

Research Article

On Convergence with respect to an Ideal and a Family of Matrices

Jan-David Hardtke

Department of Mathematics, Free University of Berlin, Arnimallee 6, 14195 Berlin, Germany

Correspondence should be addressed to Jan-David Hardtke; hardtke@math.fu-berlin.de

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P. Das et al. recently introduced and studied the notions of strong A^I -summability with respect to an Orlicz function F and A^I -statistical convergence, where A is a nonnegative regular matrix and I is an ideal on the set of natural numbers. In this paper, we will generalise these notions by replacing A with a family of matrices and F with a family of Orlicz functions or moduli and study the thus obtained convergence methods. We will also give an application in Banach space theory, presenting a generalisation of Simons' sup-limsup-theorem to the newly introduced convergence methods (for the case that the filter generated by the ideal I has a countable base), continuing some of the author's previous work.

1. Introduction

Let us begin by recalling that an ideal I on a nonempty set Y is a nonempty set of subsets of Y such that $Y \notin I$ and I is closed under the formation of subsets and finite unions. The ideal is called admissible if $\{y\} \in I$ for each $y \in Y$. For example, if Y is infinite, then the set of all finite subsets of Y forms an ideal on Y . If I is an ideal, then $\mathcal{F}(I) := \{Y \setminus A : A \in I\}$ is a filter on Y .

Now if $(x_n)_{n \in \mathbb{N}}$ is a sequence in a topological space X and I is an ideal on the set \mathbb{N} of natural numbers, then $(x_n)_{n \in \mathbb{N}}$ is said to be I -convergent to $x \in X$ if for every neighbourhood U of x the set $\{n \in \mathbb{N} : x_n \notin U\}$ belongs to I (equivalently, $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}(I)$). In a Hausdorff space the I -limit is unique if it exists. It will be denoted by $I\text{-}\lim x_n$. If I_f is the ideal of all finite subsets of \mathbb{N} , then I_f -convergence is equivalent to the usual convergence. Thus if I is admissible, the usual convergence implies I -convergence. For a normed space X the set of all I -convergent sequences in X is a subspace of $X^{\mathbb{N}}$ and the map $(x_n) \mapsto I\text{-}\lim x_n$ is linear. We refer the reader to [1–4] for more information on I -convergence.

Recall now that for a given infinite matrix $A = (a_{nk})_{n,k \in \mathbb{N}}$ with real or complex entries a sequence $s = (s_k)_{k \in \mathbb{N}}$ of (real or complex) numbers is said to be A -summable to the number a

provided that each of the series $\sum_{k=1}^{\infty} a_{nk}s_k$ is convergent and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk}s_k = a$.

The matrix A is called regular if every sequence that is convergent in the ordinary sense is also A -summable to the same limit. A well-known theorem of Toeplitz states that A is regular if and only if the following holds:

- (i) $\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$,
- (ii) $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} = 1$,
- (iii) $\lim_{n \rightarrow \infty} a_{nk} = 0 \quad \forall k \in \mathbb{N}$.

Let us suppose for the moment that A is regular and also nonnegative (i.e., $a_{nk} \geq 0$ for all $n, k \in \mathbb{N}$). We will denote by $D(s, a, \varepsilon)$ the set $\{k \in \mathbb{N} : |s_k - a| \geq \varepsilon\}$ for every $\varepsilon > 0$. Then s is said to be A -statistically convergent to a if for every $\varepsilon > 0$ we have $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} \chi_{D(s, a, \varepsilon)}(k) = 0$, where the symbol χ_K denotes the characteristic function of the set $K \subseteq \mathbb{N}$. If one takes A to be the Cesàro matrix (i.e., $a_{nk} = 1/n$ for $k \leq n$ and $a_{nk} = 0$ for $k > n$) one gets the usual notion of statistical convergence as it was introduced by Fast in [5]. Note that the set I_A of all subsets $K \subseteq \mathbb{N}$ for which $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} \chi_K(k) = 0$ holds is an ideal on \mathbb{N} and A -statistical convergence is nothing but convergence with respect to this ideal.

For any number $p > 0$ the sequence s is said to be strongly A - p -summable to a provided that $\sum_{k=1}^{\infty} a_{nk} |s_k - a|^p < \infty$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} |s_k - a|^p = 0$. The strong A - p -summability is a linear consistent summability method and the strong A - p -limit is uniquely determined whenever it exists. In [6] Connor proved that s is statistically convergent to a whenever it is strongly p -Cesàro convergent to a and the converse is true if s is bounded. Practically the same proof given in [6] still works if one replaces the Cesàro matrix by an arbitrary nonnegative regular matrix A . In particular, strong A - p -summability and A -statistical convergence are equivalent on bounded sequences (see also [7, Theorem 8]). More information on strong matrix summability can be found in [8] (for the case $p = 1$) or [9].

In [10] Maddox proposed a generalisation of strong A - p -summability by replacing the number p with a sequence $\mathbf{p} = (p_k)_{k \in \mathbb{N}}$ of positive numbers: the sequence s is strongly A - \mathbf{p} -summable to a if $\sum_{k=1}^{\infty} a_{nk} |s_k - a|^{p_k} < \infty$ for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} |s_k - a|^{p_k} = 0$.

Next, let us recall that a function $F : [0, \infty) \rightarrow [0, \infty)$ is called an Orlicz function if it is increasing, continuous, and convex and satisfies $\lim_{t \rightarrow \infty} F(t) = \infty$ as well as $F(t) = 0$ if and only if $t = 0$. If we drop the convexity and replace it by the condition $F(s + t) \leq F(s) + F(t)$ for all $s, t \geq 0$, then F is called a modulus. For example, the function F_p defined by $F_p(t) = t^p$ is an Orlicz function for $p \geq 1$ and a modulus for $0 < p \leq 1$. We will denote the set of all Orlicz functions by \mathcal{O} and the set of all moduli by \mathcal{M} .

Connor introduced another generalisation of strong matrix summability in [7]: if F is a modulus, then s is said to be strongly A -summable to the limit a with respect to F if $\sum_{k=1}^{\infty} a_{nk} F(|s_k - a|) < \infty$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} F(|s_k - a|) = 0$. It is shown in [7, Theorem 8] that strong A -summability with respect to F implies A -statistical convergence and that the converse holds for bounded sequences. In [11] Demirci replaced the modulus in Connor's definition by an Orlicz function and studied which results carry over to this setting.

Another common generalised convergence method is that of almost convergence introduced by Lorentz in [12]. For this we first recall that a Banach limit is a linear functional L on the space ℓ^∞ of all bounded *real-valued* sequences such that L is shift-invariant (i.e., $L((s_{n+1})_{n \in \mathbb{N}}) = L((s_n)_{n \in \mathbb{N}})$), positive (i.e., $L((s_n)_{n \in \mathbb{N}}) \geq 0$ if $s_n \geq 0$ for all n), and fulfills $L(1, 1, \dots) = 1$. The existence of a Banach limit can be easily proved by means of the Hahn-Banach extension theorem. A sequence $s \in \ell^\infty$ is said to be almost convergent to $a \in \mathbb{R}$ if $L(s) = a$ for every Banach limit L .

It is proved in [12] that almost convergence is equivalent to “uniform Cesàro convergence.” More precisely, a bounded sequence $s = (s_k)_{k \in \mathbb{N}}$ in \mathbb{R} is almost convergent to $a \in \mathbb{R}$ if and only if the following holds:

$$\frac{1}{n} \sum_{k=1}^n s_{k+i} \xrightarrow{n \rightarrow \infty} a \quad \text{uniformly in } i \in \mathbb{N}_0, \quad (1)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lorentz subsequently introduced and studied the notion of F_A -convergence by replacing the Cesàro matrix with an

arbitrary real-valued regular matrix A : a bounded sequence $s = (s_k)_{k \in \mathbb{N}}$ in \mathbb{R} is said to be F_A -convergent to $a \in \mathbb{R}$ provided that

$$\sum_{k=1}^{\infty} a_{nk} s_{k+i} \xrightarrow{n \rightarrow \infty} a \quad \text{uniformly in } i \in \mathbb{N}_0. \quad (2)$$

Stieglitz further generalised the notion of almost convergence in the following way (cf. [13]): consider a sequence $\mathcal{B} = (B_i)_{i \in \mathbb{N}_0} = ((b_{nk}^{(i)})_{n,k \in \mathbb{N}})_{i \in \mathbb{N}_0}$ of matrices with entries in \mathbb{R} or \mathbb{C} and a bounded sequence $s = (s_k)_{k \in \mathbb{N}}$ of real or complex numbers. Then s is said to be $F_{\mathcal{B}}$ -convergent to the number a if each of the series $\sum_{k=1}^{\infty} b_{nk}^{(i)} s_k$ with $n \in \mathbb{N}, i \in \mathbb{N}_0$ is convergent and

$$\sum_{k=1}^{\infty} b_{nk}^{(i)} s_k \xrightarrow{n \rightarrow \infty} a \quad \text{uniformly in } i \in \mathbb{N}_0. \quad (3)$$

To obtain F_A -convergence, take $b_{nk}^{(i)} = a_{nk-i}$ for $k > i$ and $b_{nk}^{(i)} = 0$ for $k \leq i$.

Maddox introduced the $F_{\mathcal{B}}$ -analogue of strong matrix summability in [14]. If each of the matrices B_i is nonnegative and $s = (s_k)_{k \in \mathbb{N}}$ is a (not necessarily bounded) sequence in \mathbb{R} or \mathbb{C} , then s is said to be strongly $F_{\mathcal{B}}$ -convergent to a provided that

$$\sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - a| \xrightarrow{n \rightarrow \infty} 0 \quad \text{uniformly in } i \in \mathbb{N}_0. \quad (4)$$

Very recently, the authors of [15] introduced the following definitions, combining matrices and ideals.

Definition 1 (cf. [15]). Let $A = (a_{nk})_{n,k \in \mathbb{N}}$ be a nonnegative regular matrix, I an ideal on \mathbb{N} , and F an Orlicz function. Let a be any real or complex number. A sequence $s = (s_k)_{k \in \mathbb{N}}$ in \mathbb{R} or \mathbb{C} is said to be

- (i) strongly A^I -summable to a with respect to F if

$$I\text{-}\lim \sum_{k=1}^{\infty} a_{nk} F(|s_k - a|) = 0, \quad (5)$$

- (ii) A^I -statistically convergent to a if

$$I\text{-}\lim \sum_{k=1}^{\infty} a_{nk} \chi_{D(s,a,\varepsilon)}(k) = 0 \quad (6)$$

for every $\varepsilon > 0$.

It is proved in [15, Theorem 2.5] that A^I -summability with respect to F implies A^I -statistical convergence (to the same limit) and the converse holds if the sequence s is bounded and F satisfies the Δ_2 -condition (i.e., there is a constant K such that $F(2t) \leq KF(t)$ for all $t \geq 0$).

We would like to propose here the following three definitions that include all the above mentioned generalised convergence methods.

First we define a sequence $(g_n)_{n \in \mathbb{N}}$ of functions from a set S into a generalised metric space (X, d) (same as a metric space except that d is allowed to take values in $[0, \infty]$; for example, $d(a, b) = |a - b|$ for $a, b \in [0, \infty)$, $d(a, \infty) = d(\infty, a) = \infty$ for all $a \in [0, \infty)$, and $d(\infty, \infty) = 0$ defines a generalised metric on $[0, \infty]$) to be uniformly convergent to the function $g : S \rightarrow X$ along the ideal I if for every $\varepsilon > 0$ there is some $E \in I$ such that for every $s \in S$

$$\{n \in \mathbb{N} : d(g_n(s), g(s)) \geq \varepsilon\} \subseteq E \tag{7}$$

or, equivalently, for every $\varepsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \sup_{s \in S} d(g_n(s), g(s)) \geq \varepsilon\right\} \in I. \tag{8}$$

If $I = I_f$, this yields the usual definition of uniform convergence. Also, this definition is a direct generalisation of the definition of μ -statistical uniform convergence given in [16]. The uniform convergence of $(g_n)_{n \in \mathbb{N}}$ to g along I clearly implies $I\text{-}\lim g_n(s) = g(s)$ for all $s \in S$.

Now we come to the main definition.

Definition 2. Let I be an ideal on \mathbb{N} and S any nonempty set. Let $\mathcal{B} = (B_i)_{i \in S} = ((b_{nk}^{(i)})_{n,k \in \mathbb{N}})_{i \in S}$ be a family of (not necessarily regular) matrices with entries in \mathbb{R} or \mathbb{C} and $\mathcal{F} = (F_k^{(i)})_{k \in \mathbb{N}, i \in S}$ a family in $\mathcal{M} \cup \mathcal{O}$. Suppose that there is some $i_0 \in S$ such that

$$\inf_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |b_{nk}^{(i_0)}| > 0. \tag{+}$$

Finally, let $s = (s_k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{R} or \mathbb{C} and $a \in \mathbb{R}$ or \mathbb{C} .

- (i) s is said to be \mathcal{B}^I -summable to a provided that each of the series $\sum_{k=1}^{\infty} b_{nk}^{(i)} s_k$ is convergent and

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} s_k = a \quad \text{uniformly in } i \in S. \tag{9}$$

- (ii) If each matrix B_i is nonnegative, then s is said to be strongly \mathcal{B}^I -summable to a with respect to \mathcal{F} if

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)} (|s_k - a|) = 0 \quad \text{uniformly in } i \in S. \tag{10}$$

- (iii) If each B_i is nonnegative, then s is said to be \mathcal{B}^I -statistically convergent to a provided that for every $\varepsilon > 0$

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\varepsilon)}(k) = 0 \quad \text{uniformly in } i \in S. \tag{11}$$

If $F_k^{(i)} = \text{id}_{[0,\infty)}$ for all $k \in \mathbb{N}, i \in S$ in (ii) we simply speak of strong \mathcal{B}^I -summability. Clearly, strong \mathcal{B}^I -summability to

a implies \mathcal{B}^I -summability to a provided that s is bounded, $\sum_{k=1}^{\infty} b_{nk}^{(i)} < \infty$ for all $k \in \mathbb{N}, i \in S$ and

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} = 1 \quad \text{uniformly in } i \in S. \tag{12}$$

Taking $B_i = A$ and $F_k^{(i)} = F \in \mathcal{O}$ for each $i \in S$ and $k \in \mathbb{N}$ in (ii) and (iii) yields the definitions of strong A^I -summability with respect to F and of A^I -statistical convergence. If we take $I = I_f$ and $S = \mathbb{N}_0$ in (i) and (ii) we obtain the definitions of $F_{\mathcal{B}}$ - and strong $F_{\mathcal{B}}$ -convergence. Setting $I = I_f, B_i = A$ for every $i \in S$ and $F_k^{(i)} = F_{p_k}$ for all $i \in S, k \in \mathbb{N}$ in (ii) gives us the definition of Maddox's strong A - \mathbf{p} -summability.

Note also that if each B_i is nonnegative, then the set $J_{\mathcal{B},I}$ of all subsets $K \subseteq \mathbb{N}$, such that

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_K(k) = 0 \quad \text{uniformly in } i \in S, \tag{13}$$

is an ideal on \mathbb{N} (the condition (+) ensures $\mathbb{N} \notin J_{\mathcal{B},I}$). The \mathcal{B}^I -statistical convergence is nothing but the convergence with respect to $J_{\mathcal{B},I}$. In the case that B_i is the infinite unit matrix for each $i \in S$ we have $J_{\mathcal{B},I} = I$.

In the next section we will start to investigate the above convergence methods.

2. Some Convergence Theorems

If not otherwise stated, we will denote by I an ideal on \mathbb{N} , by $\mathcal{B} = (B_i)_{i \in S} = ((b_{nk}^{(i)})_{n,k \in \mathbb{N}})_{i \in S}$ a family of real or complex matrices (where S is any nonempty index set) such that there is some $i_0 \in S$ with (+), and by $\mathcal{F} = (F_k^{(i)})_{k \in \mathbb{N}, i \in S}$ a family in $\mathcal{M} \cup \mathcal{O}$. Finally, $s = (s_k)_{k \in \mathbb{N}}$ denotes a sequence in \mathbb{R} or \mathbb{C} and a an element of \mathbb{R} or \mathbb{C} , as in the previous section.

The following two propositions (wherein each B_i is implicitly assumed to be nonnegative) generalise the aforementioned results from [15, Theorem 2.5]. The techniques used there followed the line of [17] while we will adopt the techniques from [6].

Proposition 3. Suppose that s is strongly \mathcal{B}^I -summable to a with respect to \mathcal{F} and that

$$L(t) := \inf \{F_k^{(i)}(t) : k \in \mathbb{N}, i \in S\} > 0 \quad \forall t > 0. \tag{14}$$

Then s is also \mathcal{B}^I -statistically convergent to a .

Proof. Let $\varepsilon, \delta > 0$ be arbitrary. By assumption there is some $E \in I$ such that for all $i \in S$

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)} (|s_k - a|) \geq \delta L(\varepsilon)\right\} \subseteq E. \tag{15}$$

But we have

$$\sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)} (|s_k - a|) \geq \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)} (|s_k - a|) \chi_{D(s,a,\varepsilon)}(k) \tag{16}$$

$$\geq L(\varepsilon) \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\varepsilon)}(k)$$

for all $i \in S, k \in \mathbb{N}$. Hence

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\varepsilon)}(k) \geq \delta \right\} \subseteq E \quad (17)$$

for every $i \in S$ and the proof is finished. \square

Proposition 4. *Suppose that s is bounded and \mathcal{B}^I -statistically convergent to a . If \mathcal{F} is equicontinuous at 0 and there exists an $A \in I$ such that*

$$M := \sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty, \quad (18)$$

as well as

$$h(t) := \sup \{ F_k^{(i)}(t) : k \in \mathbb{N}, i \in S \} < \infty \quad \forall t \geq 0, \quad (19)$$

then s is also strongly \mathcal{B}^I -summable to a with respect to \mathcal{F} .

Proof. Let $\varepsilon > 0$ be arbitrary. Take $\tau > 0$ with $\tau(M + h(\|s\|_{\infty} + |a|)) < \varepsilon$. Since \mathcal{F} is equicontinuous at 0, we can find a $\delta > 0$ such that $F_k^{(i)}(t) \leq \tau$ for all $t \in [0, \delta]$ and all $k \in \mathbb{N}, i \in S$.

Because s is \mathcal{B}^I -statistically convergent to a there is some $E \in I$ such that for every $i \in S$

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\delta)}(k) \geq \tau \right\} \subseteq E. \quad (20)$$

It follows that for every $n \in \mathbb{N} \setminus (E \cup A)$ and all $i \in S$

$$\begin{aligned} & \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)}(|s_k - a|) \\ & \leq \tau \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{\mathbb{N} \setminus D(s,a,\delta)}(k) \\ & \quad + \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)}(|s_k - a|) \chi_{D(s,a,\delta)}(k) \\ & \leq \tau M + h(\|s\|_{\infty} + |a|) \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\delta)}(k) \\ & \leq \tau (M + h(\|s\|_{\infty} + |a|)) < \varepsilon \end{aligned} \quad (21)$$

and we are done. \square

So in particular, if \mathcal{B} and \mathcal{F} meet the requirements of both Propositions 3 and 4, then \mathcal{B}^I -statistical convergence and strong \mathcal{B}^I -summability with respect to \mathcal{F} coincide on bounded sequences. Note that all the assumptions on \mathcal{F} are satisfied if $F_k^{(i)} = F_{p_{ki}}$ for a family $(p_{ki})_{k \in \mathbb{N}, i \in S}$ of positive numbers which is bounded and bounded away from zero.

If $I \subseteq J_{\mathcal{B}, I}$, in other words, if

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) = 0 \quad \text{uniformly in } i \in S \quad \forall A \in I, \quad (22)$$

then I -convergence implies \mathcal{B}^I -statistical convergence (to the same limit). Thus if \mathcal{B} and \mathcal{F} additionally satisfy the requirements of Proposition 4, then for bounded sequences I -convergence also implies strong \mathcal{B}^I -summability to the same limit. Concerning the consistency of ordinary \mathcal{B}^I -summability, we have the following sufficient conditions which are analogous to those of Toeplitz's theorem. We write d_I for the set of all bounded sequences $(a_k)_{k \in \mathbb{N}}$ for which $\{k \in \mathbb{N} : a_k \neq 0\} \in I$.

Lemma 5. *Suppose that $\sum_{k=1}^{\infty} |b_{nk}^{(i)}| < \infty$ for all $n \in \mathbb{N}, i \in S$ and*

$$\exists A \in I \quad M := \sup \left\{ \sum_{k=1}^{\infty} |b_{nk}^{(i)}| : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty, \quad (23)$$

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} a_k = 0 \quad \text{uniformly in } i \in S \quad \forall (a_k) \in d_I, \quad (24)$$

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} = 1 \quad \text{uniformly in } i \in S. \quad (25)$$

Then for every bounded sequence $s = (s_n)_{n \in \mathbb{N}}$ in \mathbb{R} or \mathbb{C} , if $I\text{-}\lim s_n = a$, then s is also \mathcal{B}^I -summable to a .

Proof. Because of (25) we may assume $a = 0$. Let $\varepsilon > 0$ be arbitrary. Since $I\text{-}\lim s_n = 0$, we have $C := \{n \in \mathbb{N} : |s_n| \geq \varepsilon\} \in I$ and hence by (24) there is some $E \in I$ such that

$$\left\{ n \in \mathbb{N} : \left| \sum_{k=1}^{\infty} b_{nk}^{(i)} s_k \chi_C(k) \right| \geq \varepsilon \right\} \subseteq E \quad \forall i \in S. \quad (26)$$

But for all $i \in S$ and all $n \in \mathbb{N} \setminus A$

$$\begin{aligned} \left| \sum_{k=1}^{\infty} b_{nk}^{(i)} s_k \right| & \leq \left| \sum_{k=1}^{\infty} b_{nk}^{(i)} s_k \chi_C(k) \right| \\ & \quad + \sum_{k=1}^{\infty} |b_{nk}^{(i)}| \chi_{\mathbb{N} \setminus C}(k) |s_k| \\ & \leq \left| \sum_{k=1}^{\infty} b_{nk}^{(i)} s_k \chi_C(k) \right| + M\varepsilon, \end{aligned} \quad (27)$$

and thus

$$\left\{ n \in \mathbb{N} : \left| \sum_{k=1}^{\infty} b_{nk}^{(i)} s_k \right| \geq \varepsilon(1 + M) \right\} \subseteq E \cup A \quad \forall i \in S \quad (28)$$

and we are done. \square

The next proposition is the direct generalisation of [18, Theorem 3.3] to our setting. Its proof is easy and moreover virtually the same as in [18] so it will be omitted.

Proposition 6. Suppose that we are given two families of non-negative matrices $\mathcal{B} = ((b_{nk}^{(i)})_{n,k \in \mathbb{N}})_{i \in S}$ and $\mathcal{A} = ((a_{nk}^{(i)})_{n,k \in \mathbb{N}})_{i \in S}$. If

$$I\text{-}\lim \sum_{k=1}^{\infty} |a_{nk}^{(i)} - b_{nk}^{(i)}| = 0 \quad \text{uniformly in } i \in S, \quad (29)$$

then $J_{\mathcal{B},I} = J_{\mathcal{A},I}$.

In [19] it was proved that a bounded (real) sequence s is statistically convergent to a if and only if s is Cesàro-summable to a and the “variance” $\sigma_n(s)^2 := 1/n \sum_{i=1}^n (a - 1/n \sum_{k=1}^n s_k)^2$ converges to 0. The proposition below is a generalisation of this result. We will use the notation

$$\sigma_{ni}^{\mathcal{B},\mathcal{F}}(s) := \sum_{k=1}^{\infty} b_{nk}^{(i)} F_{ki} (|s_k - (B_i s)(n)|), \quad (30)$$

provided that each B_i is nonnegative.

First we need the following lemma, whose proof is analogous to those of Propositions 3 and 4 and will therefore be omitted.

Lemma 7. Suppose that \mathcal{F} and \mathcal{B} fulfill the requirements of Propositions 3 and 4 and let $y = (y_{ni})_{n \in \mathbb{N}, i \in S}$ be a family in \mathbb{R} or \mathbb{C} . Put $A_{\varepsilon,ni} := D(s, y_{ni}, \varepsilon)$ for all $i \in S, n \in \mathbb{N}$ and $\varepsilon > 0$. Then

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} F_{ki} (|s_k - y_{ni}|) = 0 \quad \text{uniformly in } i \in S \quad (31)$$

implies that for every $\varepsilon > 0$

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{A_{\varepsilon,ni}}(k) = 0 \quad \text{uniformly in } i \in S \quad (32)$$

and the converse is true if s is bounded and $\sup_{i \in \mathbb{N}, n \in \mathbb{N} \setminus V} |y_{ni}| < \infty$ for some $V \in I$.

Proposition 8. Let s be bounded. Under the same hypotheses as in the previous lemma and the additional assumption that $\sum_{k=1}^{\infty} |b_{nk}^{(i)}| < \infty$ for all $n \in \mathbb{N}, i \in S$ and

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} = 1 \quad \text{uniformly in } i \in S, \quad (33)$$

s is \mathcal{B}^I -statistically convergent to the number a if and only if s is \mathcal{B}^I -summable to a and $\sigma_{ni}^{\mathcal{B},\mathcal{F}}(s)$ converges to 0 along I uniformly in $i \in S$.

Proof. In view of Lemma 7 it is enough to consider the case $F_{ki} = \text{id}_{[0,\infty)}$ for all $k \in \mathbb{N}, i \in S$. We first assume that s is \mathcal{B}^I -summable to a and that

$$\begin{aligned} & I\text{-}\lim \sigma_{ni}^{\mathcal{B},\mathcal{F}}(s) \\ &= I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - (B_i s)(n)| = 0 \quad \text{uniformly in } i \in S. \end{aligned} \quad (34)$$

Because of

$$\begin{aligned} & \sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - a| \\ & \leq \sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - (B_i s)(n)| \\ & \quad + \sum_{k=1}^{\infty} b_{nk}^{(i)} |(B_i s)(n) - a| \\ & \leq \sigma_{ni}^{\mathcal{B},\mathcal{F}}(s) + |(B_i s)(n) - a| M \quad \forall n \in \mathbb{N} \setminus A, \forall i \in S, \end{aligned} \quad (35)$$

where A and M are as in Proposition 4, it follows that s is strongly \mathcal{B}^I -summable to a and hence by Proposition 3 it is also \mathcal{B}^I -statistically convergent to a .

Conversely, let s be \mathcal{B}^I -statistically convergent to a . Then by Proposition 4 s is also strongly \mathcal{B}^I -summable to a and because of assumption (33) it follows that s is \mathcal{B}^I -summable to a . Moreover, we have

$$\begin{aligned} \sigma_{ni}^{\mathcal{B},\mathcal{F}}(s) &= \sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - (B_i s)(n)| \\ & \leq \sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - a| + \sum_{k=1}^{\infty} b_{nk}^{(i)} |a - (B_i s)(n)| \\ & \leq \sum_{k=1}^{\infty} b_{nk}^{(i)} |s_k - a| + M |a - (B_i s)(n)| \end{aligned} \quad (36)$$

$$\forall n \in \mathbb{N} \setminus A, \quad \forall i \in S$$

and hence $\sigma_{ni}^{\mathcal{B},\mathcal{F}}(s)$ converges to 0 along I uniformly in $i \in S$. \square

According to [12, Theorem 2], for any regular matrix A the F_A -convergence of a sequence implies its almost convergence to the same limit and by [12, Theorem 3] the converse is true if A satisfies $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk} - a_{nk+1}| = 0$. The following two results are generalisations of these facts. Their proofs remain virtually the same and will not be given here.

Proposition 9. Let $A = (a_{nk})_{n,k \in \mathbb{N}}$ be an infinite matrix in \mathbb{R} such that $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for all $n \in \mathbb{N}$ and $I\text{-}\lim \sum_{k=1}^{\infty} a_{nk} = 1$. Put $\mathcal{A} = ((a_{nk}^{(i)})_{n,k \in \mathbb{N}})_{i \in \mathbb{N}_0}$, where $a_{nk}^{(i)} = a_{nk-i}$ for $k > i$ and $a_{nk}^{(i)} = 0$ for $k \leq i$.

Let $s \in \ell^{\infty}$ be \mathcal{A}^I -summable to the value a . Then s is also almost convergent to a .

Theorem 10. Let A and \mathcal{A} be as in the previous proposition but assume additionally that $I\text{-}\lim a_{nk} = 0$ for every $k \in \mathbb{N}$, $\sup_{n \in \mathbb{N} \setminus V} \sum_{k=1}^{\infty} |a_{nk}| < \infty$ for some $V \in I$, and

$$I\text{-}\lim \sum_{k=1}^{\infty} |a_{nk} - a_{nk+1}| = 0. \quad (37)$$

Let C be the Cesàro matrix and suppose that the family \mathcal{C} arises from C as \mathcal{A} from A . Suppose further that the ideal I is admissible and that J is another ideal. Let $s \in \ell^\infty$ be \mathcal{C}^J -summable to the value a . Then s is also \mathcal{A}^I -summable to a .

In [4] the notion of I -Cauchy sequences in arbitrary metric spaces, which generalises the notion of statistically Cauchy sequences of Fridy (cf. [20]), was introduced. A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is said to be an I -Cauchy sequence if for every $\varepsilon > 0$ there is some $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d(x_n, x_k) \geq \varepsilon\} \in I$. For $I = I_f$ this yields just an equivalent formulation of the notion of an ordinary Cauchy sequence. Fridy's notion of statistically Cauchy sequences is obtained by taking $I = J_{C, I_f}$, where C is the Cesàro matrix. It was proved in [4] that every I -convergent sequence is I -Cauchy (cf. [4, Proposition 1]) and that, in the case of an admissible ideal I , the metric space (X, d) is complete if and only if every I -Cauchy sequence in (X, d) is I -convergent (cf. [4, Theorem 2]). The proof of [4, Theorem 2] also shows that every I -convergent sequence possesses a subsequence which is convergent in the ordinary sense.

In [20] it was proved that a sequence of numbers is statistically convergent if and only if it is statistically Cauchy, but a third equivalent condition was obtained there as well; namely, a number sequence $(s_n)_{n \in \mathbb{N}}$ is statistically convergent if and only if there is a sequence $(t_n)_{n \in \mathbb{N}}$ which is convergent in the usual sense and coincides "almost everywhere" with $(s_n)_{n \in \mathbb{N}}$, which in our notation means precisely $\{n \in \mathbb{N} : s_n \neq t_n\} \in J_{C, I_f}$.

It is clear that, for any two sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ in an arbitrary topological space, if $(y_n)_{n \in \mathbb{N}}$ is I -convergent and $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then $(x_n)_{n \in \mathbb{N}}$ is also I -convergent. For the case of \mathcal{B}^I -statistical convergence of sequences of numbers we can prove a converse result provided that $\mathcal{F}(I)$ has a countable base that fulfills a certain condition with respect to the matrix-family \mathcal{B} . The proof uses the basic ideas from [20].

Theorem 11. *Let I be an admissible ideal with $I \subseteq J_{\mathcal{B}, I}$ such that there is an increasing sequence $(B_m)_{m \in \mathbb{N}}$ in I for which $\{\mathbb{N} \setminus B_m : m \in \mathbb{N}\}$ forms a base of $\mathcal{F}(I)$ and*

$$\sup_{i \in S} \sup_{n \in B_m} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{\mathbb{N} \setminus B_m}(k) \xrightarrow{m \rightarrow \infty} 0. \quad (38)$$

Then the sequence $s = (s_n)_{n \in \mathbb{N}}$ is \mathcal{B}^I -statistically convergent to a if and only if there is a sequence $(t_n)_{n \in \mathbb{N}}$ which is I -convergent to a and fulfills $\{n \in \mathbb{N} : s_n \neq t_n\} \in J_{\mathcal{B}, I}$.

Proof. We only have to show the necessity. So let s be \mathcal{B}^I -statistically convergent to a . Put $\varepsilon_m = 2^{-m}$ and $A_m = \{k \in \mathbb{N} : |s_k - a| \geq \varepsilon_m\}$ for every $m \in \mathbb{N}$. Then for every $m \in \mathbb{N}$ there exists a set $E_m \in I$ such that

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{A_m}(k) \geq \varepsilon_m \right\} \subseteq E_m \quad \forall i \in S \quad (39)$$

and by (38) we can find a strictly increasing sequence $(M_p)_{p \in \mathbb{N}}$ in \mathbb{N} such that

$$\sup_{i \in S} \sup_{n \in B_{M_p}} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{\mathbb{N} \setminus B_{M_p}}(k) \leq \varepsilon_p \quad \forall p \in \mathbb{N}. \quad (40)$$

Next we fix a strictly increasing sequence $(p_m)_{m \in \mathbb{N}}$ in \mathbb{N} such that $E_m \subseteq B_{M_{p_m}}$ for every $m \in \mathbb{N}$. We write F_m for $B_{M_{p_m}}$. Then $F_m \subseteq F_{m+1}$ and $\bigcup_{m=1}^{\infty} F_m = \mathbb{N}$.

Let $m(k) = \min\{m \in \mathbb{N} : k \in F_m\}$ for every $k \in \mathbb{N}$ and put

$$t_k = \begin{cases} s_k & \text{if } k \notin A_{m(k)} \\ a & \text{if } k \in A_{m(k)}. \end{cases} \quad (41)$$

It is easily checked that $\{k \in \mathbb{N} : |t_k - a| \geq \varepsilon_m\} \subseteq F_m$ for every m and hence $(t_k)_{k \in \mathbb{N}}$ is I -convergent to a .

Now it remains to show $C := \{k \in \mathbb{N} : s_k \neq t_k\} \in J_{\mathcal{B}, I}$. To this end, fix $\varepsilon > 0$ and choose m such that $\sum_{l=m+1}^{\infty} \varepsilon_l \leq \varepsilon/3$ and $\varepsilon_{p_m} \leq \varepsilon/3$.

Since $I \subseteq J_{\mathcal{B}, I}$, we can find $E \in I$ with

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{E_m}(k) \geq \frac{\varepsilon}{3} \right\} \subseteq E \quad \forall i \in S. \quad (42)$$

Then $F_m \cup E \in I$ and for every $n \in \mathbb{N} \setminus (F_m \cup E)$ and each $i \in S$ we have $m(n) > m$ and

$$\begin{aligned} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_C(k) &= \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{C \cap F_m}(k) \\ &\quad + \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{C \cap (\mathbb{N} \setminus F_m)}(k) \\ &\stackrel{(42)}{<} \frac{\varepsilon}{3} + \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{C \cap (\mathbb{N} \setminus F_{m(n)})}(k) \\ &\quad + \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{C \cap (F_{m(n)} \setminus F_m)}(k) \\ &\stackrel{(40)}{\leq} \frac{\varepsilon}{3} + \varepsilon_{p_{m(n)}} \\ &\quad + \sum_{l=m+1}^{m(n)} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{C \cap (F_l \setminus F_{l-1})}(k) \\ &\leq \frac{\varepsilon}{3} + \varepsilon_{p_m} + \sum_{l=m+1}^{m(n)} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{A_l}(k) \\ &\stackrel{(39)}{\leq} \frac{2}{3} \varepsilon + \sum_{l=m+1}^{m(n)} \varepsilon_l \leq \varepsilon, \end{aligned} \quad (43)$$

which completes the proof. \square

Note that condition (38) is in particular satisfied for $B_m = \{1, \dots, m\}$ if $I = I_f$ and each B_i is a lower triangular matrix.

Making use of his aforementioned characterisation of statistical convergence, Fridy further proved in [20] the

following Tauberian theorem for statistical convergence: a statistically convergent sequence $(s_n)_{n \in \mathbb{N}}$ which satisfies $|s_n - s_{n+1}| = O(1/n)$ for $n \rightarrow \infty$ is convergent in the ordinary sense. It is not too difficult to obtain the following slightly more general result by modifying the proof from [20] accordingly (there the functions φ, ψ , and h below are simply $\varphi(x) = 1/x = \psi(x)$ and $h(x) = x(1+x)^{-1}$). For the sake of brevity, we skip the details.

Theorem 12. *Let I be an admissible ideal and $A = (a_{nk})_{n,k \geq 1}$ a lower triangular matrix such that $I\text{-lim} \sum_{k=1}^n a_{nk} = 1$ and $I\text{-lim} a_{nk} = 0$ for every $k \in \mathbb{N}$. Suppose that φ, ψ , and h are functions from $[0, \infty)$ into itself such that φ is decreasing on $(0, \infty)$, $\min_{k=1, \dots, n} a_{nk} \geq \psi(n)$ for every $n \in \mathbb{N}$, $I\text{-lim} x_n = 0$ whenever $I\text{-lim} h(x_n) = 0$, and*

$$x\psi(x+y) \geq h(x\varphi(y)) \quad \forall x, y \geq 0. \tag{44}$$

Let $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ be number sequences such that $\lim_{n \rightarrow \infty} t_n = 0$, $\{n \in \mathbb{N} : s_n \neq t_n\} \in J_{A,I}$, and $|s_n - s_{n+1}| = O(\varphi(n))$ for $n \rightarrow \infty$. Then $I\text{-lim} s_n = 0$.

Combining Theorems 11 and 12 we get the following corollary.

Corollary 13. *Under the same general hypothesis as in Theorem 12 with $I = I_f$, if $(s_n)_{n \in \mathbb{N}}$ is a sequence which is A -statistically convergent to the number a and fulfills $|s_n - s_{n+1}| = O(\varphi(n))$ for $n \rightarrow \infty$, then $(s_n)_{n \in \mathbb{N}}$ is convergent to a in the usual sense.*

3. Limit Superior and Limit Inferior

In [21] Demirci introduced the concepts of limit superior and limit inferior with respect to an ideal I on \mathbb{N} , generalising the notions of statistical limit superior and limit inferior from [22]. For a sequence $(s_n)_{n \in \mathbb{N}}$ in \mathbb{R} put

$$I\text{-lim sup } s_n := \sup \{t \in \mathbb{R} : \{n \in \mathbb{N} : s_n > t\} \notin I\}, \tag{45}$$

$$I\text{-lim inf } s_n := \inf \{t \in \mathbb{R} : \{n \in \mathbb{N} : s_n < t\} \notin I\}.$$

The same definitions were independently introduced by the authors of [3]. Note that since $(s_n)_{n \in \mathbb{N}}$ is not assumed to be bounded, it can happen that these values are ∞ or $-\infty$. If $I = I_f$ the above definitions are equivalent to the usual definitions of limit superior and limit inferior. It is proved in [21] (and in [3] as well) that $I\text{-lim inf } s_n \leq I\text{-lim sup } s_n$ and that $(s_n)_{n \in \mathbb{N}}$ is I -convergent to $a \in \mathbb{R}$ if and only if $I\text{-lim inf } s_n = a = I\text{-lim sup } s_n$ (cf. [21, Theorems 3 and 4] or [3, Theorems 3.2 and 3.4]).

Let us also remark that

$$\begin{aligned} I\text{-lim sup } s_n &= \inf_{A \in I} \sup \{s_n : n \in \mathbb{N} \setminus A\}, \\ I\text{-lim inf } s_n &= \sup_{A \in I} \inf \{s_n : n \in \mathbb{N} \setminus A\}, \end{aligned} \tag{46}$$

as is easily checked.

In [22, Lemma on p.3628] necessary and sufficient conditions for a real matrix A to satisfy the inequality $\limsup Ax \leq$

$\text{st-lim sup } x$ for all $x \in \ell^\infty$ were obtained (here, $\text{st-lim sup } x$ denotes the aforementioned statistical limit superior that was introduced in [22]; in our terminology it is nothing but the limit superior with respect to the ideal J_{C,I_f} , where C is the Cesàro matrix).

Later, Demirci gave a more general necessity result concerning the I -limit superior and the I -limit inferior (cf. [21, Corollary 1]). The following proposition is a further generalisation of this result while its proof follows the lines from [22].

Proposition 14. *Let I, J be ideals on \mathbb{N} and $A = (a_{nk})_{n,k \in \mathbb{N}}$ an infinite matrix in \mathbb{R} such that the following conditions are satisfied:*

$$\sum_{k=1}^{\infty} |a_{nk}| < \infty \quad \forall n \in \mathbb{N}, \tag{47}$$

$$I\text{-lim} \sum_{k=1}^{\infty} |a_{nk}| = 1 = I\text{-lim} \sum_{k=1}^{\infty} a_{nk}, \tag{48}$$

$$I\text{-lim} \sum_{k=1}^{\infty} |a_{nk}| \chi_E(k) = 0 \quad \forall E \in J. \tag{49}$$

Then

$$I\text{-lim sup } As \leq J\text{-lim sup } s \quad \forall s \in \ell^\infty \tag{50}$$

as well as

$$I\text{-lim inf } As \geq J\text{-lim inf } s \quad \forall s \in \ell^\infty. \tag{51}$$

Proof. Let $s = (s_n)_{n \in \mathbb{N}} \in \ell^\infty$ be arbitrary and put $b = J\text{-lim sup } s$. Since s is bounded, we have $b \in \mathbb{R}$. Also, fix an arbitrary $\varepsilon > 0$. Then by [21, Theorem 1] (or [3, Theorem 3.1]) we have $E := \{n \in \mathbb{N} : s_n > b + \varepsilon\} \in J$. We put $F = \mathbb{N} \setminus E$.

For every $a \in \mathbb{R}$ set $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$, as in [22]. Note that $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

Then for every $n \in \mathbb{N}$

$$\begin{aligned} (As)(n) &= \sum_{k=1}^{\infty} a_{nk} s_k = \sum_{k=1}^{\infty} a_{nk}^+ \chi_E(k) s_k \\ &\quad + \sum_{k=1}^{\infty} a_{nk}^+ \chi_F(k) s_k - \sum_{k=1}^{\infty} a_{nk}^- s_k \\ &\leq \|s\|_\infty \sum_{k=1}^{\infty} |a_{nk}| \chi_E(k) + (b + \varepsilon) \sum_{k=1}^{\infty} a_{nk}^+ \chi_F(k) \\ &\quad + \frac{1}{2} \|s\|_\infty \sum_{k=1}^{\infty} (|a_{nk}| - a_{nk}) \\ &= \|s\|_\infty \sum_{k=1}^{\infty} |a_{nk}| \chi_E(k) + \frac{1}{2} \|s\|_\infty \sum_{k=1}^{\infty} (|a_{nk}| - a_{nk}) \\ &\quad + \frac{b + \varepsilon}{2} \left(\sum_{k=1}^{\infty} (|a_{nk}| + a_{nk}) (1 - \chi_E(k)) \right). \end{aligned} \tag{52}$$

Because of $E \in J$ and the assumptions (48) and (49) the I -limit of the right-hand side of the above inequality is equal to $b + \varepsilon$. Together with the obvious monotonicity of I -lim sup it follows that I -lim sup $As \leq b + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the proof is finished.

The second statement follows from the first one by multiplication with -1 . \square

It was also proved in [22] that a sequence of real numbers which is bounded above and Cesàro-summable to its statistical limit superior is statistically convergent (cf. [22, Theorem 5]). It is possible to modify the proof of [22] to obtain the following more general result. We use the same notation as in the previous section.

Theorem 15. *Suppose that each B_i is nonnegative, $\sum_{k=1}^\infty b_{nk}^{(i)} < \infty$ for all $n \in \mathbb{N}, i \in S$, and*

$$I\text{-lim} \sum_{k=1}^\infty b_{nk}^{(i)} = 1 \quad \text{uniformly in } i \in S. \quad (53)$$

If $s = (s_n)_{n \in \mathbb{N}}$ is a bounded sequence of real numbers and $a \in \mathbb{R}$ such that s is \mathcal{B}^I -summable to a and $J_{\mathcal{B}, I}\text{-lim sup } s = a$ or $J_{\mathcal{B}, I}\text{-lim inf } s = a$, then s is \mathcal{B}^I -statistically convergent to a .

Proof. It is enough to prove the statement for the case $J_{\mathcal{B}, I}\text{-lim sup } s = a$. Suppose that s is not \mathcal{B}^I -statistically convergent to a . Then $J_{\mathcal{B}, I}\text{-lim inf } s < a$ and hence there must be some $t < a$ such that $E := \{n \in \mathbb{N} : s_n < t\} \notin J_{\mathcal{B}, I}$. Consequently, there exists a $d > 0$ such that

$$A := \left\{ n \in \mathbb{N} : \sup_{i \in S} \sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) \geq d \right\} \notin I. \quad (54)$$

Fix an arbitrary $\varepsilon > 0$ and put $F := \{n \in \mathbb{N} : t \leq s_n \leq a + \varepsilon\}$ and $G := \{n \in \mathbb{N} : s_n > a + \varepsilon\}$. Take $\delta \in (0, \varepsilon)$ with $\delta|a + \varepsilon| \leq \varepsilon$. By our assumption (53) we have

$$C := \left\{ n \in \mathbb{N} : \sup_{i \in S} \left| \sum_{k=1}^\infty b_{nk}^{(i)} - 1 \right| \geq \delta \right\} \in I. \quad (55)$$

It follows from [21, Theorem 1] that $G \in J_{\mathcal{B}, I}$ and hence

$$D := \left\{ n \in \mathbb{N} : \sup_{i \in S} \sum_{k=1}^\infty b_{nk}^{(i)} \chi_G(k) \geq \delta \right\} \in I. \quad (56)$$

Now let $n \in H := A \cap (\mathbb{N} \setminus (C \cup D))$ be arbitrary. Since $n \in A$, there is some $i \in S$ such that $\sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) > d/2$. Write

$M = \|s\|_\infty$. It then follows from the definitions of the sets E, F, G, C , and D and the choice of δ that

$$\begin{aligned} \sum_{k=1}^\infty b_{nk}^{(i)} s_k &= \sum_{k=1}^\infty b_{nk}^{(i)} s_k \chi_E(k) \\ &\quad + \sum_{k=1}^\infty b_{nk}^{(i)} s_k \chi_F(k) + \sum_{k=1}^\infty b_{nk}^{(i)} s_k \chi_G(k) \\ &\leq t \sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) \\ &\quad + (a + \varepsilon) \sum_{k=1}^\infty b_{nk}^{(i)} \chi_F(k) + M\delta \\ &= t \sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) + M\delta \\ &\quad + (a + \varepsilon) \sum_{k=1}^\infty b_{nk}^{(i)} (1 - \chi_E(k) - \chi_G(k)) \\ &\leq t \sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) + M\delta \\ &\quad + (a + \varepsilon) \left(1 - \sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) \right) \\ &\quad + |a + \varepsilon| \left(\sum_{k=1}^\infty b_{nk}^{(i)} \chi_G(k) + \left| \sum_{k=1}^\infty b_{nk}^{(i)} - 1 \right| \right) \\ &\leq a + \varepsilon + M\varepsilon + (t - a - \varepsilon) \\ &\quad \times \sum_{k=1}^\infty b_{nk}^{(i)} \chi_E(k) + 2|a + \varepsilon|\delta \\ &< a + \varepsilon(M + 3) + (t - a - \varepsilon) \frac{d}{2}. \end{aligned} \quad (57)$$

Thus we have

$$\begin{aligned} \sup_{i \in S} \left| a - \sum_{k=1}^\infty b_{nk}^{(i)} s_k \right| & \\ &> \frac{d}{2} (a + \varepsilon - t) - \varepsilon(M + 3) \quad \forall n \in H. \end{aligned} \quad (58)$$

Suppose that

$$\begin{aligned} h := I\text{-lim sup} \sup_{i \in S} \left| a - \sum_{k=1}^\infty b_{nk}^{(i)} s_k \right| & \\ &< \frac{d}{2} (a + \varepsilon - t) - \varepsilon(M + 3). \end{aligned} \quad (59)$$

Then it would follow that $H \in I$. But $C, D \in I$ and hence

$$A = H \cup (C \cap A) \cup (D \cap A) \in I, \quad (60)$$

contradicting (54).

Thus $h \geq (d/2)(a + \varepsilon - t) - \varepsilon(M + 3)$ and since $\varepsilon > 0$ was arbitrary, we get $h \geq (a - t)d/2 > 0$ and hence s is not B^I -summable to a . \square

We conclude this section with a lemma that will be needed later and may also be of independent interest. First we need one more definition: a number sequence $(s_n)_{n \in \mathbb{N}}$ is called I -bounded if there is a constant $K > 0$ such that $\{n \in \mathbb{N} : |s_n| > K\} \in I$. Note that I -convergent sequences are I -bounded and that the I -boundedness of $(s_n)_{n \in \mathbb{N}}$ implies that $I\text{-lim sup } s_n$ and $I\text{-lim inf } s_n$ are finite.

Lemma 16. For any ideal I on \mathbb{N} and all I -bounded sequences $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R} the inequalities

$$\begin{aligned} I\text{-lim sup } (s_n + t_n) &\leq I\text{-lim sup } s_n + I\text{-lim sup } t_n, \\ I\text{-lim sup } (s_n + t_n) &\geq I\text{-lim sup } s_n + I\text{-lim sup } t_n \end{aligned} \tag{61}$$

hold. If one of the sequences is I -convergent, then equality holds.

Proof. It is enough to prove the statement for the I -lim sup. Let $a = I\text{-lim sup } s_n$ and $b = I\text{-lim sup } t_n$. If $u, v \in \mathbb{R}$ such that $u > a$ and $v > b$ then $A := \{n \in \mathbb{N} : s_n > u\} \in I$ and $B := \{n \in \mathbb{N} : t_n > v\} \in I$. Hence $A \cup B \in I$. But

$$C := \{n \in \mathbb{N} : s_n + t_n > u + v\} \subseteq A \cup B, \tag{62}$$

and thus $C \in I$.

If $I\text{-lim sup}(s_n + t_n) > u + v$, then there would be some $\eta > u + v$ such that $\{n \in \mathbb{N} : s_n + t_n > \eta\} \notin I$, which would imply $C \notin I$. Thus we must have $I\text{-lim sup}(s_n + t_n) \leq u + v$. Since $u > a$ and $v > b$ were arbitrary, it follows that $I\text{-lim sup}(s_n + t_n) \leq a + b$.

Now suppose that $(s_n)_{n \in \mathbb{N}}$ is I -convergent to a and fix an arbitrary $\varepsilon > 0$. Put $D := \{n \in \mathbb{N} : s_n + t_n > a + b - \varepsilon\}$, $E := \{n \in \mathbb{N} : s_n > a - \varepsilon/2\}$, and $F := \{n \in \mathbb{N} : t_n > b - \varepsilon/2\}$.

By [21, Theorem 1] $F \notin I$ and because of $I\text{-lim } s_n = a$ we have $\mathbb{N} \setminus E \in I$, that is, $E \in \mathcal{F}(I)$.

If $E \cap F \in I$, then $(\mathbb{N} \setminus E) \cup (\mathbb{N} \setminus F) \in \mathcal{F}(I)$ and hence $(\mathbb{N} \setminus F) \cap E = ((\mathbb{N} \setminus E) \cup (\mathbb{N} \setminus F)) \cap E \in \mathcal{F}(I)$; thus $\mathbb{N} \setminus F \in \mathcal{F}(I)$, contradicting the fact that $F \notin I$.

So we must have $E \cap F \notin I$ and since $E \cap F \subseteq D$, it follows that $D \notin I$, which implies $I\text{-lim sup}(s_n + t_n) \geq a + b - \varepsilon$. Letting $\varepsilon \rightarrow 0$ completes the proof. \square

4. Cluster Points

Fridy [23] defined and studied statistical cluster points and statistical limit points of a sequence. These concepts were later generalised by the authors of [1] to an arbitrary admissible ideal I . Consider a sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) . An element $x \in X$ is called an I -cluster point of $(x_n)_{n \in \mathbb{N}}$ if $\{n \in \mathbb{N} : d(x_n, x) < \varepsilon\} \notin I$ for every $\varepsilon > 0$ and it is called an I -limit point of $(x_n)_{n \in \mathbb{N}}$ if there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $\{n_k : k \in \mathbb{N}\} \notin I$ that converges to x . For $I = I_f$, both notions are equivalent to the usual notion of cluster points. Every I -limit point is also an I -cluster point of $(x_n)_{n \in \mathbb{N}}$ (cf. [1, Proposition 4.1]) but the converse is not true in general. It was shown in [3, Theorem 3.5] that a bounded sequence

$(s_n)_{n \in \mathbb{N}}$ in \mathbb{R} always possesses an I -cluster point and that the I -lim sup and the I -lim inf of the sequence are the greatest and the smallest of them, respectively. It is easily observed that the same proof still works if the sequence is only I -bounded.

Concerning $J_{\mathcal{B}, I}$ -cluster points, we can give the following characterisation.

Proposition 17. Suppose that $\sup_{n \in \mathbb{N}, i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} < \infty$ and

$$I\text{-lim } \sum_{k=1}^{\infty} b_{nk}^{(i)} = 1 \quad \text{uniformly in } i \in S. \tag{63}$$

Then a is a $J_{\mathcal{B}, I}$ -cluster point of $s = (s_n)_{n \in \mathbb{N}}$ if and only if for every $\varepsilon > 0$

$$I\text{-lim inf } \inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s, a, \varepsilon)}(k) < 1. \tag{64}$$

Proof. Put $A_\varepsilon = D(s, a, \varepsilon)$ and $B_\varepsilon = \mathbb{N} \setminus A_\varepsilon$ for every $\varepsilon > 0$. By definition, a is a $J_{\mathcal{B}, I}$ -cluster point of s if and only if $B_\varepsilon \notin J_{\mathcal{B}, I}$ for every $\varepsilon > 0$ which is the case if and only if

$$I\text{-lim sup } \sup_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{B_\varepsilon}(k) > 0. \tag{65}$$

But $\sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{B_\varepsilon}(k) = \sum_{k=1}^{\infty} b_{nk}^{(i)} - \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{A_\varepsilon}(k)$, so because of (63) and Lemma 16 it follows that a is a $J_{\mathcal{B}, I}$ -cluster point of s if and only if

$$\begin{aligned} I\text{-lim sup } \sup_{i \in S} \left(1 - \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{A_\varepsilon}(k) \right) &> 0 \\ \iff 1 - I\text{-lim inf } \inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{A_\varepsilon}(k) &> 0 \end{aligned} \tag{66}$$

and the proof is finished. \square

This characterisation yields the following sufficient condition for a $J_{\mathcal{B}, I}$ -cluster point.

Corollary 18. Under the same assumptions as in the previous proposition, if $\mathcal{F} = (F_k^{(i)})_{k \in \mathbb{N}, i \in S}$ is a family in $\mathcal{M} \cup \mathcal{O}$ such that

$$\begin{aligned} L(t) := \inf \{F_k^{(i)}(t) : k \in \mathbb{N}, i \in S\} &> 0 \quad \forall t > 0, \\ I\text{-lim inf } \inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)}(|s_k - a|) &= 0, \end{aligned} \tag{67}$$

then a is a $J_{\mathcal{B}, I}$ -cluster point of s .

Proof. For every $\varepsilon > 0$ and all $i \in S, n \in \mathbb{N}$ we have

$$\begin{aligned} &\sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)}(|s_k - a|) \\ &\geq \sum_{k=1}^{\infty} b_{nk}^{(i)} F_k^{(i)}(|s_k - a|) \chi_{D(s, a, \varepsilon)}(k) \\ &\geq L(\varepsilon) \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s, a, \varepsilon)}(k) \end{aligned} \tag{68}$$

and thus it follows from the assumptions that

$$I\text{-}\lim \inf \inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\varepsilon)}(k) = 0 < 1 \quad \forall \varepsilon > 0. \quad (69)$$

Hence by the previous proposition, a is a $J_{\mathcal{B},I}$ -cluster point of s . \square

5. Pre-Cauchy Sequences

The authors of [24] introduced the notion of statistically pre-Cauchy sequences. The sequence $s = (s_k)_{k \in \mathbb{N}}$ is called a statistically pre-Cauchy sequence if $\lim_{n \rightarrow \infty} 1/n^2 |\{(i, j) \in \{1, \dots, n\}^2 : |s_i - s_j| \geq \varepsilon\}| = 0$ for every $\varepsilon > 0$. They show that a statistically convergent sequence is statistically pre-Cauchy and that the converse is not true in general but under certain additional assumptions. It is further proved that s is statistically pre-Cauchy if

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n |s_i - s_j| = 0 \quad (70)$$

and that the converse is true if s is bounded (cf. [24, Theorem 3]).

We propose the following generalisation of the definition of statistically pre-Cauchy sequences to our setting.

Definition 19. If each B_i is nonnegative, a sequence $s = (s_k)_{k \in \mathbb{N}}$ of real or complex numbers is called a \mathcal{B}^I -statistically pre-Cauchy sequence if for every $\varepsilon > 0$

$$I\text{-}\lim \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} \chi_{D(s,\varepsilon)}(k, l) = 0 \quad \text{uniformly in } i \in S, \quad (71)$$

where $D(s, \varepsilon) := \{(k, l) \in \mathbb{N}^2 : |s_k - s_l| \geq \varepsilon\}$.

First we show that, under an additional assumption on \mathcal{B} , \mathcal{B}^I -statistically convergent sequences are \mathcal{B}^I -statistically pre-Cauchy.

Lemma 20. *Suppose that s is \mathcal{B}^I -statistically convergent and*

$$\exists A \in IM := \sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty. \quad (72)$$

Then s is a \mathcal{B}^I -statistically pre-Cauchy sequence.

Proof. Say s is \mathcal{B}^I -statistically convergent to a . For every $\varepsilon > 0$ and all $n \in \mathbb{N} \setminus A$ we have

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} \chi_{D(s,\varepsilon)}(k, l) \\ & \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} (\chi_{D(s,a,\varepsilon/2)}(k) + \chi_{D(s,a,\varepsilon/2)}(l)) \\ & \leq 2M \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_{D(s,a,\varepsilon/2)}(k) \longrightarrow 0 \quad \text{along } I \text{ uniformly in } i \in S. \end{aligned} \quad (73)$$

\square

The next two propositions are the analogues of [24, Theorem 3]. Since their proofs parallel very much those of Proposition 3 and Proposition 4, respectively, they will be omitted. In the formulation of both propositions, we differ from our usual notation and allow $\mathcal{F} = (F_{kl}^{(i)})_{k,l \in \mathbb{N}, i \in S}$ to be a family in $\mathcal{M} \cup \mathcal{O}$ with index set $\mathbb{N} \times \mathbb{N} \times S$ instead of $\mathbb{N} \times S$.

Proposition 21. *Suppose that*

$$\begin{aligned} I\text{-}\lim \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} F_{kl}^{(i)} (|s_k - s_l|) &= 0 \quad \text{uniformly in } i \in S, \\ L(t) := \inf \{ F_{kl}^{(i)}(t) : k, l \in \mathbb{N}, i \in S \} &> 0 \quad \forall t > 0. \end{aligned} \quad (74)$$

Then s is \mathcal{B}^I -statistically pre-Cauchy.

Proposition 22. *Suppose that s is bounded and \mathcal{B}^I -statistically pre-Cauchy. If \mathcal{F} is equicontinuous at 0 and*

$$\exists A \in IM := \sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty, \quad (75)$$

as well as

$$h(t) := \sup \{ F_{kl}^{(i)}(t) : k, l \in \mathbb{N}, i \in S \} < \infty \quad \forall t \geq 0, \quad (76)$$

then we also have

$$I\text{-}\lim \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} F_{kl}^{(i)} (|s_k - s_l|) = 0 \quad \text{uniformly in } i \in S. \quad (77)$$

It was proved in [24] that a statistically pre-Cauchy sequence $(s_n)_{n \in \mathbb{N}}$ which possesses a convergent subsequence $(s_{n_k})_{k \in \mathbb{N}}$ such that the set of indices $\{n_k : k \in \mathbb{N}\}$ is “large” in the sense that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} |\{n_k : k \in \mathbb{N}, n_k \leq n\}| > 0 \quad (78)$$

is statistically convergent. This result can be generalised in the following way.

Theorem 23. *Suppose that $I \subseteq J_{\mathcal{B},I}$ and*

$$\sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N}, i \in S \right\} < \infty. \quad (79)$$

Let a be any real or complex number. Let $s = (s_n)_{n \in \mathbb{N}}$ be a \mathcal{B}^I -statistically pre-Cauchy sequence and let $W \subseteq \mathbb{N}$ be such that for every $\varepsilon > 0$ the set $\{n \in W : |s_n - a| \geq \varepsilon\}$ belongs to I and furthermore

$$w := I\text{-}\lim \inf \inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_W(k) > 0. \quad (80)$$

Then s is \mathcal{B}^I -statistically convergent to a .

Proof. Take $\varepsilon, \delta > 0$ arbitrary. Then $V := \{k \in W : |s_k - a| \geq \varepsilon/2\} \in I$, by assumption. Put $A := \{k \in W : |s_k - a| < \varepsilon/2\}$, $B := \{k \in \mathbb{N} : |s_k - a| \geq \varepsilon\}$, and $C := \{(k, l) \in \mathbb{N}^2 : |s_k - s_l| \geq \varepsilon/2\}$. Then $A \times B \subseteq C$.

Let us also fix $\tau \in (0, w)$ such that $\tau(w - \tau)^{-1} \leq \delta$. Since s is \mathcal{B}^I -statistically pre-Cauchy, there is some $E \in I$ such that

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} \chi_C(k, l) \geq \tau \right\} \subseteq E \quad \forall i \in S. \quad (81)$$

But we have

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(i)} \chi_C(k, l) \\ & \geq \left(\sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) \right) \left(\sum_{l=1}^{\infty} b_{nl}^{(i)} \chi_B(l) \right) \end{aligned} \quad (82)$$

and thus

$$\left\{ n \in \mathbb{N} : \left(\sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) \right) \left(\sum_{l=1}^{\infty} b_{nl}^{(i)} \chi_B(l) \right) \geq \tau \right\} \subseteq E \quad \forall i \in S. \quad (83)$$

Since $V \in I \subseteq J_{\mathcal{B}, I}$, it follows that

$$I\text{-lim sup}_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_V(k) = 0. \quad (84)$$

Because of Lemma 16 this implies

$$\begin{aligned} w &= I\text{-lim inf}_{i \in S} \inf_{k=1}^{\infty} \sum_{k=1}^{\infty} b_{nk}^{(i)} (\chi_A(k) + \chi_V(k)) \\ &\leq I\text{-lim inf} \left(\inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) + \sup_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_V(k) \right) \\ &= I\text{-lim inf}_{i \in S} \inf_{k=1}^{\infty} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) =: r. \end{aligned} \quad (85)$$

By [21, Theorem 2] we have

$$F := \left\{ n \in \mathbb{N} : \inf_{i \in S} \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) < r - \tau \right\} \in I. \quad (86)$$

If $n \in \mathbb{N} \setminus (E \cup F)$, then $\sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_B(k) < \tau(r - \tau)^{-1} \leq \tau(w - \tau)^{-1} \leq \delta$ for every $i \in S$.

Thus $E \cup F \in I$ with

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_B(k) \geq \delta \right\} \subseteq E \cup F \quad \forall i \in S \quad (87)$$

and the proof is finished. \square

By [24, Theorem 5] a bounded statistically pre-Cauchy sequence in \mathbb{R} whose set of cluster points is nowhere dense is statistically convergent. To obtain an analogous result in our setting, we introduce the following strengthening of the notion of \mathcal{B}^I -statistically pre-Cauchy sequences.

Definition 24. If each B_i is nonnegative, a sequence $s = (s_k)_{k \in \mathbb{N}}$ of real or complex numbers is called a \mathcal{B}^I_+ -statistically pre-Cauchy sequence if for every $\varepsilon > 0$

$$I\text{-lim} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{nk}^{(i)} b_{nl}^{(j)} \chi_{D(s, \varepsilon)}(k, l) = 0 \quad \text{uniformly in } i, j \in S. \quad (88)$$

For \mathcal{B}^I_+ -statistically pre-Cauchy sequences, Lemma 20, Proposition 21, and Proposition 22 hold accordingly (with the obvious modifications, one can even take a family $\mathcal{F} = (F_{kl}^{(i,j)})_{k,l \in \mathbb{N}, i,j \in S}$ in $\mathcal{M} \cup \mathcal{O}$ with index set $\mathbb{N}^2 \times S^2$ in this case).

The next lemma generalises [24, Lemma 4] while its proof follows the same lines.

Lemma 25. *Let I be an admissible ideal. Suppose that $\sum_{k=1}^{\infty} b_{nk}^{(i)} < \infty$ for all $n \in \mathbb{N}, i \in S$ and*

$$\exists A \in I \quad M := \sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty, \quad (89)$$

$$I\text{-lim} \sum_{k=1}^{\infty} b_{nk}^{(i)} = 1 \quad \text{uniformly in } i \in S. \quad (90)$$

Let \mathcal{W} be a basis for $\mathcal{F}(I)$ such that for every $\{n_1 < n_2 < \dots < n_k < n_{k+1} < \dots\} \in \mathcal{W}$ the following holds:

$$\exists k_0 \in \mathbb{N}, \forall k \geq k_0 \inf_{i \in S} \sum_{l=1}^{\infty} |b_{n_k l}^{(i)} - b_{n_{k+1} l}^{(i)}| < \frac{1}{3}. \quad (91)$$

Let $s = (s_n)_{n \in \mathbb{N}}$ be a \mathcal{B}^I_+ -statistically pre-Cauchy sequence in \mathbb{R} and $\alpha < \beta$ such that $H := \{n \in \mathbb{N} : s_n \in (\alpha, \beta)\} \in J_{\mathcal{B}, I}$.

Then $X := \{n \in \mathbb{N} : s_n \leq \alpha\} \in J_{\mathcal{B}, I}$ or $Y := \{n \in \mathbb{N} : s_n \geq \beta\} \in J_{\mathcal{B}, I}$.

Proof. Let us put $t_n = s_n$ if $n \notin H$ and $t_n = \alpha$ if $n \in H$. Since $H \in J_{\mathcal{B}, I}$, it is not difficult to see that $t = (t_n)_{n \in \mathbb{N}}$ is also \mathcal{B}^I_+ -statistically pre-Cauchy. Put $P := \{n \in \mathbb{N} : t_n \leq \alpha\}$ and $Q := \{n \in \mathbb{N} : t_n \geq \beta\}$. Then $X \subseteq P \cup H$ and $Y \subseteq Q \cup H$; thus it suffices to show $P \in J_{\mathcal{B}, I}$ or $Q \in J_{\mathcal{B}, I}$. Note also that $t_n \notin (\alpha, \beta)$ for all $n \in \mathbb{N}$ and hence $Q = \mathbb{N} \setminus P$.

For the sake of brevity, we define for $n \in \mathbb{N}$ and $i \in S$

$$D_{ni}(K) := \sum_{k=1}^{\infty} b_{nk}^{(i)} \chi_A(k) \quad \forall K \subseteq \mathbb{N}. \quad (92)$$

We claim that

$$I\text{-lim} D_{ni}(P) (1 - D_{nj}(P)) = 0 \quad \text{uniformly in } i, j \in S. \quad (93)$$

To see this, fix an arbitrary $\varepsilon > 0$ and note that $P \times Q \subseteq D(t, \beta - \alpha)$. So, since t is \mathcal{B}^I_+ -statistically pre-Cauchy, there is some $E \in I$ such that

$$\left\{ n \in \mathbb{N} : D_{ni}(P) D_{nj}(Q) \geq \frac{\varepsilon}{2} \right\} \subseteq E \quad \forall i, j \in S. \quad (94)$$

By (90) there exists $F \in I$ such that

$$\left\{ n \in \mathbb{N} : |D_{ni}(\mathbb{N}) - 1| \geq \frac{\varepsilon}{2M} \right\} \subseteq F \quad \forall i \in S. \quad (95)$$

Because of (89) and $D_{ni}(Q) = D_{ni}(\mathbb{N}) - D_{ni}(P)$ this easily implies

$$\{n \in \mathbb{N} : |D_{ni}(P)(1 - D_{nj}(P))| \geq \varepsilon\} \subseteq E \cup F \cup A \quad \forall i, j \in S, \tag{96}$$

proving our claim. In particular, we can find $C \in \mathcal{W}$ with

$$|D_{ni}(P)(1 - D_{nj}(P))| < \frac{1}{9} \quad \forall n \in C, \forall i, j \in S. \tag{97}$$

Then for every $n \in C$ we must have

$$\sup_{i \in S} D_{ni}(P) \leq \frac{1}{3} \quad \text{or} \quad \inf_{j \in S} D_{nj}(P) \geq \frac{2}{3}. \tag{98}$$

Write $C = \{n_1 < n_2 \dots n_k < n_{k+1} \dots\}$ and choose k_0 according to (91). Suppose first that $\sup_{i \in S} D_{n_{k_0}i}(P) \leq 1/3$. Then the same must hold for every $k > k_0$; for otherwise we could find a minimal $k > k_0$ with $\inf_{i \in S} D_{n_ki}(P) \geq 2/3$ which would imply

$$\sum_{l=1}^{\infty} |b_{n_k l}^{(i)} - b_{n_{k-1} l}^{(i)}| \geq D_{n_k i}(P) - D_{n_{k-1} i}(P) \geq \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \tag{99}$$

for all $i \in S$, contradicting the choice of k_0 .

So we have $D_{n_k i}(P) \leq 1/3$ for all $k \geq k_0$ and all $i \in S$. Now fix again an arbitrary $\varepsilon > 0$. By (93) there is $G \in I$ such that

$$\{n \in \mathbb{N} : |D_{ni}(P)(1 - D_{nj}(P))| \geq \frac{2}{3}\varepsilon\} \subseteq G \quad \forall i, j \in S. \tag{100}$$

Since I is admissible, $R := G \cup (\mathbb{N} \setminus \{n_k : k \geq k_0\})$ is again an element of I and we have

$$\{n \in \mathbb{N} : D_{ni}(P) \geq \varepsilon\} \subseteq R \quad \forall i \in S. \tag{101}$$

Thus we have shown that $D_{ni}(P)$ converges along I to zero uniformly in $i \in S$, which means exactly that $P \in J_{\mathcal{B}, I}$.

In the second case, $\inf_{i \in S} D_{n_{k_0}i}(P) \geq 2/3$, one can show analogously that $Q \in J_{\mathcal{B}, I}$. \square

Note that if $I = I_f$ and $\inf_{i \in S} \sum_{l=1}^{\infty} |b_{n_l}^{(i)} - b_{n_{l+1}}^{(i)}| < 1/3$ for all but finitely many $n \in \mathbb{N}$, then we can take $\mathcal{W} = \{n \in \mathbb{N} : n \geq m\} : m \in \mathbb{N}\}$ and condition (91) is satisfied. For the Cesàro matrix C we even have $\lim_{n \rightarrow \infty} \sum_{l=1}^{\infty} |c_{nl} - c_{n+1l}| = 0$.

As in [24], we can now use the above lemma to obtain a sufficient condition for \mathcal{B}^I -statistical convergence.

Theorem 26. *Under the same general hypotheses as in the previous lemma, if $s = (s_n)_{n \in \mathbb{N}}$ is a $J_{\mathcal{B}, I}$ -bounded \mathcal{B}_+^I -statistically pre-Cauchy sequence in \mathbb{R} such that the set Z of all $J_{\mathcal{B}, I}$ -cluster points of s is nowhere dense (note that Z is closed (cf. [1, Theorem 4.1(i)]), so “ Z nowhere dense” just means that Z has empty interior) in \mathbb{R} , then s is \mathcal{B}^I -statistically convergent.*

Proof. Suppose that s is $J_{\mathcal{B}, I}$ -bounded and \mathcal{B}_+^I -statistically pre-Cauchy but not \mathcal{B}^I -statistically convergent.

As mentioned before, the $J_{\mathcal{B}, I}$ -boundedness assures that there is some $a \in Z$. Since s is not \mathcal{B}^I -statistically convergent

there is an $\varepsilon > 0$ such that $\{n \in \mathbb{N} : s_n \leq a - \varepsilon\} \notin J_{\mathcal{B}, I}$ or $\{n \in \mathbb{N} : s_n \geq a + \varepsilon\} \notin J_{\mathcal{B}, I}$. Without loss of generality, we assume the former.

As in [24], we will show that $(a - \varepsilon, a) \subseteq Z$. If not, there would be an open interval $(\alpha, \beta) \subseteq (a - \varepsilon, a)$ such that $\{n \in \mathbb{N} : s_n \in (\alpha, \beta)\} \in J_{\mathcal{B}, I}$.

It follows from Lemma 25 that $X = \{n \in \mathbb{N} : s_n \leq \alpha\} \in J_{\mathcal{B}, I}$ or $Y := \{n \in \mathbb{N} : s_n \geq \beta\} \in J_{\mathcal{B}, I}$.

Since $X \supseteq \{n \in \mathbb{N} : s_n \leq a - \varepsilon\} \notin J_{\mathcal{B}, I}$, we would have $Y \in J_{\mathcal{B}, I}$. But we can find $\delta > 0$ with $\beta < a - \delta$ and because of $a \in Z$ the set $\{n \in \mathbb{N} : s_n > a - \delta\}$ cannot belong to $J_{\mathcal{B}, I}$ where on the other hand it is contained in Y .

Thus Z has nonempty interior and the proof is finished. \square

As an immediate consequence of Theorem 26 we get the following corollary.

Corollary 27. *Under the same general assumptions as in Lemma 25, if s is a \mathcal{B}_+^I -statistically pre-Cauchy sequence in \mathbb{R} whose range is finite, then s is \mathcal{B}^I -statistically convergent.*

6. A Sup-Limsup-Theorem

In this section we will present the generalisation of Simons’ equality that was announced in the abstract, but first we need to recall some definitions: a boundary for a real Banach space X is a subset H of B_{X^*} (for every Banach space Y we denote by B_Y its closed unit ball and by S_Y its unit sphere) such that for every $x \in X$ there is some $x^* \in H$ with $x^*(x) = \|x\|$. By the Hahn-Banach-theorem, S_{X^*} is always a boundary for X . It easily follows from the Krein-Milman-theorem that $\text{ex } B_{X^*}$, the set of extreme points of B_{X^*} , is also a boundary for X .

A famous theorem due to Rainwater (cf. [25]) states that a bounded sequence in X which is convergent to some $x \in X$ under every functional from $\text{ex } B_{X^*}$ is weakly convergent to x .

Later Simons (cf. [26, 27]) generalised this result to an arbitrary boundary H by proving that for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in X the equality

$$\sup_{x^* \in H} \limsup x^*(x_n) = \sup_{x^* \in B_{X^*}} \limsup x^*(x_n), \tag{102}$$

which is nowadays known as Simons’ equality, holds.

An easy separation argument shows that every boundary H satisfies $B_{X^*} = \overline{\text{co}}^{w^*} H$, but $B_{X^*} = \overline{\text{co}} H$ is not true in general (here $\text{co } A$ denotes the convex hull, \overline{A}^{w^*} the weak*-closure, and \overline{A} the norm-closure of $A \subseteq X^*$).

In [28] Fonf and Lindenstrauss introduced the following intermediate notion. Consider a convex weak*-compact subset K of X^* (where X is a real or complex Banach space). A subset H of K is said to (I)-generate K provided that whenever H is written as a countable union $H = \bigcup_{m=1}^{\infty} H_m$, then

$$\overline{\text{co}} \left(\bigcup_{m=1}^{\infty} \overline{\text{co}}^{w^*} H_m \right) = K \tag{103}$$

or equivalently, whenever H is written as a countable union $H = \bigcup_{m=1}^{\infty} H_m$ with $H_m \subseteq H_{m+1}$, then

$$\overline{\bigcup_{m=1}^{\infty} \overline{\text{co}}^{w^*} H_m} = K. \tag{104}$$

Clearly, $K = \overline{\text{co}}H$ implies that $H(I)$ -generates K which in turn implies $K = \overline{\text{co}}^{w^*} H$, but the converses are not true in general as was shown in [28]. It was also proved in [28] that, for a real Banach space, every boundary of K (I)-generates K (the set H is called a boundary of K if $\max\{x^*(x) : x^* \in H\} = \sup\{x^*(x) : x^* \in K\}$ for every $x \in X$). In this terminology, H is a boundary for X if and only if it is a boundary of B_{X^*} .

Nygaard proved in [29] that Rainwater’s theorem holds true for every (I)-generating subset of B_{X^*} and the authors of [30] showed that Simons’ equality is equivalent to the (I)-generation property (cf. [30, Theorem 2.2]; see also [31, Lemma 2.1 and Remark 2.2]).

In [32] the author investigated the possibility to generalise the Rainwater-Simons-convergence theorem for (I)-generating sets to some generalised convergence methods such as strong A - \mathbf{p} -summability and almost convergence by proving a general Simons-like inequality for (I)-generating sets (cf. [32, Theorem 3.1]). We will continue this work here, using similar arguments as in [32] to generalise Simons’ equality to the $J_{\mathcal{B},I}$ -limsup for the case that $\mathcal{F}(I)$ has a countable base, and obtain some related convergence results.

First we need the following lemma, whose proof is—once more—analogue to those of Propositions 3 and 4. Therefore, the details will be skipped.

Lemma 28. *Let each B_i be nonnegative. Define $f : \mathbb{R} \rightarrow [0, \infty)$ by $f(t) = t$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Put $A(s, a, \varepsilon) := \{k \in \mathbb{N} : s_k > a + \varepsilon\}$ for every $\varepsilon > 0$. Then*

$$I\text{-}\lim \sum_{k=1}^{\infty} b_{nk}^{(i)} f(s_k - a) = 0 \quad \text{uniformly in } i \in S \tag{105}$$

$$\implies A(s, a, \varepsilon) \in J_{\mathcal{B},I} \quad \forall \varepsilon > 0$$

and the converse is true if the sequence s is bounded and

$$\sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty \tag{106}$$

for some $A \in I$.

Now we turn to the generalisation of Simons’ equality.

Theorem 29. *Let X be a real Banach space, $K \subseteq X^*$ a convex weak*-compact subset, and $H \subseteq K$ an (I)-generating set for K . Let the ideal I be such that the filter $\mathcal{F}(I)$ has a countable base. Assume that each B_i is nonnegative and that there exists an $A \in I$ such that*

$$M := \sup \left\{ \sum_{k=1}^{\infty} b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty. \tag{107}$$

Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in X . Then the equality

$$\sup_{x^* \in H} J_{\mathcal{B},I}\text{-}\lim \sup x^*(x_n) = \sup_{x^* \in K} J_{\mathcal{B},I}\text{-}\lim \sup x^*(x_n) \tag{108}$$

holds.

Proof. Denote the left-hand supremum by c and the right-hand supremum by d . We only have to show $d \leq c$. Let $R = \sup_{n \in \mathbb{N}} \|x_n\|$. Let $(C_n)_{n \in \mathbb{N}}$ be a countable base for $\mathcal{F}(I)$. Without loss of generality we may assume $C_{n+1} \subseteq C_n$ for all n . Take $x^* \in K$ and $\varepsilon > 0$ arbitrary and put

$$E_m = \left\{ y^* \in K : \sum_{k=1}^{\infty} b_{nk}^{(i)} f(y^*(x_k) - c) \leq \varepsilon \forall i \in S, n \in C_m \right\}, \tag{109}$$

$$H_m = E_m \cap H \quad \forall m \in \mathbb{N},$$

where f is as in the previous lemma. Then $H_m \subseteq H_{m+1}$ for every $m \in \mathbb{N}$. It follows from [21, Theorem 1] that $\{n \in \mathbb{N} : y^*(x_n) > c + \delta\} \in J_{\mathcal{B},I}$ for every $\delta > 0$. Together with the previous lemma this easily implies $\bigcup_{m=1}^{\infty} H_m = H$.

Since $H(I)$ -generates K , we get that

$$K = \overline{\bigcup_{m=1}^{\infty} \overline{\text{co}}^{w^*} H_m}. \tag{110}$$

Thus we can find $m \in \mathbb{N}$ and $y^* \in \overline{\text{co}}^{w^*} H_m$ with $\|x^* - y^*\| \leq \varepsilon$. It is easily checked that E_m is convex and weak*-closed; hence $y^* \in E_m$. But for every $k \in \mathbb{N}$

$$\begin{aligned} f(x^*(x_k) - c) &\leq f(x^*(x_k) - y^*(x_k)) \\ &\quad + f(y^*(x_k) - c) \\ &\leq \|x^* - y^*\| \|x_k\| + f(y^*(x_k) - c) \\ &\leq R\varepsilon + f(y^*(x_k) - c). \end{aligned} \tag{111}$$

It follows that

$$\begin{aligned} \sum_{k=1}^{\infty} b_{nk}^{(i)} f(x^*(x_k) - c) &\leq MR\varepsilon + \sum_{k=1}^{\infty} b_{nk}^{(i)} f(y^*(x_k) - c) \\ &\leq \varepsilon(MR + 1) \end{aligned} \tag{112}$$

for every $i \in S$ and every $n \in C_m \cap (\mathbb{N} \setminus A)$. Since $C_m \cap (\mathbb{N} \setminus A) \in \mathcal{F}(I)$ and $\varepsilon > 0$ was arbitrary, we conclude with Lemma 28 that $\{n \in \mathbb{N} : x^*(x_n) > c + \delta\} \in J_{\mathcal{B},I}$ for every $\delta > 0$, whence $J_{\mathcal{B},I}\text{-}\lim \sup x^*(x_n) \leq c$. \square

As a corollary, we get the following convergence result.

Corollary 30. *Under the same hypotheses as in Theorem 29 with $K = B_{X^*}$, if $x \in X$ is such that $(x^*(x_n))_{n \in \mathbb{N}}$ is \mathcal{B}^I -statistically convergent to $x^*(x)$ for every $x^* \in H$ then the same holds true for every $x^* \in X^*$; that is, $(x_n)_{n \in \mathbb{N}}$ is “weakly \mathcal{B}^I -statistically convergent to x .”*

Moreover, for every family $\mathcal{F} = (F_k^{(i)})_{k \in \mathbb{N}, i \in S}$ in $\mathcal{M} \cup \mathcal{O}$ which is equicontinuous at 0 and satisfies

$$\inf \{F_k^{(i)}(t) : k \in \mathbb{N}, i \in S\} > 0 \quad \forall t > 0, \tag{113}$$

$$\sup \{F_k^{(i)}(t) : k \in \mathbb{N}, i \in S\} < \infty \quad \forall t \geq 0,$$

$(x^*(x_n))_{n \in \mathbb{N}}$ is strongly \mathcal{B}^I -summable to $x^*(x)$ with respect to \mathcal{F} for every $x^* \in X^*$ whenever this statement holds for every $x^* \in H$.

Proof. The first statement follows directly from Theorem 29 and the second follows from the first one via Propositions 3 and 4. \square

It is clear that this convergence result carries over to complex Banach spaces (note that if X is a complex Banach space and $H(I)$ -generates B_{X^*} ; then $\{\operatorname{Re} x^* : x^* \in H\}$ (I)-generates $\{\operatorname{Re} x^* : x^* \in B_{X^*}\}$, the unit ball of the underlying real space).

In particular, if we take each B_i to be the infinite unit matrix, we get that, for every ideal I such that $\mathcal{F}(I)$ has a countable base, $I\text{-}\lim x^*(x_n) = x^*(x)$ for every $x^* \in X^*$ whenever this is true for every x^* in an (I)-generating subset of B_{X^*} (in particular, in a boundary for X). We can also prove an analogous convergence result for \mathcal{B}^I -summability.

Proposition 31. *Let X be a real or complex Banach space and $H \subseteq B_{X^*}$ an (I)-generating set for B_{X^*} . Suppose that $\mathcal{F}(I)$ has a countable base, $\sum_{k=1}^\infty |b_{nk}^{(i)}| < \infty$ for all $n \in \mathbb{N}, i \in S$, and moreover*

$$M := \sup \left\{ \sum_{k=1}^\infty |b_{nk}^{(i)}| : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty \tag{114}$$

for some $A \in I$.

Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in X and $x \in X$ such that $(x^*(x_n))_{n \in \mathbb{N}}$ is \mathcal{B}^I -summable to $x^*(x)$ for every $x^* \in H$. Then the same is true for every $x^* \in X^*$.

Proof. Let $(C_n)_{n \in \mathbb{N}}$ be a decreasing countable basis for $\mathcal{F}(I)$. Let $R \geq \sup_{n \in \mathbb{N}} \|x_n\|$ and $R \geq \|x\|$. Take any $x^* \in B_{X^*}$ and fix an arbitrary $\varepsilon > 0$. Define

$$E_m := \left\{ y^* \in B_{X^*} : \sup_{i \in S} \left| \sum_{k=1}^\infty b_{nk}^{(i)} y^*(x_k) - y^*(x) \right| \leq \varepsilon \forall n \in C_m \right\}, \tag{115}$$

$$H_m := E_m \cap H \quad \forall m \in \mathbb{N}.$$

Then $H_m \nearrow H$ and since $H(I)$ -generates B_{X^*} , we can find $m \in \mathbb{N}$ and $y^* \in \overline{\operatorname{co}}^w H_m$ such that $\|x^* - y^*\| \leq \varepsilon$.

It is not too hard to see that E_m is convex and weak*-closed and thus $y^* \in E_m$. Consequently, for all $i \in S$ and $n \in C_m \cap (\mathbb{N} \setminus A)$ we have

$$\begin{aligned} \left| \sum_{k=1}^\infty b_{nk}^{(i)} x^*(x_k) - x^*(x) \right| &\leq \left| \sum_{k=1}^\infty b_{nk}^{(i)} (x^*(x_k) - y^*(x_k)) \right| \\ &\quad + \left| \sum_{k=1}^\infty b_{nk}^{(i)} y^*(x_k) - y^*(x) \right| \\ &\quad + |y^*(x) - x^*(x)| \\ &\leq M \|x^* - y^*\| R + \varepsilon + \|x^* - y^*\| R \\ &\leq \varepsilon (R(M + 1) + 1). \end{aligned} \tag{116}$$

Since $C_m \cap (\mathbb{N} \setminus A) \in \mathcal{F}(I)$ and $\varepsilon > 0$ was arbitrary, we are done. \square

The next result concerning \mathcal{B}^I -statistically pre-Cauchy sequences is a generalisation of [32, Corollary 3.5]. Using Propositions 21 and 22 with $F_{kl}^{(i)} = \operatorname{id}_{[0, \infty)}$ for all $k, l \in \mathbb{N}$ and $i \in S$, its proof can be carried out analogously to that of Proposition 31. The details will be omitted.

Proposition 32. *Let X be a real or complex Banach space and $H \subseteq B_{X^*}$ an (I)-generating set for B_{X^*} . Suppose that $\mathcal{F}(I)$ has a countable base, that each B_i is nonnegative, and that there is some $A \in I$ such that*

$$\sup \left\{ \sum_{k=1}^\infty b_{nk}^{(i)} : n \in \mathbb{N} \setminus A, i \in S \right\} < \infty. \tag{117}$$

Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in X such that $(x^*(x_n))_{n \in \mathbb{N}}$ is \mathcal{B}^I -statistically pre-Cauchy and \mathcal{B}_+^I -statistically pre-Cauchy, respectively, for every $x^* \in H$. Then the same is true for every $x^* \in X^*$.

Finally, let us give characterisations of weak-compactness and reflexivity that generalise [32, Corollaries 3.7 and 3.8].

Corollary 33. *Let M be a bounded subset of the Banach space X and B an (I)-generating set for B_{X^*} . Then M is weakly relatively compact if (and only if) for every sequence $(x_n)_{n \in \mathbb{N}}$ in M there is an element $x \in X$, an ideal I on \mathbb{N} such that $\mathcal{F}(I)$ admits a countable base, and a nonnegative matrix $A = (a_{nk})_{n, k \geq 1}$ such that*

$$\exists C \in I \quad \sup_{n \in \mathbb{N} \setminus C} \sum_{k=1}^\infty a_{nk} < \infty, \tag{118}$$

$$I\text{-}\lim a_{nk} = 0 \quad \forall k \in \mathbb{N} \tag{119}$$

and $(x^*(x_n))_{n \in \mathbb{N}}$ is A^I -statistically convergent to $x^*(x)$ for every $x^* \in B$.

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in M and fix x, I , and A as above. By Corollary 30 $(x^*(x_n))_{n \in \mathbb{N}}$ is A^I -statistically convergent to $x^*(x)$ for every $x^* \in X^*$. Thus,

given finitely many functionals $x_1^*, \dots, x_m^* \in X^*$, the sequence $(\sum_{j=1}^m |x_j^*(x_n - x)|)_{n \in \mathbb{N}}$ is A^I -statistically convergent to zero. Hence for any $\varepsilon > 0$ the set $D_\varepsilon = \{n \in \mathbb{N} : \sum_{j=1}^m |x_j^*(x_n - x)| < \varepsilon\}$ does not belong to $J_{A,I}$.

By (119), $J_{A,I}$ is admissible; therefore, D_ε must be infinite for every $\varepsilon > 0$, which shows that x is a weak-cluster point of $(x_n)_{n \in \mathbb{N}}$.

So M is weakly relatively and countably compact and by the Eberlein-Shmulyan theorem, it must be also weakly relatively compact. \square

Corollary 34. *If B_X is an (I) -generating set for $B_{X^{**}}$ (we consider X canonically embedded into its bidual), then X is reflexive if (and only if) for every sequence $(x_n^*)_{n \in \mathbb{N}}$ in B_{X^*} there is a functional $x^* \in X^*$, an ideal I on \mathbb{N} such that $\mathcal{F}(I)$ admits a countable base, and a nonnegative matrix A such that (118) and (119) are satisfied and $(x_n^*(x))_{n \in \mathbb{N}}$ is A^I -statistically convergent to $x^*(x)$ for every $x \in X$.*

Proof. By the previous corollary, B_{X^*} is weakly compact; thus X^* and hence also X are reflexive. \square

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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