Research Article

Fekete-Szegő Type Coefficient Inequalities for Certain Subclass of Analytic Functions and Their Applications Involving the Owa-Srivastava Fractional Operator

Serap Bulut

Civil Aviation College, Kocaeli University, Arslanbey Campus, 41285 İzmit, Kocaeli, Turkey

Correspondence should be addressed to Serap Bulut; bulutserap@yahoo.com

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A new subclass of analytic functions is introduced. For this class, firstly the Fekete-Szegő type coefficient inequalities are derived. Various known or new special cases of our results are also pointed out. Secondly some applications of our main results involving the Owa-Srivastava fractional operator are considered. Thus, as one of these applications of our result, we obtain the Fekete-Szegő type inequality for a class of normalized analytic functions, which is defined here by means of the Hadamard product (or convolution) and the Owa-Srivastava fractional operator.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}. \quad (2)$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

Fekete and Szegő [1] proved a noticeable result that the estimate

$$|a_3 - \lambda a_2^2| \leq \begin{cases} -4\lambda + 3, & \lambda \leq 0 \\ 1 + 2 \exp\left(\frac{-2\lambda}{1-\lambda}\right), & 0 \leq \lambda \leq 1 \\ 4\lambda - 3, & \lambda \geq 1 \end{cases} \quad (3)$$

holds for $f \in \mathcal{S}$. The result is sharp in the sense that for each λ there is a function in the class under consideration for which equality holds.

The coefficient functional

$$\phi_\lambda(f) = a_3 - \lambda a_2^2 = \frac{1}{6} \left(f'''(0) - \frac{3\lambda}{2} (f''(0))^2 \right) \quad (4)$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation; namely,

$$\phi_\lambda(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta} \phi_\lambda(f) \quad (\theta \in \mathbb{R}). \quad (5)$$

In fact, rather than the simplest case when

$$\phi_0(f) = a_3, \quad (6)$$

we have several important ones. For example,

$$\phi_1(f) = a_3 - a_2^2 \quad (7)$$

represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2. \quad (8)$$

Moreover, the first two nontrivial coefficients of the n th root transform

$$(f(z^n))^{1/n} = z + c_{n+1} z^{n+1} + c_{2n+1} z^{2n+1} + \dots \quad (9)$$

of f with the power series (1) are written by

$$\begin{aligned} c_{n+1} &= \frac{a_2}{n}, \\ c_{2n+1} &= \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n^2}, \end{aligned} \quad (10)$$

so that

$$a_3 - \lambda a_2^2 = n(c_{2n+1} - \mu c_{n+1}^2), \quad (11)$$

where

$$\mu = \lambda n + \frac{n-1}{2}. \quad (12)$$

Thus it is quite natural to ask about inequalities for ϕ_λ corresponding to subclasses of $\mathcal{S}$. This is called Fekete-Szegő problem. Actually, many authors have considered this problem for typical classes of univalent functions (see, e.g., [1-12]).

For two functions f and g , analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and we write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \quad (13)$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}), \quad (14)$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \quad (15)$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \quad (16)$$

Let $\varphi(z)$ be an analytic function with

$$\varphi(0) = 1, \quad \varphi'(0) > 0, \quad \operatorname{Re}\{\varphi(z)\} > 0, \quad (z \in \mathbb{U}), \quad (17)$$

which maps the open unit disk $\mathbb{U}$ onto a star-like region with respect to 1 and is symmetric with respect to the real axis.

This paper contains analogues of (3) for the following classes of analytic functions.

Definition 1. Let

$$\alpha > 0, \quad 0 \leq \gamma \leq 1, \quad b \in \mathbb{C} \setminus \{0\}. \quad (18)$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_b(\alpha, \gamma; \varphi)$ if it satisfies the following subordination condition:

$$\begin{aligned} 1 + \frac{1}{b} \left((1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) \right. \\ \left. + \gamma z f''(z) - 1 \right) < \varphi(z), \end{aligned} \quad (19)$$

where $\varphi(z)$ is defined to be the same as above for $z \in \mathbb{U}$.

Remark 2. (i) If we set

$$\alpha = 2\gamma + 1 \quad (20)$$

in Definition 1, then we have the class

$$\mathcal{R}_b(2\gamma + 1, \gamma; \varphi) = \mathcal{R}_\gamma^b(\varphi) \quad (21)$$

which consists of functions satisfying

$$1 + \frac{1}{b} (f'(z) + \gamma z f''(z) - 1) \prec \varphi(z) \quad (z \in \mathbb{U}). \quad (22)$$

This class was introduced by Bansal [13].

(ii) If we set

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; \quad z \in \mathbb{U}) \quad (23)$$

in Definition 1, then we have a new class

$$\mathcal{R}_b\left(\alpha, \gamma; \frac{1 + Az}{1 + Bz}\right) = \mathcal{R}_b(\alpha, \gamma; A, B) \quad (24)$$

which consists of functions satisfying

$$\begin{aligned} 1 + \frac{1}{b} \left((1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) \right. \\ \left. + \gamma z f''(z) - 1 \right) < \frac{1 + Az}{1 + Bz}. \end{aligned} \quad (25)$$

Taking

$$\alpha = 2\gamma + 1 \quad (26)$$

in (25), we have the class

$$\mathcal{R}_b(2\gamma + 1, \gamma; A, B) = \mathcal{R}_\gamma^b(A, B) \quad (27)$$

which consists of functions satisfying

$$\begin{aligned} 1 + \frac{1}{b} (f'(z) + \gamma z f''(z) - 1) < \frac{1 + Az}{1 + Bz} \\ (-1 \leq B < A \leq 1; \quad z \in \mathbb{U}). \end{aligned} \quad (28)$$

This class was introduced by Bansal [14].

(iii) If we set

$$\gamma = 0, \quad b = 1, \quad A = 1, \quad B = -1 \quad (29)$$

in (25), then we have the class

$$\mathcal{R}_1(\alpha, 0; 1, -1) = \mathcal{R}(\alpha) \quad (30)$$

which consists of functions satisfying

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) \right\} > 0. \quad (31)$$

This class was introduced by Murugusundaramoorthy and Magesh [15]. The subclass $\mathcal{R}_1(1, 0; 1, -1) = \mathcal{R}(1) = \mathcal{R}$ was studied by MacGregor [16].

We denote by $\mathcal{P}$ the class of the analytic functions in $\mathbb{U}$ with

$$p(0) = 1, \quad \operatorname{Re}\{p(z)\} > 0. \quad (32)$$

We will need the following lemmas.

Lemma 3 (see [12]). If $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, then

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \nu \leq 0 \\ 2, & 0 \leq \nu \leq 1 \\ 4\nu - 2, & \nu \geq 1. \end{cases} \quad (33)$$

When $\nu < 0$ or $\nu > 1$, equality holds true if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then equality holds true if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, then the equality holds true if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right) \frac{1-z}{1+z} \quad (0 \leq \eta \leq 1) \quad (34)$$

or one of its rotations. If $\nu = 1$, then the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when $\nu = 0$.

Although the above upper bound is sharp, in the case when $0 < \nu < 1$, it can be further improved as follows:

$$\begin{aligned} |c_2 - \nu c_1^2| + \nu |c_1|^2 &\leq 2 \quad \left(0 < \nu \leq \frac{1}{2}\right), \\ |c_2 - \nu c_1^2| + (1-\nu) |c_1|^2 &\leq 2 \quad \left(\frac{1}{2} < \nu \leq 1\right). \end{aligned} \quad (35)$$

Lemma 4 (see [17]). Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$. Then for any complex number ν

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}, \quad (36)$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}. \quad (37)$$

2. Fekete-Szegő Problem for the Function

Class $\mathcal{R}_b(\alpha, \gamma; \varphi)$

By making use of Lemma 3, we first prove the Fekete-Szegő type inequalities asserted by Theorem 5 below.

Theorem 5. Let

$$\alpha > 0, \quad 0 \leq \gamma \leq 1, \quad b > 0. \quad (38)$$

Also let

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (39)$$

where

$$B_1 > 0, \quad B_2 \geq 0. \quad (40)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_b(\alpha, \gamma; \varphi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 b}{(1+2\alpha+2\gamma)} \left(\frac{B_2}{B_1} - \mu \frac{B_1 b (1+2\alpha+2\gamma)}{(1+\alpha)^2} \right), & \mu \leq \sigma_1 \\ \frac{B_1 b}{(1+2\alpha+2\gamma)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{B_1 b}{(1+2\alpha+2\gamma)} \left(-\frac{B_2}{B_1} + \mu \frac{B_1 b (1+2\alpha+2\gamma)}{(1+\alpha)^2} \right), & \mu \geq \sigma_2, \end{cases} \quad (41)$$

where

$$\begin{aligned} \sigma_1 &= \frac{(B_2 - B_1)(1+\alpha)^2}{B_1^2 b (1+2\alpha+2\gamma)}, \\ \sigma_2 &= \frac{(B_2 + B_1)(1+\alpha)^2}{B_1^2 b (1+2\alpha+2\gamma)}, \\ \sigma_3 &= \frac{B_2(1+\alpha)^2}{B_1^2 b (1+2\alpha+2\gamma)}. \end{aligned} \quad (42)$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &+ \frac{(1+\alpha)^2}{B_1 b (1+2\alpha+2\gamma)} \\ &\times \left[1 - \frac{B_2}{B_1} + \mu \frac{B_1 b (1+2\alpha+2\gamma)}{(1+\alpha)^2} \right] |a_2|^2 \\ &\leq \frac{B_1 b}{(1+2\alpha+2\gamma)}. \end{aligned} \quad (43)$$

Furthermore, if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &+ \frac{(1+\alpha)^2}{B_1 b (1+2\alpha+2\gamma)} \\ &\times \left[1 + \frac{B_2}{B_1} - \mu \frac{B_1 b (1+2\alpha+2\gamma)}{(1+\alpha)^2} \right] |a_2|^2 \\ &\leq \frac{B_1 b}{(1+2\alpha+2\gamma)}. \end{aligned} \quad (44)$$

Each of these results is sharp.

Proof. Since $f \in \mathcal{R}_b(\alpha, \gamma; \varphi)$, we have

$$h(z) < \varphi(z), \quad (45)$$

where

$$\begin{aligned} h(z) &= 1 + \frac{1}{b} \left((1-\alpha+2\gamma) \frac{f(z)}{z} + (\alpha-2\gamma) f'(z) \right. \\ &\quad \left. + \gamma z f''(z) - 1 \right) \\ &= 1 + h_1 z + h_2 z^2 + \dots \end{aligned} \quad (46)$$

From (46), we have

$$h_1 = \frac{1}{b} (1 + \alpha) a_2, \quad h_2 = \frac{1}{b} (1 + 2\alpha + 2\gamma) a_3. \quad (47)$$

Since $\varphi(z)$ is univalent and $h(z) \prec \varphi(z)$, the function

$$p_1(z) = \frac{1 + \varphi^{-1}(h(z))}{1 - \varphi^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots \quad (48)$$

is analytic and has a positive real part in $\mathbb{U}$. Also we have

$$\begin{aligned} h(z) &= \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \\ &= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \dots \end{aligned} \quad (49)$$

Thus by (47) and (49) we get

$$\begin{aligned} a_2 &= \frac{B_1 c_1 b}{2(1 + \alpha)}, \\ a_3 &= \frac{b}{(1 + 2\alpha + 2\gamma)} \left[\frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} \right]. \end{aligned} \quad (50)$$

Taking into account (50), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{b}{(1 + 2\alpha + 2\gamma)} \\ &\times \left[\frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} \right] - \mu \frac{B_1^2 c_1^2 b^2}{4(1 + \alpha)^2} \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} (c_2 - \delta c_1^2), \end{aligned} \quad (51)$$

where

$$\delta = \frac{1}{2} \left(1 - \frac{B_2}{B_1} + \frac{\mu B_1 b (1 + 2\alpha + 2\gamma)}{(1 + \alpha)^2} \right). \quad (52)$$

The assertion of Theorem 5 now follows by an application of Lemma 3. On the other hand, using (51) for the values of $\sigma_1 \leq \mu \leq \sigma_3$, we have

$$\begin{aligned} &|a_3 - \mu a_2^2| + (\mu - \sigma_1) |a_2|^2 \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} |c_2 - \delta c_1^2| \\ &\quad + (\mu - \sigma_1) \frac{B_1^2 b^2 |c_1|^2}{4(1 + \alpha)^2} \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} |c_2 - \delta c_1^2| \\ &\quad + \left(\mu - \frac{(B_2 - B_1)(1 + \alpha)^2}{B_1^2 b (1 + 2\alpha + 2\gamma)} \right) \frac{B_1^2 b^2 |c_1|^2}{4(1 + \alpha)^2} \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} \{ |c_2 - \delta c_1^2| + \delta |c_1|^2 \} \\ &\leq \frac{B_1 b}{(1 + 2\alpha + 2\gamma)}. \end{aligned} \quad (53)$$

Similarly, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we get

$$\begin{aligned} &|a_3 - \mu a_2^2| + (\sigma_2 - \mu) |a_2|^2 \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} |c_2 - \delta c_1^2| \\ &\quad + (\sigma_2 - \mu) \frac{B_1^2 b^2 |c_1|^2}{4(1 + \alpha)^2} \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} |c_2 - \delta c_1^2| \\ &\quad + \left(\frac{(B_2 + B_1)(1 + \alpha)^2}{B_1^2 b (1 + 2\alpha + 2\gamma)} - \mu \right) \frac{B_1^2 b^2 |c_1|^2}{4(1 + \alpha)^2} \\ &= \frac{B_1 b}{2(1 + 2\alpha + 2\gamma)} \{ |c_2 - \delta c_1^2| + (1 - \delta) |c_1|^2 \} \\ &\leq \frac{B_1 b}{(1 + 2\alpha + 2\gamma)}. \end{aligned} \quad (54)$$

To show that the bounds asserted by Theorem 5 are sharp, we define the following functions:

$$K_{\varphi_n}(z) \quad (n = 2, 3, \dots), \quad (55)$$

with

$$K_{\varphi_n}(0) = 0 = K'_{\varphi_n}(0) - 1, \quad (56)$$

by

$$\begin{aligned} &1 + \frac{1}{b} \left((1 - \alpha + 2\gamma) \frac{K_{\varphi_n}(z)}{z} + (\alpha - 2\gamma) K'_{\varphi_n}(z) \right. \\ &\quad \left. + \gamma z K''_{\varphi_n}(z) - 1 \right) = \varphi(z^{n-1}), \end{aligned} \quad (57)$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ ($0 \leq \eta \leq 1$), with

$$F_\eta(0) = 0 = F'_\eta(0) - 1, \quad G_\eta(0) = 0 = G'_\eta(0) - 1, \quad (58)$$

by

$$\begin{aligned} 1 + \frac{1}{b} \left((1 - \alpha + 2\gamma) \frac{F_\eta(z)}{z} + (\alpha - 2\gamma) F'_\eta(z) \right. \\ \left. + \gamma z F''_\eta(z) - 1 \right) = \varphi \left(\frac{z(z + \eta)}{1 + \eta z} \right), \\ 1 + \frac{1}{b} \left((1 - \alpha + 2\gamma) \frac{G_\eta(z)}{z} + (\alpha - 2\gamma) G'_\eta(z) \right. \\ \left. + \gamma z G''_\eta(z) - 1 \right) = \varphi \left(-\frac{z(z + \eta)}{1 + \eta z} \right), \end{aligned} \quad (59)$$

respectively. Then, clearly, the functions K_{φ_n} , F_η , and $G_\eta \in \mathcal{R}_b(\alpha, \gamma; \varphi)$. We also write

$$K_\varphi = K_{\varphi_2}. \quad (60)$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality in Theorem 5 holds true if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds true if and only if f is K_{φ_3} or one of its rotations. If $\mu = \sigma_1$, then the equality holds true if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if f is G_η or one of its rotations. $\square$

By making use of Lemma 4, we immediately obtain the following Fekete-Szegő type inequality.

Theorem 6. *Let*

$$\alpha > 0, \quad 0 \leq \gamma \leq 1, \quad b \in \mathbb{C} \setminus \{0\}. \quad (61)$$

Also let

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (62)$$

where

$$B_1 > 0, \quad B_2 \geq 0. \quad (63)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_b(\alpha, \gamma; \varphi)$, then for any complex number μ

$$\begin{aligned} |a_3 - \mu a_2^2| \\ \leq \frac{B_1 |b|}{(1 + 2\alpha + 2\gamma)} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{\mu B_1 b (1 + 2\alpha + 2\gamma)}{(1 + \alpha)^2} \right| \right\}. \end{aligned} \quad (64)$$

The result is sharp.

Remark 7. The coefficient bounds for $|a_2|$ and $|a_3|$ are special cases of those asserted by Theorem 5.

Taking $\alpha = 2\gamma + 1$ in Theorem 6, we have the following corollary.

Corollary 8 (see [13]). *Let*

$$0 \leq \gamma \leq 1, \quad b \in \mathbb{C} \setminus \{0\}. \quad (65)$$

Also let

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (66)$$

where

$$B_1 > 0, \quad B_2 \geq 0. \quad (67)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_\gamma^b(\varphi)$, then for any complex number μ

$$\begin{aligned} |a_3 - \mu a_2^2| \\ \leq \frac{B_1 |b|}{3(1 + 2\gamma)} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{3\mu B_1 b (1 + 2\gamma)}{4(1 + \gamma)^2} \right| \right\}. \end{aligned} \quad (68)$$

The result is sharp.

If we set

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}) \quad (69)$$

in Theorem 6, then we have

$$B_1 = A - B, \quad B_2 = -B(A - B). \quad (70)$$

So we get the following corollary.

Corollary 9. *Let*

$$\alpha > 0, \quad 0 \leq \gamma \leq 1, \quad b \in \mathbb{C} \setminus \{0\}. \quad (71)$$

Also let

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}). \quad (72)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_b(\alpha, \gamma; A, B)$, then for any complex number μ

$$\begin{aligned} |a_3 - \mu a_2^2| \leq \frac{(A - B) |b|}{(1 + 2\alpha + 2\gamma)} \\ \times \max \left\{ 1, \left| B + \frac{\mu(A - B)b(1 + 2\alpha + 2\gamma)}{(1 + \alpha)^2} \right| \right\}. \end{aligned} \quad (73)$$

The result is sharp.

Putting $\alpha = 2\gamma + 1$ in Corollary 9, we obtain the following corollary.

Corollary 10 (see [13]). *Let*

$$0 \leq \gamma \leq 1, \quad b \in \mathbb{C} \setminus \{0\}. \quad (74)$$

Also let

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}). \quad (75)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_\gamma^b(A, B)$, then for any complex number μ

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A-B)|b|}{3(1+2\gamma)} \\ &\times \max \left\{ 1, \left| B + \frac{3\mu(A-B)b(1+2\gamma)}{4(1+\gamma)^2} \right| \right\}. \end{aligned} \quad (76)$$

The result is sharp.

Also putting $\gamma = 0$, $b = 1$, $A = 1$, and $B = -1$ in Corollary 9, we obtain the following corollary.

Corollary 11. Let $\alpha > 0$. If $f(z)$ given by (1) belongs to the function class $\mathcal{R}(\alpha)$, then for any complex number μ

$$|a_3 - \mu a_2^2| \leq \frac{2}{(1+2\alpha)} \max \left\{ 1, \left| 1 - \frac{2\mu(1+2\alpha)}{(1+\alpha)^2} \right| \right\}. \quad (77)$$

3. Applications to Analytic Functions Defined by Using Fractional Calculus Operators and Convolution

The subject of fractional calculus (i.e., calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so. For the applications of the results given in the preceding sections, we first introduce the class $\mathcal{R}_b^\rho(\alpha, \gamma; \varphi)$, which is defined by means of the Hadamard product (or convolution) and a certain operator of fractional calculus, known as the Owa-Srivastava operator (see, e.g., [18, 19]).

Definition 12. The fractional integral of order ρ is defined, for a function $f(z)$, by

$$D_z^{-\rho} f(z) = \frac{1}{\Gamma(\rho)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\rho}} d\zeta \quad (\rho > 0), \quad (78)$$

where the function $f(z)$ is analytic in a simply connected domain of the complex z -plane containing the origin, and the multiplicity of $(z-\zeta)^{\rho-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 13. The fractional derivative of order ρ is defined, for a function $f(z)$, by

$$D_z^\rho f(z) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\rho} d\zeta \quad (0 \leq \rho < 1), \quad (79)$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\rho}$ is removed, as in Definition 12.

Definition 14. Under the hypotheses of Definition 13, the fractional derivative of order $n + \rho$ is defined, for a function $f(z)$, by

$$D_z^{n+\rho} f(z) = \frac{d^n}{dz^n} (D_z^\rho f(z)), \quad (80)$$

$$(0 \leq \rho < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Using Definitions 12, 13, and 14, fractional derivatives, and fractional integrals, Owa and Srivastava [20] introduced the operator $\Omega^\rho : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$(\Omega^\rho f)(z) = \Gamma(2-\rho) z^\rho D_z^\rho f(z), \quad \rho \neq 2, 3, 4, \dots \quad (81)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\rho)}{\Gamma(n-\rho+1)} a_n z^n. \quad (82)$$

This operator is known as the Owa-Srivastava operator. In terms of the Owa-Srivastava operator Ω^ρ defined by (81), we now introduce the function class $\mathcal{R}_b^\rho(\alpha, \gamma; \varphi)$ in the following way:

$$\mathcal{R}_b^\rho(\alpha, \gamma; \varphi) = \{f \in \mathcal{A} : \Omega^\rho f \in \mathcal{R}_b(\alpha, \gamma; \varphi)\}. \quad (83)$$

Note that the function class $\mathcal{R}_b^\rho(\alpha, \gamma; \varphi)$ is a special case of the function class $\mathcal{R}_b^g(\alpha, \gamma; \varphi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\rho)}{\Gamma(n-\rho+1)} z^n. \quad (84)$$

Now suppose that

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0). \quad (85)$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}_b^g(\alpha, \gamma; \varphi) \quad (86)$$

$$\iff (f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathcal{R}_b(\alpha, \gamma; \varphi),$$

we can obtain the coefficient estimates for functions in the class $\mathcal{R}_b^g(\alpha, \gamma; \varphi)$ from the corresponding estimates for functions in the class $\mathcal{R}_b(\alpha, \gamma; \varphi)$. By applying Theorem 5 to the following Hadamard product (or convolution):

$$(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots, \quad (87)$$

we get the following theorem after an obvious change of the parameter μ .

Theorem 15. Let

$$\alpha > 0, \quad 0 \leq \gamma \leq 1, \quad b > 0. \quad (88)$$

Also let

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (89)$$

where

$$B_1 > 0, \quad B_2 \geq 0, \quad g_n > 0 \quad (n = 3, 4, \dots). \quad (90)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_b^g(\alpha, \gamma; \varphi)$, then

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & \leq \begin{cases} \frac{B_1 b}{(1 + 2\alpha + 2\gamma) g_3} \left(\frac{B_2}{B_1} - \mu \frac{g_3}{g_2^2} \frac{B_1 b (1 + 2\alpha + 2\gamma)}{(1 + \alpha)^2} \right), & \mu \leq \sigma_1 \\ \frac{B_1 b}{(1 + 2\alpha + 2\gamma) g_3}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{B_1 b}{(1 + 2\alpha + 2\gamma) g_3} \left(-\frac{B_2}{B_1} + \mu \frac{g_3}{g_2^2} \frac{B_1 b (1 + 2\alpha + 2\gamma)}{(1 + \alpha)^2} \right), & \mu \geq \sigma_2, \end{cases} \end{aligned} \quad (91)$$

where

$$\begin{aligned} \sigma_1 &= \frac{g_2^2 (B_2 - B_1) (1 + \alpha)^2}{g_3 B_1^2 b (1 + 2\alpha + 2\gamma)}, \\ \sigma_2 &= \frac{g_2^2 (B_2 + B_1) (1 + \alpha)^2}{g_3 B_1^2 b (1 + 2\alpha + 2\gamma)}. \end{aligned} \quad (92)$$

Each of these results is sharp.

When g corresponds to the Owa-Srivastava operator given in (82), we obtain

$$g_2 = \frac{\Gamma(3) \Gamma(2 - \rho)}{\Gamma(3 - \rho)} = \frac{2}{2 - \rho}, \quad (93)$$

$$g_3 = \frac{\Gamma(4) \Gamma(2 - \rho)}{\Gamma(4 - \rho)} = \frac{6}{(2 - \rho)(3 - \rho)}. \quad (94)$$

For g_2 and g_3 given by (93) and (94), respectively, Theorem 15 reduces to the following result.

Theorem 16. Let

$$\alpha > 0, \quad 0 \leq \gamma \leq 1, \quad b > 0. \quad (95)$$

Also let

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (96)$$

where

$$B_1 > 0, \quad B_2 \geq 0. \quad (97)$$

If $f(z)$ given by (1) belongs to the function class $\mathcal{R}_b^g(\alpha, \gamma; \varphi)$, then

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & \leq \begin{cases} \frac{(2 - \rho)(3 - \rho) B_1 b}{6(1 + 2\alpha + 2\gamma)} \left(\frac{B_2}{B_1} - \mu \frac{3(2 - \rho) B_1 b (1 + 2\alpha + 2\gamma)}{2(3 - \rho)(1 + \alpha)^2} \right), & \mu \leq \sigma_1 \\ \frac{(2 - \rho)(3 - \rho) B_1 b}{6(1 + 2\alpha + 2\gamma)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{(2 - \rho)(3 - \rho) B_1 b}{6(1 + 2\alpha + 2\gamma)} \left(-\frac{B_2}{B_1} + \mu \frac{3(2 - \rho) B_1 b (1 + 2\alpha + 2\gamma)}{2(3 - \rho)(1 + \alpha)^2} \right), & \mu \geq \sigma_2, \end{cases} \end{aligned} \quad (98)$$

where

$$\begin{aligned} \sigma_1 &= \frac{2(3 - \rho)(B_2 - B_1)(1 + \alpha)^2}{3(2 - \rho) B_1^2 b (1 + 2\alpha + 2\gamma)}, \\ \sigma_2 &= \frac{2(3 - \rho)(B_2 + B_1)(1 + \alpha)^2}{3(2 - \rho) B_1^2 b (1 + 2\alpha + 2\gamma)}. \end{aligned} \quad (99)$$

Each of these results is sharp.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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