Research Article

Generalized Growth of Special Monogenic Functions

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Received 6 January 2014; Accepted 2 March 2014; Published 1 April 2014

Academic Editor: Yan Xu

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We study the generalized growth of special monogenic functions. The characterizations of generalized order, generalized lower order, generalized type, and generalized lower type of special monogenic functions have been obtained in terms of their Taylor's series coefficients.

1. Introduction

Clifford analysis offers the possibility of generalizing complex function theory to higher dimensions. It considers Clifford algebra valued functions that are defined in open subsets of $\mathbb{R}^n$ for arbitrary finite $n \in \mathbb{N}$ and that are solutions of higher-dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions.

In order to make calculations more concise, we use the following notations, where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ is n -dimensional multi-index and $\mathbf{x} \in \mathbb{R}^n$:

$$\begin{aligned} \mathbf{x}^{\mathbf{m}} &= x_1^{m_1}, \dots, x_n^{m_n}, \\ \mathbf{m}! &= m_1! \dots m_n!, \\ |\mathbf{m}| &= m_1 + \dots + m_n. \end{aligned} \quad (1)$$

Following Almeida and Kraußhar [1] and Constaes et al. [2, 3], we give some definitions and associated properties.

By $\{e_1, e_2, \dots, e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The associated real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo $\mathbf{x}^2 = -||\mathbf{x}||^2 e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n} . In the Clifford algebra Cl_{0n} , the following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, 2, \dots, n, \quad (2)$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, 2, \dots, n\}\}$ with $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$, where $1 \leq i_1 < i_2 < \dots < i_r \leq n$, $e_\emptyset = e_0 = 1$.

Each $a \in \text{Cl}_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. The conjugation in Clifford algebra Cl_{0n} is defined by $\bar{a} = \sum_A a_A \bar{e}_A$, where $\bar{e}_A = \bar{e}_{i_r} \bar{e}_{i_{r-1}} \dots \bar{e}_{i_1}$ and $\bar{e}_j = -e_j$ for $j = 1, 2, \dots, n$, $\bar{e}_0 = e_0 = 1$. The linear subspace $\text{span}_{\mathbb{R}}\{1, e_1, \dots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset \text{Cl}_{0n}$ is the so-called space of paravectors $z = x_0 + x_1 e_1 + x_2 e_2 + \dots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. Here, $x_0 = \text{Sc}(z)$ is scalar part and $\mathbf{x} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n = \text{Vec}(z)$ is vector part of paravector z . The Clifford norm of an arbitrary $a = \sum_A a_A e_A$ is given by

$$\|a\| = \left(\sum_A |a_A|^2 \right)^{1/2}. \quad (3)$$

Also, for $b \in \text{Cl}_{0n}$, we have $\|ab\| \leq 2^{n/2} \|a\| \|b\|$. Each paravector $z \in \mathbb{R}^{n+1} \setminus \{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ which can be represented in the form $z^{-1} = \bar{z}/||z||^2$. In order to make calculations more concise, we use the following notation:

$$\begin{aligned} k_{\mathbf{m}} &= \binom{n + |\mathbf{m}| - 1}{|\mathbf{m}|} = \frac{(n)_{\mathbf{m}}}{\mathbf{m}!}, \\ (n)_{\mathbf{m}} &= n(n+1) \dots (n + |\mathbf{m}| - 1). \end{aligned} \quad (4)$$

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}. \quad (5)$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, then a function $g : U \rightarrow \text{Cl}_{0n}$ is called left (right) monogenic at a point $z \in U$ if $Dg(z) = 0$ ($gD(z) = 0$). The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Following Abul-Ez and Constaes [4], we consider the class of monogenic polynomials $p_{\mathbf{m}}$ of degree $|\mathbf{m}|$, defined as

$$p_{\mathbf{m}}(z) = \sum_{i+j=|\mathbf{m}|} \frac{((n-1)/2)_i}{i!} \frac{((n+1)/2)_j}{j!} (\bar{z})^i z^j. \quad (6)$$

Let w_n be n -dimensional surface area of $n+1$ -dimensional unit ball and let S^n be n -dimensional sphere. Then, the class of monogenic polynomials described in (6) satisfies (see [5], pp. 1259)

$$\frac{1}{w_{\mathbf{m}}} \int_{S^n} \overline{p_{\mathbf{m}}(z)} p_{\mathbf{l}}(z) dS_z = k_{\mathbf{m}} \delta_{|\mathbf{m}||\mathbf{l}|}. \quad (7)$$

Also following Abul-Ez and De Almeida [5], we have

$$\max_{\|z\|=r} \|p_{\mathbf{m}}(z)\| = k_{\mathbf{m}} r^{\mathbf{m}}. \quad (8)$$

2. Preliminaries

Now following Abul-Ez and De Almeida [5], we give some definitions which will be used in the next section.

Definition 1. Let Ω be a connected open subset of $\mathbb{R}^{n+1}$ containing the origin and let $g(z)$ be monogenic in Ω . Then, $g(z)$ is called special monogenic in Ω , if and only if its Taylor's series near zero has the form (see [5], pp. 1259)

$$g(z) = \sum_{|\mathbf{m}|=0}^{\infty} p_{\mathbf{m}}(z) c_{\mathbf{m}}, \quad c_{\mathbf{m}} \in \text{Cl}_{0n}. \quad (9)$$

Definition 2. Let $g(z) = \sum_{|\mathbf{m}|=0}^{\infty} p_{\mathbf{m}}(z) c_{\mathbf{m}}$ be a special monogenic function defined on a neighborhood of the closed ball $B(0, r)$. Then,

$$\|c_{\mathbf{m}}\| \leq \frac{1}{\sqrt{k_{\mathbf{m}}}} M(r, g) r^{-\mathbf{m}}, \quad (10)$$

where $M(r, g) = \max_{\|z\|=r} \|g(z)\|$ is the maximum modulus of $g(z)$ (see [5], pp. 1260).

Definition 3. Let $g : \mathbb{R}^{n+1} \rightarrow \text{Cl}_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). Then, for $r > 0$ the maximum term of this special monogenic function is given by (see [5], pp. 1260)

$$\mu(r) = \mu(r, g) = \max_{|\mathbf{m}| \geq 0} \{\|a_{\mathbf{m}}\| k_{\mathbf{m}} r^{\mathbf{m}}\}. \quad (11)$$

Also the index $\mathbf{m}$ with maximal length $|\mathbf{m}|$ for which maximum term is achieved is called the central index and is denoted by (see [5], pp. 1260)

$$\nu(r) = \nu(r, f) = \mathbf{m}. \quad (12)$$

Definition 4. Let $g : \mathbb{R}^{n+1} \rightarrow \text{Cl}_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). Then, the order ρ and lower order λ of $g(z)$ are defined as (see [5], pp. 1263)

$$\frac{\rho}{\lambda} = \lim_{r \rightarrow \infty} \sup \inf \frac{\log \log M(r, g)}{\log r}. \quad (13)$$

Definition 5. Let $g : \mathbb{R}^{n+1} \rightarrow \text{Cl}_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). Then, the type σ and lower type ω of special monogenic function $g(z)$ having nonzero finite generalized order are defined as (see [5], pp. 1270)

$$\frac{\sigma}{\omega} = \lim_{r \rightarrow \infty} \sup \inf \frac{\log M(r, g)}{r^{\rho}}. \quad (14)$$

For generalization of the classical characterizations of growth of entire functions, Seremeta [6] introduced the concept of the generalized order and generalized type with the help of general growth functions as follows.

Let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

- (i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, and differentiable, and it tends to ∞ as $x \rightarrow \infty$,
- (ii) $\lim_{x \rightarrow \infty} (h[\{1 + 1/\psi(x)\}x]/h(x)) = 1$,
for every function $\psi(x)$ such that $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

- (i) $\lim_{x \rightarrow \infty} (h(cx)/h(x)) = 1$,
- (ii) for every $c > 0$; that is, $h(x)$ is slowly increasing.

Following Srivastava and Kumar [7] and Kumar and Bala ([8–10]), here we give definitions of generalized order, generalized lower order, generalized type, and generalized lower type of special monogenic functions. For special monogenic function $g(z)$ and functions $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, we define the generalized order $\rho(\alpha, \beta, g)$ and generalized lower order $\lambda(\alpha, \beta, g)$ of $g(z)$ as

$$\frac{\rho(\alpha, \beta, g)}{\lambda(\alpha, \beta, g)} = \lim_{r \rightarrow \infty} \sup \inf \frac{\alpha[\log M(r, g)]}{\beta(\log r)}. \quad (15)$$

If in above equation we put $\alpha(r) = \log r$ and $\beta(r) = r$, then we get definitions of order and lower order as defined by Abul-Ez and De Almeida (see [5], pp. 1263). Hence, their definitions of order and lower order are special cases of our definitions.

Further, for $\alpha(x), \beta^{-1}(x), \gamma(x) \in L^0$, we define the generalized type $\sigma(\alpha, \beta, \rho, g)$ and generalized lower type of special monogenic function $g(z)$ having nonzero finite generalized order as

$$\frac{\sigma(\alpha, \beta, \rho, g)}{\omega(\alpha, \beta, \rho, g)} = \lim_{r \rightarrow \infty} \sup \inf \frac{\alpha[\log M(r, g)]}{\beta[\{\gamma(r)\}^{\rho}]}. \quad (16)$$

If in above equation we put $\alpha(r) = r$, $\beta(r) = r$, and $\gamma(r) = r$, then we get definitions of type and lower type as defined by Abul-Ez and De Almeida (see [5], pp. 1270). Hence, their definitions of type and lower type are special cases of our definitions.

Abul-Ez and De Almeida [5] have obtained the characterizations of order, lower order, type, and lower type of special monogenic functions in terms of their Taylor's series coefficients. In the present paper we have obtained the characterizations of generalized order, generalized lower order, generalized type and generalized lower type of special monogenic functions in terms of their Taylor's series coefficients. The results obtained by Abul-Ez and De Almeida [5] are special cases of our results.

3. Main Results

We now prove the following.

Theorem 6. Let $g : \mathbb{R}^{n+1} \rightarrow Cl_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). If $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, then the generalized order ρ of $g(z)$ is given as

$$\rho = \rho(\alpha, \beta, g) = \lim_{|\mathbf{m}| \rightarrow \infty} \sup \frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}}. \quad (17)$$

Proof. Write

$$\theta = \lim_{|\mathbf{m}| \rightarrow \infty} \sup \frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}}. \quad (18)$$

Now, first we prove that $\rho \geq \theta$. The coefficients of a monogenic Taylor's series satisfy Cauchy's inequality; that is,

$$\|c_{\mathbf{m}}\| \leq \frac{1}{\sqrt{k_{\mathbf{m}}}} M(r, g) r^{-|\mathbf{m}|}. \quad (19)$$

Also from (15), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$M(r, g) \leq \exp \left[\alpha^{-1} \{ \bar{\rho} \beta(\log r) \} \right], \quad \bar{\rho} = \rho + \varepsilon. \quad (20)$$

Now, from inequality (19), we get

$$\|c_{\mathbf{m}}\| \leq \frac{1}{\sqrt{k_{\mathbf{m}}}} r^{-|\mathbf{m}|} \exp \left[\alpha^{-1} \{ \bar{\rho} \beta(\log r) \} \right]. \quad (21)$$

Since $(1/\sqrt{k_{\mathbf{m}}}) \leq 1$, (see [11], pp. 148) so the above inequality reduces to

$$\|c_{\mathbf{m}}\| \leq r^{-|\mathbf{m}|} \exp \left[\alpha^{-1} \{ \bar{\rho} \beta(\log r) \} \right]. \quad (22)$$

Putting $r = \exp[\beta^{-1}\{\alpha(|\mathbf{m}|)/\bar{\rho}\}]$ in the above inequality, we get, for all large values of $|\mathbf{m}|$,

$$\|c_{\mathbf{m}}\| \leq \exp \left[|\mathbf{m}| - |\mathbf{m}| \beta^{-1} \left\{ \frac{\alpha(|\mathbf{m}|)}{\bar{\rho}} \right\} \right] \quad (23)$$

or

$$\beta^{-1} \{ \alpha(|\mathbf{m}|) / \bar{\rho} \} \leq 1 - \frac{1}{|\mathbf{m}|} \{ \log \|c_{\mathbf{m}}\| \} \quad (24)$$

or

$$\frac{\alpha(|\mathbf{m}|)}{\beta \left\{ 1 + \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}} \leq \bar{\rho} \quad (25)$$

or

$$\frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}} \leq \bar{\rho} \frac{\beta \left\{ 1 + \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}}. \quad (26)$$

Since $\beta(x) \in L^0$, $\beta(1+x) \approx \beta(x)$. Hence, proceeding to limits as $|\mathbf{m}| \rightarrow \infty$, we get

$$\theta = \lim_{|\mathbf{m}| \rightarrow \infty} \sup \frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}} \leq \bar{\rho}. \quad (27)$$

Since $\varepsilon > 0$ is arbitrarily small, so finally we get

$$\theta \leq \rho. \quad (28)$$

Now, we will prove that $\theta \geq \rho$. If $\theta = \infty$, then there is nothing to prove. So let us assume that $0 \leq \theta < \infty$. Therefore, for a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for all multi-indices $\mathbf{m}$ with $|\mathbf{m}| > n_0$, we have

$$0 \leq \frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}} < \theta + \varepsilon = \bar{\theta} \quad (29)$$

or

$$\|c_{\mathbf{m}}\| \leq \exp \left[-|\mathbf{m}| \beta^{-1} \left\{ \frac{\alpha(|\mathbf{m}|)}{\bar{\theta}} \right\} \right]. \quad (30)$$

Now, from the property of maximum modulus (see [11], pp. 148), we have

$$M(r, g) \leq \sum_{|\mathbf{m}|=0}^{\infty} \|c_{\mathbf{m}}\| k_{\mathbf{m}} r^{|\mathbf{m}|} \quad (31)$$

or

$$M(r, g) \leq \sum_{|\mathbf{m}|=0}^{\infty} k_{\mathbf{m}} r^{|\mathbf{m}|} \exp \left[-|\mathbf{m}| \beta^{-1} \left\{ \frac{\alpha(|\mathbf{m}|)}{\bar{\theta}} \right\} \right]. \quad (32)$$

On the lines of the proof of the theorem given by Srivastava and Kumar (see [7], Theorem 2.1, pp. 666), we get

$$\rho \leq \theta. \quad (33)$$

Combining this with inequality (28), we get (17). Hence, Theorem 6 is proved. $\square$

Next, we prove the following.

Theorem 7. Let $g : \mathbb{R}^{n+1} \rightarrow Cl_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). Also let $\alpha(x), \beta(x), \gamma(x) \in L^0$ and $0 < \rho < \infty$; then the generalized type σ of $g(z)$ is given as

$$\sigma = \sigma(\alpha, \beta, \rho, g) = \lim_{|\mathbf{m}| \rightarrow \infty} \sup \frac{\alpha(|\mathbf{m}|/\rho)}{\beta \left[\left\{ \gamma \left(e^{1/\rho} \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right) \right\}^{\rho} \right]}. \quad (34)$$

Proof. Write

$$\eta = \lim_{|\mathbf{m}| \rightarrow \infty} \sup \frac{\alpha(|\mathbf{m}|/\rho)}{\beta \left[\left\{ \gamma \left(e^{1/\rho} \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right) \right\}^\rho \right]}. \quad (35)$$

Now, first we prove that $\sigma \geq \eta$. From (16), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$M(r, g) \leq \exp \left[\alpha^{-1} \left\{ \bar{\sigma} \beta \left[\left\{ \gamma(r) \right\}^\rho \right] \right\} \right], \quad (36)$$

where $\bar{\sigma} = \sigma + \varepsilon$. Now, using (19), we get

$$\|c_{\mathbf{m}}\| \leq \frac{1}{\sqrt{k_{\mathbf{m}}}} r^{-|\mathbf{m}|} \exp \left[\alpha^{-1} \left\{ \bar{\sigma} \beta \left[\left\{ \gamma(r) \right\}^\rho \right] \right\} \right]. \quad (37)$$

Now, as in the proof of Theorem 6, here this inequality reduces to

$$\|c_{\mathbf{m}}\| \leq r^{-|\mathbf{m}|} \exp \left[\alpha^{-1} \left\{ \bar{\sigma} \beta \left[\left\{ \gamma(r) \right\}^\rho \right] \right\} \right]. \quad (38)$$

Putting $r = \gamma^{-1} \left(\left[\beta^{-1} \left\{ (1/\bar{\sigma}) \alpha(|\mathbf{m}|/\rho) \right\} \right]^{1/\rho} \right)$, we get

$$\begin{aligned} \|c_{\mathbf{m}}\| &\leq \left\{ \gamma^{-1} \left(\left[\beta^{-1} \left\{ \frac{1}{\bar{\sigma}} \alpha \left(\frac{|\mathbf{m}|}{\rho} \right) \right\} \right]^{1/\rho} \right) \right\}^{-|\mathbf{m}|} \\ &\quad \times \exp \left(\frac{|\mathbf{m}|}{\rho} \right) \end{aligned} \quad (39)$$

or

$$\|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \geq \left\{ \gamma^{-1} \left(\left[\beta^{-1} \left\{ \frac{1}{\bar{\sigma}} \alpha \left(\frac{|\mathbf{m}|}{\rho} \right) \right\} \right]^{1/\rho} \right) \right\} \exp \left(\frac{-1}{\rho} \right) \quad (40)$$

or

$$\frac{\alpha(|\mathbf{m}|/\rho)}{\beta \left[\left\{ \gamma \left(e^{1/\rho} \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right) \right\}^\rho \right]} \leq \bar{\sigma}. \quad (41)$$

Now, proceeding to limits as $|\mathbf{m}| \rightarrow \infty$, we get

$$\eta \leq \bar{\sigma}. \quad (42)$$

Since $\varepsilon > 0$ is arbitrarily small, so finally we get

$$\eta \leq \sigma. \quad (43)$$

Now, we will prove that $\eta \geq \sigma$. If $\eta = \infty$, then there is nothing to prove. So let us assume that $0 \leq \eta < \infty$. Therefore, for a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for all multi-indices $\mathbf{m}$ with $|\mathbf{m}| > n_0$, we have

$$0 \leq \frac{\alpha(|\mathbf{m}|/\rho)}{\beta \left[\left\{ \gamma \left(e^{1/\rho} \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right) \right\}^\rho \right]} \leq \eta + \varepsilon = \bar{\eta} \quad (44)$$

or

$$\|c_{\mathbf{m}}\| \leq \left(\gamma^{-1} \left\{ \left[\beta^{-1} \left\{ \frac{1}{\bar{\eta}} \alpha \left(\frac{|\mathbf{m}|}{\rho} \right) \right\} \right]^{1/\rho} \right\} \right)^{-|\mathbf{m}|} e^{|\mathbf{m}|/\rho}. \quad (45)$$

Now, from the property of maximum modulus (see [11], pp. 148), we have

$$M(r, g) \leq \sum_{|\mathbf{m}|=0}^{\infty} \|c_{\mathbf{m}}\| k_{\mathbf{m}} r^{|\mathbf{m}|} \quad (46)$$

or

$$\begin{aligned} M(r, g) &\leq \sum_{|\mathbf{m}|=0}^{\infty} k_{\mathbf{m}} r^{|\mathbf{m}|} \\ &\quad \times \left(\gamma^{-1} \left\{ \left[\beta^{-1} \left\{ \frac{1}{\bar{\eta}} \alpha \left(\frac{|\mathbf{m}|}{\rho} \right) \right\} \right]^{1/\rho} \right\} \right)^{-|\mathbf{m}|} e^{|\mathbf{m}|/\rho}. \end{aligned} \quad (47)$$

On the lines of the proof of the theorem given by Srivastava and Kumar (see [7], Theorem 2.2, pp. 670), we get

$$\sigma \leq \eta. \quad (48)$$

Combining this with (43), we get (34). Hence, Theorem 7 is proved. $\square$

Next, we have the following.

Theorem 8. Let $g : \mathbb{R}^{n+1} \rightarrow Cl_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). If $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$ then the generalized lower order λ of $g(z)$ satisfies

$$\lambda = \lambda(\alpha, \beta, g) \geq \lim_{|\mathbf{m}| \rightarrow \infty} \inf \frac{\alpha(|\mathbf{m}|)}{\beta \left\{ \log \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right\}}. \quad (49)$$

Further, if

$$\psi(t) = \max_{|\mathbf{m}'|=t} \left\{ \frac{\|c_{\mathbf{m}}\| k_{\mathbf{m}}}{\|c_{\mathbf{m}'}\| k_{\mathbf{m}'}}}, |\mathbf{m}'| = |\mathbf{m}| + 1 \right\} \quad (50)$$

is a nondecreasing function of t , then equality holds in (49).

Proof. The proof of the above theorem follows on the lines of the proof of Theorem 6 and [7] Theorem 2.4 (pp. 674). Hence, we omit the proof. $\square$

Next, we have the following.

Theorem 9. Let $g : \mathbb{R}^{n+1} \rightarrow Cl_{0n}$ be a special monogenic function whose Taylor's series representation is given by (9). Also let $(x), \beta(x), \gamma(x) \in L^0$ and $0 < \rho < \infty$; then the generalized lower type ω of $g(z)$ satisfies

$$\omega = \omega(\alpha, \beta, \rho, g) \geq \lim_{|\mathbf{m}| \rightarrow \infty} \inf \frac{\alpha(|\mathbf{m}|/\rho)}{\beta \left[\left\{ \gamma \left(e^{1/\rho} \|c_{\mathbf{m}}\|^{-1/|\mathbf{m}|} \right) \right\}^\rho \right]}. \quad (51)$$

Further, if

$$\psi(t) = \max_{|\mathbf{m}'|=t} \left\{ \frac{\|c_{\mathbf{m}}\| k_{\mathbf{m}}}{\|c_{\mathbf{m}'}\| k_{\mathbf{m}'}}}, |\mathbf{m}'| = |\mathbf{m}| + 1 \right\} \quad (52)$$

is a nondecreasing function of t , then equality holds in (51).

Proof. The proof of the above theorem follows on the lines of the proof of Theorem 7 and [7] Theorem 2.4 (pp. 674). Hence, we omit the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author is very thankful to the referee for the valuable comments and observations which helped in improving the paper.

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