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Research Article

Growth Analysis of Composite Entire Functions Related to Slowly Changing Functions Oriented Relative Order and Relative Type

Sanjib Kumar Datta,[1] Tanmay Biswas,[2] and Sarmila Bhattacharyya[3]

[1] *Department of Mathematics, University of Kalyani, Kalyani, Nadia District, West Bengal 741235, India*
[2] *Rajbari, Rabindrapalli, R. N. Tagore Road, Krishnagar, Kotwali, Nadia District, West Bengal 741101, India*
[3] *Jhorehat Fakir Chandra High School for Girls, Jhorehat, Sankrail, Howrah District, West Bengal 711302, India*

Correspondence should be addressed to Sanjib Kumar Datta; sanjib_kr_datta@yahoo.co.in

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Some results on comparative growth properties of maximum terms and maximum moduli of composite entire functions on the basis of relative L^*-order and relative L^*-type are proved in this paper.

1. Introduction, Definitions, and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r, f) = \max(|a_n| r^n)$ and the maximum modulus $M(r, f)$ of f on $|z| = r$ is defined as $M(r, f) = \max_{|z|=r} |f(z)|$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly, that is, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker [1] defined it in the following way.

Definition 1 (see [1]). A positive continuous function $L(r)$ is called a slowly changing function if, for ε (>0),

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \quad \text{for } r \geq r(\varepsilon) \tag{1}$$

and uniformly for k (≥ 1).

If, further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = 0. \tag{2}$$

Somasundaram and Thamizharasi [2] introduced the notions of L-order and L-type for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly, that is, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". The more generalized concepts for L-order and L-type for entire functions are L^*-order and L^*-type. Their definitions are as follows.

Definition 2 (see [2]). The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}, \qquad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}, \tag{3}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

Using the inequalities $\mu_f(r) \leq M_f(r) \leq (R/(R - r))\mu_f(R)$ {*cf.* [3]}, for $0 \leq r < R$, one may verify that

$$\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]}, \qquad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]}. \tag{4}$$

Definition 3 (see [2]). The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty. \tag{5}$$

If an entire function g is nonconstant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\log_{s\to\infty} M_g^{-1}(s) = \infty$.

Bernal [4] introduced the definition of relative order of an entire function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(r^\mu) \ \ \forall r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned} \tag{6}$$

The definition coincides with the classical one [5] if $g(z) = \exp z$.

Similarly, one can define the relative lower order of an entire function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \tag{7}$$

Datta and Maji [6] gave an alternative definition of relative order and relative lower order in terms of maximum term of an entire function with respect to another entire in the following way.

Definition 4 (see [6]). The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function f with respect to an entire function g are defined as follows:

$$\begin{aligned}\rho_g(f) &= \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}, \\ \lambda_g(f) &= \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.\end{aligned} \tag{8}$$

In the line of Somasundaram and Thamizharasi [2] and Bernal [4], one may define the relative L^*-order of an entire function in the following manner.

Definition 5 (see [7, 8]). The relative L^*-order $\rho_g^{L^*}(f)$ and relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as

$$\begin{aligned}\rho_g^{L^*}(f) &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}, \\ \lambda_g^{L^*}(f) &= \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}.\end{aligned} \tag{9}$$

Datta et al. [9] also gave an alternative definition of L^*-order and relative L^*-lower order in terms of maximum term of an entire function which are as follows.

Definition 6 (see [9]). The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to g are as follows:

$$\begin{aligned}\rho_g^{L^*}(f) &= \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]}, \\ \lambda_g^{L^*}(f) &= \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]}.\end{aligned} \tag{10}$$

To determine the relative growth of two entire functions having same nonzero finite relative L^*-order with respect to another entire function, one may introduce the concept of the relative L^*-type in the following way.

Definition 7. The relative L^*-type $\sigma_g^{L^*}(f)$ of an entire function f with respect to g is defined as follows:

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}, \quad 0 < \rho_g^{L^*}(f) < \infty. \tag{11}$$

Considering $g = \exp z$, one may easily verify that Definition 7 coincides with the classical Definition 3.

In the paper we study some comparative growth properties of maximum term and maximum modulus of composition of entire functions corresponding to its left or right factors on the basis of relative L^*-order and relative L^*-type. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [10].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 8 (see [11]). *If f and g are any two entire functions then, for all sufficiently large values of r,*

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \le M_{f\circ g}(r) \le M_f\left(M_g(r)\right). \tag{12}$$

Lemma 9 (see [12]). *Let f and g be any two entire functions. Then, for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \le \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g(R)\right). \tag{13}$$

Lemma 10 (see [12]). *Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \ge \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right). \tag{14}$$

Lemma 11 (see [4]). *Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then, for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r). \tag{15}$$

Lemma 12 (see [6]). *If f is entire and $\alpha > 1$, $0 < \beta < \alpha$, then, for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r). \tag{16}$$

Lemma 13 (see [13]). *Let f and h be any two entire functions. Then for any $\alpha > 1$*

$$\begin{aligned} &(i)\ M_h^{-1} M_f(r) \leq \mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right], \\ &(ii)\ \mu_h^{-1}\mu_f(r) \leq \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)}M_f(r)\right]. \end{aligned} \tag{17}$$

3. Theorems

In this section we present the main results of the paper.

Theorem 14. *Let f and h be any two entire functions such that $\rho_h^{L^*}(f)$ is finite and positive. Also let g be an entire function with finite nonzero L^* order. Then, for each $\delta \in (-\infty, \infty)$ and for $A > (1+\delta)\rho_g^{L^*}$,*

$$\liminf_{r\to\infty} \frac{\left\{\log \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta}}{\log \mu_h^{-1}\mu_f(\exp(r^A))} = 0. \tag{18}$$

Proof. If $1+\delta < 0$ then the theorem is trivial. So we take $1+\delta > 0$. Now taking $R = \beta r$ in Lemma 9 and in view of Lemma 12 we have for all sufficiently large values of r that

$$\begin{aligned} &\mu_{f\circ g}(r) \leq \left(\frac{\alpha}{\alpha-1}\right)\mu_f\left(\frac{\alpha\beta}{(\beta-1)}\mu_g(\beta r)\right), \\ &\text{i.e., } \mu_{f\circ g}(r) \leq \mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right). \end{aligned} \tag{19}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from above for all sufficiently large values of r that

$$\begin{aligned} &\mu_h^{-1}\mu_{f\circ g}(r) \leq \mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right), \\ &\text{i.e., } \log \mu_h^{-1}\mu_{f\circ g}(r) \\ &\qquad \leq \log \mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right), \\ &\text{i.e., } \log \mu_h^{-1}\mu_{f\circ g}(r) \\ &\qquad \leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\log \mu_g(\beta r)\, e^{L(\mu_g(\beta r))}\right] + O(1), \\ &\text{i.e., } \log \mu_h^{-1}\mu_{f\circ g}(r) \\ &\qquad \leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left[\beta r e^{L(r)}\right]^{(\rho_g^{L^*}+\varepsilon)} \\ &\qquad\quad + \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\mu_g(\beta r)\right) + O(1), \\ &\text{i.e., } \left\{\log \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta} \\ &\qquad \leq \left[\left[\beta r e^{L(r)}\right]^{(\rho_g^{L^*}+\varepsilon)}\left(\rho_h^{L^*}(f)+\varepsilon\right)\right. \\ &\qquad\quad \left. + \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\mu_g(\beta r)\right) + O(1)\right]^{1+\delta}. \end{aligned} \tag{20}$$

Again we have for a sequence of r tending to infinity and for ε (>0),

$$\begin{aligned} &\log \mu_h^{-1}\mu_f\left(\exp\left(r^A\right)\right) \\ &\qquad \geq \left(\rho_h^{L^*}(f)-\varepsilon\right)\log\left[\exp\left(r^A\right)\exp\left\{L\left(\exp\left(r^A\right)\right)\right\}\right], \\ &\text{i.e., } \log \mu_h^{-1}\mu_f\left(\exp\left(r^A\right)\right) \\ &\qquad \geq \left(\rho_h^{L^*}(f)-\varepsilon\right)\left[r^A + L\left(\exp\left(r^A\right)\right)\right]. \end{aligned} \tag{21}$$

So from (20) and (21) we get for a sequence of r tending to infinity that

$$\begin{aligned} &\frac{\left\{\log \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta}}{\log \mu_h^{-1}\mu_f(\exp(r^A))} \\ &\quad \leq \left(\left[\left[\beta r e^{L(r)}\right]^{(\rho_g^{L^*}+\varepsilon)}\left(\rho_h^{L^*}(f)+\varepsilon\right)\right.\right. \\ &\qquad \left.\left. + \left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\mu_g(\beta r)\right) + O(1)\right]^{1+\delta}\right) \\ &\qquad \times \left(\left(\rho_h^{L^*}(f)-\varepsilon\right)\left[r^A + L\left(\exp\left(r^A\right)\right)\right]\right)^{-1}. \end{aligned} \tag{22}$$

Let

$$\begin{aligned} &\left[\beta e^{L(r)}\right]^{(\rho_g^{L^*}+\varepsilon)}\left(\rho_h^{L^*}(f)+\varepsilon\right) = k_1, \\ &\left(\rho_h^{L^*}(f)+\varepsilon\right)L\left(\mu_g(\beta r)\right) = k_2, \\ &\left(\rho_h^{L^*}(f)-\varepsilon\right) = k_3, \\ &\left(\rho_h^{L^*}(f)-\varepsilon\right)L\left(\exp\left(r^A\right)\right) = k_4. \end{aligned} \tag{23}$$

Then from (22) we obtain for a sequence of r tending to infinity

$$\begin{aligned} &\frac{\left\{\log \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta}}{\log \mu_h^{-1}\mu_f(\exp(r^A))} \leq \frac{\left[r^{(\rho_g^{L^*}+\varepsilon)}k_1 + k_2\right]^{1+\delta}}{k_3 r^A + k_4}, \\ &\text{i.e., } \frac{\left\{\log \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta}}{\log \mu_h^{-1}\mu_f(\exp(r^A))} \\ &\qquad \leq \frac{r^{(\rho_g^{L^*}+\varepsilon)(1+\delta)}\left[k_1 + k_2/r^{(\rho_g^{L^*}+\varepsilon)}\right]^{1+\delta}}{k_3 r^\beta + k_4}, \end{aligned} \tag{24}$$

where k_1, k_2, k_3, and k_4 are finite.

Since $(\rho_g^{L^*}+\varepsilon)(1+\delta)<A$, therefore

$$\liminf_{r\to\infty}\frac{\left\{\log\mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta}}{\log\mu_h^{-1}\mu_f\left(\exp\left(r^A\right)\right)}=0, \tag{25}$$

where we choose ε (>0) such that

$$0<\varepsilon<\min\left\{\rho_h^{L^*}(f),\frac{A}{1+\delta}-\rho_g^{L^*}\right\}. \tag{26}$$

This proves the theorem. □

Theorem 15. *Let f and h be any two entire functions with $\rho_h^{L^*}(f)$ is finite. Also let g be an entire function with finite nonzero L^* order.Then for each $\delta\in(-\infty,\infty)$ and for $A>(1+\delta)\rho_g^{L^*}$,*

$$\liminf_{r\to\infty}\frac{\left\{\log\mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\delta}}{\log\mu_h^{-1}\mu_g\left\{\exp\left(r^A\right)\right\}}=0, \tag{27}$$

where $\rho_h^{L^*}(g)>0$.

We omit the proof of Theorem 15 as it follows from Theorem 14 and the following inequality in place of (21)

$$\log\mu_h^{-1}\mu_g\left(\exp\left(r^A\right)\right)\geq\left(\rho_h^{L^*}(g)-\varepsilon\right)\left[r^A+L\left(\exp\left(r^A\right)\right)\right] \tag{28}$$

for a sequence of values of r tending to infinity.

In the line of Theorems 14 and 15, the following two theorems can be proved by using Lemmas 8 and 11 and hence their proofs are omitted.

Theorem 16. *Let f and h be any two entire functions such that $\rho_h^{L^*}(f)$ is finite and positive. Also let g be an entire function with finite nonzero L^* order.Then, for each $\delta\in(-\infty,\infty)$ and for $A>(1+\delta)\rho_g^{L^*}$,*

$$\liminf_{r\to\infty}\frac{\left\{\log M_h^{-1}M_{f\circ g}(r)\right\}^{1+\delta}}{\log M_h^{-1}M_f\left\{\exp\left(r^A\right)\right\}}=0. \tag{29}$$

Theorem 17. *Let f and h be any two entire functions with $\rho_h^{L^*}(f)$ is finite. Also let g be an entire function with finite nonzero L^* order. Then, for each $\delta\in(-\infty,\infty)$ and for $A>(1+\delta)\rho_g^{L^*}$,*

$$\liminf_{r\to\infty}\frac{\left\{\log M_h^{-1}M_{f\circ g}(r)\right\}^{1+\delta}}{\log M_h^{-1}M_g\left\{\exp\left(r^A\right)\right\}}=0, \tag{30}$$

where $\rho_h^{L^*}(g)>0$.

Remark 18. In Theorems 14 and 16, if we take the condition "$0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$" instead of "$\rho_h^{L^*}(f)$ is finite and positive" the theorems remain true with "limit" in place of "limit inferior".

Remark 19. In Theorems 15 and 17 if we consider the condition "$\lambda_h^{L^*}(g)>0$" instead of "$\rho_h^{L^*}(g)>0$" the theorems remain true with "limit" in place of "limit inferior".

Theorem 20. *Let f, g, and h be three entire functions such that $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$, $\rho_g^{L^*}>0$, and $g(0)=0$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_f(r)+L\left((1/24)\mu_g(r/4)\right)}\geq\frac{\rho_g^{L^*}}{\rho_h^{L^*}(f)}. \tag{31}$$

Proof. In view of Lemmas 10 and 12 we have

$$\mu_{f\circ g}(r)\geq\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right). \tag{32}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from above for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{f\circ g}(r)\geq\mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right),$$

$$\begin{aligned}
\text{i.e., } &\log\mu_h^{-1}\mu_{f\circ g}(r)\\
&\geq\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left\{\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right\}+L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right),\\
\text{i.e., } &\log\mu_h^{-1}\mu_{f\circ g}(r)\geq\left(\lambda_h^{L^*}(f)-\varepsilon\right)\\
&\qquad\times\left[\log\left\{\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right\}\right.\\
&\qquad\left.+L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right],\\
\text{i.e., } &\log\mu_h^{-1}\mu_{f\circ g}(r)\geq\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\mu_g\left(\frac{r}{4}\right)\\
&\quad\cdot\left\{\frac{\log\mu_g(r/4)+L\left((1/24)\mu_g(r/4)\right)+O(1)}{\log\mu_g(r/4)}\right\},\\
\text{i.e., } &\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\\
&\geq\log^{[2]}\mu_g\left(\frac{r}{4}\right)+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\\
&\quad-\log\left[\exp\left\{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}\right]\\
&\quad+\log\left\{\left(\left[\log\mu_g\left(\frac{r}{4}\right)+L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right]+O(1)\right)\right.\\
&\qquad\left.\times\left(\log\mu_g\left(\frac{r}{4}\right)\right)^{-1}\right\},
\end{aligned}$$

$$\begin{aligned}\text{i.e., } &\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\\ &\geq \log^{[2]}\mu_g\left(\frac{r}{4}\right)\\ &\quad+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\\ &\quad+\log\left\{\left(\left[\log\mu_g\left(\frac{r}{4}\right)+L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right]+O(1)\right)\right.\\ &\qquad\times\left(\exp\left\{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}\right.\\ &\qquad\left.\left.\times\log\mu_g\left(\frac{r}{4}\right)\right)^{-1}\right\},\end{aligned}\tag{33}$$

$$\begin{aligned}\text{i.e., } &\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\\ &\geq \log^{[2]}\mu_g\left(\frac{r}{4}\right)+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right).\end{aligned}\tag{34}$$

Now from (34) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned}\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq \left(\rho_g^{L^*}-\varepsilon\right)\log\left\{\frac{r}{4}e^{L(r/4)}\right\}\\ &\quad+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right).\end{aligned}\tag{35}$$

Now we get for all sufficiently large values of r that

$$\begin{aligned}&\log\mu_h^{-1}\mu_f(r)\leq\left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{re^{L(r)}\right\},\\ &\text{i.e., } \log\mu_h^{-1}\mu_f(r)\leq\left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{\frac{r}{4}e^{L(r/4)}\right\}+\log 4.\end{aligned}\tag{36}$$

Hence from (35) and (36) it follows for all sufficiently large values of r that

$$\begin{aligned}&\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\\ &\quad\geq\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)\left(\log\mu_h^{-1}\mu_f(r)-\log 4\right)\\ &\qquad+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right),\end{aligned}$$

$$\begin{aligned}\text{i.e., } &\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)\\ &\quad\geq\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)\left[\log\mu_h^{-1}\mu_f(r)+L\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right]\\ &\qquad-\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)\log 4,\\ \text{i.e., } &\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_f(r)+L\left((1/24)\mu_g(r/4)\right)}\\ &\quad\geq\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)\\ &\qquad-\frac{\left(\left(\rho_g^{L^*}-\varepsilon\right)/\left(\rho_h^{L^*}(f)+\varepsilon\right)\right)\log 4}{\log\mu_h^{-1}\mu_f(r)+L\left((1/24)\mu_g(r/4)\right)}.\end{aligned}\tag{37}$$

Since ε (>0) is arbitrary, it follows from (37) that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_f(r)+L\left((1/24)\mu_g(r/4)\right)}\geq\frac{\rho_g^{L^*}}{\rho_h^{L^*}(f)}.\tag{38}$$

This proves the theorem. □

In the line of Theorem 20, the following theorem can be proved.

Theorem 21. *Let f, g, and h be any three entire functions with $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$, $\lambda_g^{L^*}>0$, and $g(0)=0$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_f(r)+L\left((1/24)\mu_g(r/4)\right)}\geq\frac{\lambda_g^{L^*}}{\rho_h^{L^*}(f)}.\tag{39}$$

The proof is omitted.

Theorem 22. *Let f, g, and h be any three entire functions such that $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$, $\rho_g^{L^*}>0$, and $g(0)=0$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[2]}M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r)+L\left((1/8)M_g(r/2)\right)}\geq\frac{\rho_g^{L^*}}{\rho_h^{L^*}(f)}.\tag{40}$$

Theorem 23. *Let f, g, and h be any three entire functions with $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$, $\lambda_g^{L^*}>0$, and $g(0)=0$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[2]}M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r)+L\left((1/8)M_g(r/2)\right)}\geq\frac{\lambda_g^{L^*}}{\rho_h^{L^*}(f)}.\tag{41}$$

We omit the proofs of Theorems 22 and 23 because those can be carried out in the line of Theorems 20 and 21, respectively, and with the help of Lemmas 8 and 11.

Theorem 24. *Let f, g, and h be any three entire functions such that $\rho_h^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \infty$. Then*

$$\lim_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f(r)} = \infty. \tag{42}$$

Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log \mu_h^{-1}\mu_{f\circ g}(r) \le \beta \log \mu_h^{-1}\mu_f(r). \tag{43}$$

Again from the definition of $\rho_h^{L^*}(f)$ it follows for all sufficiently large values of r that

$$\begin{gathered} \log \mu_h^{-1}\mu_f(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left(re^{L(r)}\right), \\ \text{i.e., } \log \mu_h^{-1}\mu_f(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left(re^{L(r)}\right). \end{gathered} \tag{44}$$

Thus from (43) and (44) we have for a sequence of values of r tending to infinity that

$$\begin{gathered} \log \mu_h^{-1}\mu_{f\circ g}(r) \le \beta\left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left(re^{L(r)}\right), \\ \text{i.e., } \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log\left(re^{L(r)}\right)} \le \frac{\beta\left(\rho_h^{L^*}(f) + \varepsilon\right)\log\left(re^{L(r)}\right)}{\log\left(re^{L(r)}\right)}, \\ \text{i.e., } \liminf_{r\to\infty} \frac{\log^{[m]}\mu(r, f\circ g)}{\log\left(re^{L(r)}\right)} = \lambda_h^{L^*}(f\circ g) < \infty. \end{gathered} \tag{45}$$

This is a contradiction.

This proves the theorem. □

Remark 25. Theorem 24 is also valid with "limit superior" instead of "limit" if $\lambda_h^{L^*}(f\circ g) = \infty$ is replaced by $\rho_h^{L^*}(f\circ g) = \infty$ and the other conditions remain the same.

In the line of Theorem 24 the following theorem can also be proved.

Theorem 26. *Let f, g, and h be any three entire functions with $\rho_h^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f\circ g) = \infty$. Then*

$$\lim_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r)} = \infty. \tag{46}$$

Further, if $\rho_h^{L^*}(f\circ g) = \infty$ instead of $\lambda_h^{L^*}(f\circ g) = \infty$ then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r)} = \infty. \tag{47}$$

Corollary 27. *Under the assumptions of Theorem 24 or Remark 25 and Theorem 26,*

$$\begin{gathered} \lim_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} = \infty, \\ \lim_{r\to\infty} \frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)} = \infty. \end{gathered} \tag{48}$$

Proof. By Theorem 24 or Remark 25 we obtain, for all sufficiently large values of r and for $K > 1$,

$$\begin{gathered} \log \mu_h^{-1}\mu_{f\circ g}(r) > K \log \mu_h^{-1}\mu_f(r), \\ \text{i.e., } \mu_h^{-1}\mu_{f\circ g}(r) > \left\{\mu_h^{-1}\mu_f(r)\right\}^K, \end{gathered} \tag{49}$$

from which the first part of the corollary follows.

Similarly from Theorem 26, the second part of the corollary is established. □

Theorem 28. *Let f, g, and h be any three entire functions such that (i) $0 < \rho_h^{L^*}(f) < \infty$, (ii) $0 < \sigma_h^{L^*}(f) < \infty$, (iii) $\rho_h^{L^*}(f\circ g) = \rho_h^{L^*}(f)$, and (iv) $\sigma_h^{L^*}(f\circ g) < \infty$. Then, for any $\alpha > 1$,*

$$\begin{gathered} \liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \le \frac{(2\alpha-1)^{2\rho_h^{L^*}(f)} \cdot \alpha^{\rho_h^{L^*}(f)+1}}{(\alpha-1)^{2\rho_h^{L^*}(f)}} \cdot \frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(f)}, \\ \frac{(\alpha-1)^{2\rho_h^{L^*}(f)}}{(2\alpha-1)^{2\rho_h^{L^*}(f)}\alpha^{\rho_h^{L^*}(f)+1}} \cdot \frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(f)} \le \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)}. \end{gathered} \tag{50}$$

Proof. From the definition of relative L^*-type and and in view of Lemmas 12 and 13 we obtain for all sufficiently large values of r that

$$\begin{aligned} &\mu_h^{-1}\mu_{f\circ g}(r) \\ &\quad \le \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)} M_{f\circ g}(r)\right] \\ &\quad \le \alpha M_h^{-1}\left[M_{f\circ g}\left(\left(\frac{2\alpha-1}{\alpha-1}\right) r\right)\right], \end{aligned} \tag{51}$$

$$\begin{aligned} &\text{i.e., } \mu_h^{-1}\mu_{f\circ g}(r) \\ &\quad \le \alpha\left(\sigma_h^{L^*}(f\circ g) + \varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right) re^{L(r)}\right\}^{\rho_h^{L^*}(f\circ g)}, \end{aligned} \tag{52}$$

$$\mu_h^{-1}\mu_f(r) \le \alpha\left(\sigma_h^{L^*}(f) + \varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right) re^{L(r)}\right\}^{\rho_h^{L^*}(f)}. \tag{53}$$

Also we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} &\mu_h^{-1}\mu_{f\circ g}(r) \ge M_h^{-1}M_f\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha} r\right), \\ &\text{i.e., } \mu_h^{-1}\mu_{f\circ g}(r) \\ &\quad \ge \left(\sigma_h^{L^*}(f\circ g) - \varepsilon\right)\left\{\left(\frac{(\alpha-1)r}{(2\alpha-1)\alpha}\right) e^{L(r/\beta)}\right\}^{\rho_h^{L^*}(f\circ g)}, \end{aligned} \tag{54}$$

$$\begin{aligned} &\text{i.e., } \mu_h^{-1}\mu_{f\circ g}(r) \ge \left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\rho_h^{L^*}(f\circ g)} \\ &\qquad \cdot \left(\sigma_h^{L^*}(f\circ g) - \varepsilon\right) \cdot \left\{re^{L(r)}\right\}^{\rho_h^{L^*}(f\circ g)}, \end{aligned} \tag{55}$$

$$\mu_h^{-1}\mu_f(r) \geq \left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\rho_h^{L^*}(f)} \cdot \left(\sigma_h^{L^*}(f)-\varepsilon\right)\cdot\left\{re^{L(r)}\right\}^{\rho_h^{L^*}(f)}. \tag{56}$$

Now from (51) and (56) it follows for a sequence of values of r tending to infinity that

$$\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{\alpha\left(\sigma_h^{L^*}(f\circ g)+\varepsilon\right)\left\{((2\alpha-1)/(\alpha-1))\,re^{L(r)}\right\}^{\rho_h^{L^*}(f\circ g)}}{((\alpha-1)/((2\alpha-1)\alpha))^{\rho_h^{L^*}(f)}.\left(\sigma_h^{L^*}(f)-\varepsilon\right)\cdot\left\{re^{L(r)}\right\}^{\rho_h^{L^*}(f)}}. \tag{57}$$

In view of the condition (iii), we get from (57) that

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{\alpha\left(\sigma_h^{L^*}(f\circ g)+\varepsilon\right)((2\alpha-1)/(\alpha-1))^{\rho_h^{L^*}(f\circ g)}}{((\alpha-1)/((2\alpha-1)\alpha))^{\rho_h^{L^*}(f)}.\left(\sigma_h^{L^*}(f)-\varepsilon\right)}. \tag{58}$$

As ε (>0) is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{(2\alpha-1)^{2\rho_h^{L^*}(f)}\cdot\alpha^{\rho_h^{L^*}(f)+1}}{(\alpha-1)^{2\rho_h^{L^*}(f)}}\cdot\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(f)}. \tag{59}$$

Again from (53) and (55), we get for a sequence of values of r tending to infinity that

$$\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \geq \left(\left(\frac{(\alpha-1)}{(2\alpha-1)\alpha}\right)^{\rho_h^{L^*}(f\circ g)}\cdot\left(\sigma_h^{L^*}(f\circ g)-\varepsilon\right)\cdot\left\{re^{L(r)}\right\}^{\rho_h^{L^*}(f\circ g)}\right) \times\left(\alpha\left(\sigma_h^{L^*}(f)+\varepsilon\right)\left\{\left(\frac{2\alpha-1}{\alpha-1}\right)re^{L(r)}\right\}^{\rho_h^{L^*}(f)}\right)^{-1}. \tag{60}$$

Since $\rho_h^{L^*}(f\circ g)=\rho_h^{L^*}(f)$, we obtain from (60) that

$$\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{((\alpha-1)/((2\alpha-1)\alpha))^{\rho_h^{L^*}(f)}}{((2\alpha-1)/(\alpha-1))^{\rho_h^{L^*}(f)}}\cdot\frac{\left(\sigma_h^{L^*}(f\circ g)-\varepsilon\right)}{\alpha\left(\sigma_h^{L^*}(f)+\varepsilon\right)}. \tag{61}$$

As ε (>0) is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{(\alpha-1)^{2\rho_h^{L^*}(f)}}{(2\alpha-1)^{2\rho_h^{L^*}(f)}\alpha^{\rho_h^{L^*}(f)_{+1}}}\cdot\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(f)}. \tag{62}$$

Thus the theorem follows from (59) and (62). □

In the line of Theorem 28, we may state the following theorem without its proof.

Theorem 29. *Let f, g, and h be any three entire functions with (i) $0<\rho_h^{L^*}(g)<\infty$, (ii) $0<\sigma_h^{L^*}(g)<\infty$, (iii) $\rho_h^{L^*}(f\circ g)=\rho_h^{L^*}(g)$, and (iv) $\sigma_h^{L^*}(f\circ g)<\infty$. Then for any $\alpha>1$*

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_g(r)} \leq \frac{(2\alpha-1)^{2\rho_h^{L^*}(g)}\cdot\alpha^{\rho_h^{L^*}(g)+1}}{(\alpha-1)^{2\rho_h^{L^*}(g)}}\cdot\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(g)},$$
$$\frac{(\alpha-1)^{2\rho_h^{L^*}(g)}}{(2\alpha-1)^{2\rho_h^{L^*}(g)}\alpha^{\rho_h^{L^*}(g)_{+1}}}\cdot\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(g)} \leq \limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_g(r)}. \tag{63}$$

Theorem 30. *Let f, g, and h be any three entire functions such that (i) $0<\rho_h^{L^*}(f)<\infty$, (ii) $0<\sigma_h^{L^*}(f)<\infty$, (iii) $\rho_h^{L^*}(f\circ g)=\rho_h^{L^*}(f)$, and (iv) $\sigma_h^{L^*}(f\circ g)<\infty$. Then*

$$\liminf_{r\to\infty}\frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)} \leq \frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(f)} \leq \limsup_{r\to\infty}\frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f(r)}. \tag{64}$$

Theorem 31. *Let f, g, and h be any three entire functions with (i) $0<\rho_h^{L^*}(g)<\infty$, (ii) $0<\sigma_h^{L^*}(g)<\infty$, (iii) $\rho_h^{L^*}(f\circ g)=\rho_h^{L^*}(g)$, and (iv) $\sigma_h^{L^*}(f\circ g)<\infty$. Then*

$$\liminf_{r\to\infty}\frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_g(r)} \leq \frac{\sigma_h^{L^*}(f\circ g)}{\sigma_h^{L^*}(g)} \leq \limsup_{r\to\infty}\frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_g(r)}. \tag{65}$$

The proofs of Theorems 30 and 31 are omitted because those can be carried out in the line of Theorems 28 and 29, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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