order $n$ with a central path $P$: $a_1a_2\cdots a_{2p-1}a_{2p}$, and satisfy*

(i) $m \le p - \sqrt{p-1}$;

(ii) $deg(a_i) = 2$, *for* $i = 2,3,\ldots,m, 2p-m+1, 2p-m+2,\ldots,2p$;

(iii) $deg(a_j) = deg(a_{j+p-m}) = s_j + 2$, $s_j \ge 0$, $j = m+1, m+2,\ldots,p$;

(iv) *for every $a_l$, $l = m+1, m+2,\ldots,2p-m$, the number of vertices at distance $i$ and lying on a branch incident on $a_l$, $2 \le i \le m$, is a constant say $t_i$ ($t_i \ge 0$).*

*Then* $rn(T) = 2(p-m)[(p+m)\sum_{i=2}^{m} t_i - \sum_{i=2}^{m} 2it_i] + [2(p+m)-4]\sum_{i=m+1}^{p} s_i + 2p(p-1) + 3$.

*Proof.* The idea is to define a radio coloring $f$ of $T$ and show that $f$ is minimal by Theorem 4. We first give an algorithm to order the vertices of $T$.

*Algorithm 6*

*Step I.* We make an ordering $x_1, x_2, \ldots, x_{2p}$ of the vertices on the central path as $a_p, a_{2p}, a_{p-1}, a_{2p-1}, a_{p-2}, a_{2p-2}, \ldots, a_1, a_{p+1}$, that is, $x_1 = a_p$, $x_2 = a_{2p}, \ldots, x_{2p} = a_{p+1}$.

*Step II.* Let $x_{l_1}, x_{l_2}, \ldots, x_{l_{2p-2m}}$ be equal to the vertices $a_p$, $a_{2p-m}$, $a_{p-1}$, $a_{2p-(m+1)}, \ldots, a_{m+1}$, $a_{p+1}$, respectively. Let $B = (b_1^{(i)}, b_2^{(i)}, \ldots, b_{2p-2m}^{(i)})$ be an ordered tuple of vertices in $T \setminus P$ such that $b_j^{(i)}$ is at distance $i$ from $P$ and lies on a branch incident on $x_{l_j}$, $i = 2, 3, \ldots, m$, and $j = 1, 2, \ldots, 2p-2m$. So for any $i$, there are $t_i$ disjoint such tuples, $i = 2, 3, \ldots, m$. Consider the sequence $S$: $x_{2p+1}, x_{2p+2}, \ldots, x_r$, where $r = 2p + 2(p-m)\sum_{i=2}^{m} t_i$.

*Step III.* We take $i = m$. In this case, there are $t_m$ disjoint tuples $B$. Select an arbitrary such tuple $B$ and use the first $2p-2m$ terms of the sequence $S$ to name the vertices in $B$ in order. For the next tuple $B$ we use the next $2p-2m$ terms of the sequence $S$ to name the vertices in $B$ in order. We proceed like this until we cover all the $t_m$ disjoint tuples.

*Step IV.* We name the vertices in $T \setminus P$ which are at distance $m-1$ from $P$, in the similar manner. We proceed like this until we name all the vertices in $T \setminus P$ and are of distance 2 from $P$.

*Step V.* Consider the sequence $S_1 : x_{r+1}, x_{r+2}, \ldots, x_n$. The terms in the beginning of $S_1$ are assigned (or named) to the distance one vertices in $T \setminus P$ adjacent to $x_{l_1}$ and $x_{l_2}$ alternately (the number of distance one vertices in $T \setminus P$ adjacent to $x_{l_i}$ and $x_{l_{i+1}}$, $1 \le i \le 2p-2m-1$ are equal). The next terms of $S_1$ are assigned to the distance one vertices in $T \setminus P$ adjacent to $x_{l_3}$ and $x_{l_4}$ alternately, and so on, till we name all the vertices of $T$. Observe that $n = 2p + (2p-2m)\sum_{i=2}^{m} t_i + 2\sum_{i=m+1}^{p} s_i$.

Now $x_1, x_2, \ldots, x_n$ is an ordering of all the vertices of $T$.

Let $f$ be a coloring to the vertices of $T$ defined by

$$f(x_1) = 1$$
$$f(x_i) = f(x_{i-1}) + 2p - d(x_i, x_{i-1}), \quad 2 \le i \le n. \tag{2}$$

Before we prove that $f$ is a minimal radio coloring of $T$, we give an illustration of $f$.

*Example 7.* In this example, we illustrate the above coloring $f$ by considering the 3-distant tree $T$ given in Figure 2

Here $p = 6$, the central path $P$: $a_1 a_2 \cdots a_{12}$, $m = 3$, $t_2 = 1$, $t_3 = 2$, $s_4 = 3$, $s_5 = 1$, $s_6 = 2$, $x_1 = a_6$, $x_2 = a_{12}$, $x_3 = a_5$, $x_4 = a_{11}$, $x_5 = a_4$, $x_6 = a_{10}$, $x_7 = a_3$, $x_8 = a_9$, $x_9 = a_2$, $x_{10} = a_8$, $x_{11} = a_1$, $x_{12} = a_7$, $x_{l_1} = a_6$, $x_{l_2} = a_9$, $x_{l_3} = a_5$, $x_{l_4} = a_8$, $x_{l_5} = a_4$, $x_{l_6} = a_7$, $S$: $x_{13}, x_{14}, \ldots, x_{30}$, and $S_1$: $x_{31}, x_{32}, \ldots, x_{42}$. So the ordering of the vertices of this 3-distant tree is in Figure 3.

The assignment $f$ is given in Figure 4.

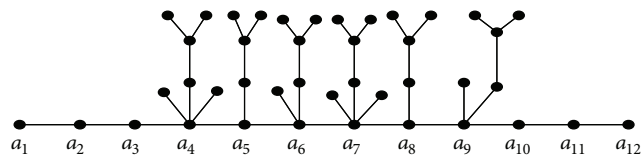


Figure 2: Tree $T$ to illustrate Theorem 5.

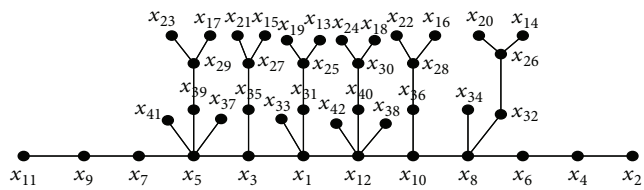


Figure 3: Ordering of vertices of $T$ obtained from Theorem 5.

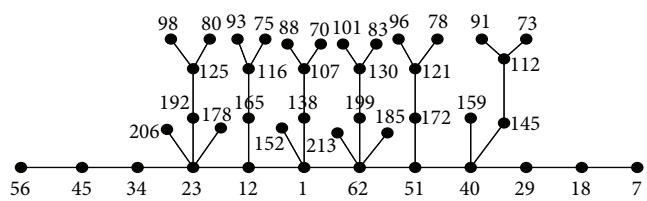


Figure 4: Radio coloring of $T$ obtained from Theorem 5.

*Continuation of the Proof of Theorem.* We first show that $f$ is a radio coloring. We need to check that $|f(x_i) - f(x_j)| \ge 1 + (2p-1) - d(x_i, x_j)$, $1 \le i \ne j \le n$ (we call this as radio condition). From the definition of $f$, radio condition holds true for pair of vertices $x_i$ and $x_{i-1}$, $2 \le i \le n$. For $1 \le i \le 2p$,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f(x_{i+2}) - f(x_{i+1}) + f(x_{i+1}) - f(x_i) \\ &= 2p - d(x_{i+2}, x_{i+1}) + 2p - d(x_{i+1}, x_i) \\ &= 4p - \{d(x_{i+2}, x_{i+1}) + d(x_{i+1}, x_i)\} \\ &= 4p - \{2p+1\} \\ &= 2p - 1 \\ &\ge 1 + 2p - d(x_i, x_{i+2}). \end{aligned} \tag{3}$$

Since

$$\begin{aligned} f(x_{2p+1}) - f(x_{2p-1}) &= f(x_{2p+1}) - f(x_{2p}) \\ &\quad + f(x_{2p}) - f(x_{2p-1}) \\ &= 2p - (m+1) + 2p - (p) \\ &= 3p - (m+1) \\ &\ge 2p - 1, \end{aligned} \tag{4}$$

radio condition holds true for all the pair of vertices $x_i$ and $x_j$, where one of them is on the central path $P$ and the other in $T \setminus P$.

Now,

$$\begin{aligned} f\left(x_{2p+3}\right)-f\left(x_{2p+1}\right) &= f\left(x_{2p+3}\right)-f\left(x_{2p+2}\right) \\ &\quad + f\left(x_{2p+2}\right)-f\left(x_{2p+1}\right) \\ &= 2p-d\left(x_{2p+2},x_{2p+3}\right)+2p \\ &\quad - d\left(x_{2p+1},x_{2p+2}\right) \\ &= 2p-\{p-m+1+2m\}+2p \\ &\quad -\{p-m+2m\} \\ &= 4p-\{(p-m+2m) \\ &\qquad +(p-m+1+2m)\} \\ &= 2p-(2m+1) \\ &= 2p-d\left(x_{2p+1},x_{2p+3}\right). \end{aligned} \tag{5}$$

Since $x_{2p+4}, x_{2p+5}, \ldots, x_{2p+2p-2m}$ are at least $2m+1$ distance apart from $x_{2p+1}$ and have colors greater than $f(x_{2p+3})$, radio condition automatically holds true, and

$$\begin{aligned} & f\left(x_{2p+2(p-m)}\right)-f\left(x_{2p+1}\right) \\ &\quad = \sum_{i=2}^{2(p-m)} f\left(x_{2p+i}\right)-f\left(x_{2p+i-1}\right) \\ &\quad = 2p\left(2\left(p-m\right)-1\right) \\ &\qquad - \sum_{i=2}^{2(p-m)} d\left(x_{2p+i},x_{2p+i-1}\right) \\ &\quad = 4p^2-4pm-2p \\ &\qquad - \left[\left(\frac{2\left(p-m\right)-2}{2}\right)\left(p+m+p+m+1\right)+p+m\right] \\ &\quad = 2p^2-(4m+2)\,p+2m\,(m+1)+1. \end{aligned} \tag{6}$$

If $x_{2p+2(p-m)+1}$ is at a distance $m$ from $P$, then

$$\begin{aligned} & f\left(x_{2p+2(p-m)+1}\right)-f\left(x_{2p+1}\right) \\ &\quad = f\left(x_{2p+2(p-m)+1}\right)-f\left(x_{2p+2(p-m)}\right) \\ &\qquad + 2p^2-(4m+2)\,p+2m\,(m+1)+1 \\ &\quad = 2p-(2m+1)+2p^2-(4m+2)\,p \\ &\qquad + 2m\,(m+1)+1 \\ &\quad = 2\left(p-m\right)^2 \\ &\quad \geq 2\left(p-\left(p-\sqrt{p-1}\right)\right)^2 \\ &\quad = 2p-2 \\ &\quad = 2p-d\left(x_{2p+1},x_{2p+2(p-m)+1}\right). \end{aligned} \tag{7}$$

Since $f(x_{2p+2(p-m)+1})-f(x_{2p+1}) \geq 2p-2$, radio condition holds true between $x_{2p+1}$ and all other vertices.

Let $i = 2p+(2p-2m)t_m$. Then $x_i$ is the last vertex at distance $m$ from $P$. Since, from the definition of $f$, radio condition holds true for $x_i$ and $x_{i+1}$, and $d(x_i, x_{i+1}) = m+1+m-1 = 2m \leq d(x_i, x_{i+j})$, $2 \leq j \leq 2p-2m-1$, radio condition automatically holds true between $x_i$ and $x_{i+j}$, $2 \leq j \leq 2p-2m-1$.

Since $f(x_{i+1})-f(x_i) = 2p-\{m+(m-1)+1\} = 2(p-m)$, $f(x_{i+2})-f(x_{i+1}) = 2p-\{2(m-1)+p-m\} = p-m+2$, and $f(x_{i+3})-f(x_{i+2}) = 2p-\{2(m-1)+p-m+1\} = p-m+1$, we have

$$\begin{aligned} & f\left(x_{i+2p-2m}\right)-f\left(x_i\right) \\ &\quad = \sum_{l=2}^{2p-2m}\left[f\left(x_{i+l}\right)-f\left(x_{i+l-1}\right)\right]+f\left(x_{i+1}\right)-f\left(x_i\right) \\ &\quad = \frac{2p-2m}{2}\left(p-m+2\right)+\frac{2p-2m-2}{2}\left(p-m+1\right) \\ &\qquad + 2\left(p-m\right) \\ &\quad = 2\left(p-m\right)^2+4\left(p-m\right)-1 \\ &\quad \geq 2p-1 \\ &\quad = 2p-d\left(x_i,x_{i+2p-2m}\right). \end{aligned} \tag{8}$$

In a similar manner, one can check the radio condition for the remaining pair of vertices. Therefore, $f$ is a radio coloring. Next, we show that $f$ is minimal. Let $x$ and $y$ be any two vertices of $T$. Then we get vertices $a_i$ and $a_j$ on the central path $P$ such that $x = a_i$ or $x$ is on a branch incident on $a_i$, and $y = a_j$ or $y$ is on a branch incident on $a_j$. Then

$$d\left(x,y\right) = \begin{cases} \left|j-i\right|, \\ \quad \text{if } x=a_i \text{ and } y=a_j \\ \left|j-i\right|+j_1, \\ \quad \text{if } x=a_i \text{ and } y \text{ lies on a branch} \\ \quad \text{and is } j_1 \text{ distance from } a_j \\ \left|j-i\right|+j_2, \\ \quad \text{if } y=a_j \text{ and } x \text{ lies on a branch} \\ \quad \text{and is } j_2 \text{ distance from } a_i \\ \left|j-i\right|+j_1+j_2, \\ \quad \text{if } x \text{ and } y \text{ lie on branches and are} \\ \quad j_2 \text{ and } j_1 \text{ distances from } a_i \text{ and } a_j \text{ respectively.} \end{cases} \tag{9}$$

So, each term in the distance sum of a radio coloring contains two indices from $\{1, 2, \ldots, 2p\}$ with different signs because from (9), distance between every pair of vertices contains

$|j-i|$, $i,j \in \{1,2,\ldots,2p\}$. Since there are $n-1$ terms in the distance sum, it contains $2(n-1)$ elements from the set $\{1,2,\ldots,2p\}$ with half positive and half negative sign. Also, the indices $1,2,3,\ldots,m$ and $2p-(m-1), 2p-(m-2),\ldots,2p-1,2p$ occur at most twice, and the index $j \in \{m+1,m+2,\ldots,2p-m\}$ occur at most $2(\sum_{i=2}^{m} t_i + s_j + 2)$ times, where $deg(a_j) = deg(a_{j+p-m}) = s_j + 2$. So, the possible maximum distance sum is $2(\sum_{i=2}^{m} t_i + s_{m+2} + 1)[(p+2)-(m+2)] + 2(\sum_{i=2}^{m} t_i + s_{m+3} + 1)[(p+3)-(m+3)] + \cdots + 2(\sum_{i=2}^{m} t_i + s_{p-1} + 1)[(2p-(m+1))-(p-1)] + 2(\sum_{i=2}^{m} t_i + s_p + 1)(2p-m) - 2(\sum_{i=2}^{m} t_i + s_p + (1/2))(p) + 2(\sum_{i=2}^{m} t_i + s_{m+1} + (1/2))(p+1) - 2(\sum_{i=2}^{m} t_i + s_{m+1} + 1)(m+1) + 2(2p+2p-1+2p-2+\cdots+2p-(m-1)) - 2(1+2+3+\cdots+m) + 2(p-m)\sum_{i=2}^{m} 2it_i + 2\sum_{i=m+1}^{p} 2s_i - 1$.

Now, the distance sum

$$\begin{aligned} d(f) &= \sum_{i=2}^{n} d(x_i, x_{i-1}) \\ &= \sum_{i=2}^{2p} d(x_i, x_{i-1}) + \sum_{i=2p+1}^{2p+(2p-2m)t_m} d(x_i, x_{i-1}) \\ &\quad + \sum_{i=2p+(2p-2m)t_m+1}^{2p+(2p-2m)t_m+(2p-2m)t_{m-1}} d(x_i, x_{i-1}) + \cdots \\ &\quad + \sum_{i=2p+(2p-2m)t_m+(2p-2m)t_{m-1}+\cdots+(2p-2m)t_3+1}^{2p+(2p-2m)t_m+(2p-2m)t_{m-1}+\cdots+(2p-2m)t_3+(2p-2m)t_2} d(x_i, x_{i-1}) \\ &\quad + \sum_{i=2p+(2p-2m)t_m+(2p-2m)t_{m-1}+\cdots+(2p-2m)t_3+(2p-2m)t_2+1}^{n} d(x_i, x_{i-1}), \end{aligned} \tag{10}$$

where $n = 2p + \sum_{i=2}^{m}(2p-2m)t_i + 2\sum_{i=m+1}^{p} s_i$ coincides with the possible maximum distance sum above and is equal to $2(p-m)[(p-m)\sum_{i=2}^{m} t_i + \sum_{i=2}^{m} 2it_i] + [2(p-m)+4]\sum_{i=m+1}^{p} 2s_i + 2p^2 - 2$.

One observes that $\sum_{i=2}^{n} \epsilon_i^f = 0$. Therefore, from Theorem 4, the radio coloring $f$ is minimal and

$$\begin{aligned} rc_{2p-1}(T) &= rc_{2p-1}(f) \\ &= (n-1)(1+2p-1) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + 1 \\ &= \left[2p + 2(p-m)\left(\sum_{i=2}^{m} t_i\right) + 2\sum_{i=m+1}^{p} s_i - 1\right](2p) \\ &\quad - \sum_{i=2}^{n} d(x_i, x_{i-1}) + 1 \\ &= 2(p-m)\left[(p+m)\sum_{i=2}^{m} t_i - \sum_{i=2}^{m} 2it_i\right] \\ &\quad + [2(p+m)-4]\sum_{i=m+1}^{p} s_i + 2p(p-1) + 3. \end{aligned} \tag{11}$$

□

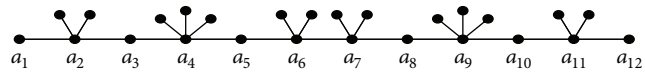


Figure 5: A caterpillar to illustrate Corollary 8.

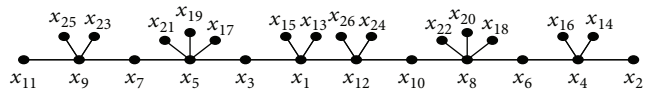


Figure 6: Ordering of vertices of the caterpillar given in Figure 5.

As a consequence of the above theorem, we determine the radio number of a class of caterpillars (1-distant trees). In the corollary below, we find radio number of caterpillars of odd diameter in which the degrees of every pair of nonpendant vertices on the central path lying at distance $p-1$ apart have the same degree (where $2p$ is the total number of vertices on the central path).

**Corollary 8.** *Let $C$ be a caterpillar of order $n$ and with a central path $P: a_1a_2a_3\cdots a_{2p}$. If $deg(a_i) = deg(a_{p+i-1}) = s_i+2$, $s_i \geq 0$, $i = 2,3,\ldots,p$, then*

$$rn(C) = 2\left[(p-1)\sum_{i=2}^{p} s_i + p(p-1)\right] + 3. \tag{12}$$

*Proof.* This is $m = 1$ case of Theorem 5. The ordering of vertices in this case includes Step I and Step V of algorithm in the proof of Theorem 5 with only variation that if a vertex on the central path is not adjacent to any pendant vertex, then we move to the next possible vertex. □

*Example 9.* In this example, we illustrate Corollary 8 by considering the caterpillar given in Figure 5.

Here $p = 6$, the central path $P$: $a_1a_2\cdots a_{12}$ and $s_2 = 2$, $s_3 = 0$, $s_4 = 3$, $s_5 = 2$. So the ordering of vertices of the caterpillar is illustrated in Figure 6 and the coloring $f$ is given in Figure 7.

The corollary below is also a consequence of Theorem 5 in which we find the radio number of caterpillars of odd diameter in which all nonpendant vertices on the central path are of the same degree.

**Corollary 10.** *Let $C$ be a caterpillar of order $n$ with a central path $P$: $a_1a_2a_3\cdots a_{2p}$. If $deg(a_i) = s+2$, $i = 2,3,\ldots,2p-1$, $s \geq 0$, then $rn(C) = 2[s(p-1)^2 + p(p-1)] + 3$.*

*Proof.* One can prove this result by substituting $s_i = s$, $i = 2,3,\ldots,p$ in Corollary 8. □

*Example 11.* In this example we illustrate Corollary 10 by considering the caterpillar given in Figure 8.

Here $p = 6$, the central path $P$: $a_1a_2\cdots a_{12}$ and $t = 2$. So the ordering of vertices of the caterpillar is illustrated in Figure 9 and the coloring $f$ is given in Figure 10.

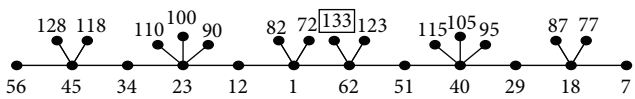


Figure 7: Radio coloring $f$ of the caterpillar given in Figure 5.

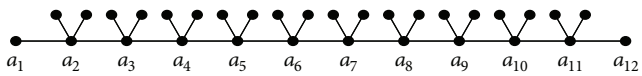


Figure 8: A caterpillar for illustration of Corollary 10.

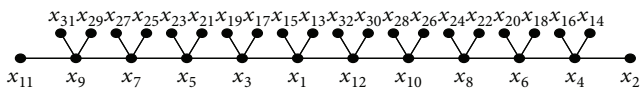


Figure 9: Ordering of vertices of the caterpillar given in Figure 8.

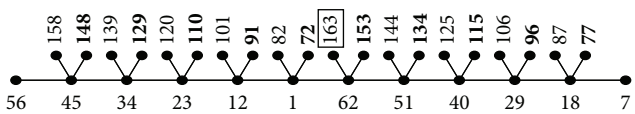


Figure 10: Minimal radio coloring of the caterpillar given in Figure 8.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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