# Multiresolution Analysis Based on Coalescence Hidden-Variable Fractal Interpolation Functions 

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#### Abstract

Multiresolution analysis arising from Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) is developed. The availability of a larger set of free variables and constrained variables with CHFIF in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_{2}(\mathbb{R})$ than that provided by multiresolution analysis based only on Affine Fractal Interpolation Functions (AFIFs). Our approach consists of introduction of the vector space of CHFIFs, determination of its dimension and construction of Riesz bases of vector subspaces $\mathbb{V}_{k}, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_{2}(\mathbb{R}) \cap C_{0}(\mathbb{R})$.


## 1. Introduction

The theory of multiresolution analysis provides a powerful method to construct wavelets having far reaching applications in analyzing signals and images [1, 2]. They permit efficient representation of functions at multiple levels of detail; that is, a function $f \in L_{2}(\mathbb{R})$, the space of real valued functions $g$ satisfying $\|g\|_{L^{2}}=\int_{\mathbb{R}}|g(x)|^{2} d x<\infty$, could be written as limit of successive approximations, each of which is smoothed version of $f$. The multiresolution analysis was first introduced by Mallat [3] and Meyer [4] using a single function. The multiresolution analysis based upon several functions was developed in [5-7]. In [8], multiresolution analysis of $L_{2}(\mathbb{R})$ was generated from certain classes of Affine Fractal Interpolation Functions (AFIFs). Such results were then generalized to several dimensions in [9, 10]. In [11], orthonormal basis for the vector space of AFIFs was explicitly constructed. A few years later, Donovan et al. [12] constructed orthogonal compactly supported continuous wavelets using multiresolution analysis arising from AFIFs. Bouboulis [13] generated multiresolution analysis of $L_{2}\left(\mathbb{R}^{d}\right)$ using AFIF on $[0,1]^{d}$ and constructed multiwavelets which are orthonormal but discontinuous. The interrelations among AFIFs, multiresolution analysis and wavelets are treated in [14]. In [15],
multiresolution analysis is developed for a Hilbert space constructed from Hausdorff measures $\mathscr{H}^{s}, 0<s<1$ on $\mathbb{R}$ and, in particuluar, on a Cantor set using linear contraction. This development was later improved in [16] by employing a nonlinear fractal system to construct wavelets with Fourier basis with respect to some fractal measure. The details on implementation of recent work on multiresolution analysis with AFIF bases can be found in [17]. It is desirable [18] that the wavelet function should reflect the features present in the original function but AFIF based wavelets generally cannot exhibit satisfactorily the features of functions simulating natural objects or outcome of scientific experiments that are partly self-affine and partly non-self-affine. The Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) introduced in [19] are ideally suited for such purposes. However, multiresolution analysis of $L_{2}(\mathbb{R})$ based on CHFIFs has hitherto remained unexplored. In the present work, such a multiresolution analysis using CHFIFs as basis functions is developed. The availability of a larger set of free variables and constrained variables in our multiresolution analysis based on CHFIFs additionally provides more control in reconstruction of functions in $L_{2}(\mathbb{R})$ than that provided by multiresolution analysis based only on affine FIFs.

The organization of the paper is as follows. The construction of a CHFIF is briefly summarized in Section 2. The vector space of CHFIFs is introduced in Section 3 and a few auxiliary results, including a result on determination of dimension of this vector space, are found in this section. In Section 4, first Riesz bases of vector subspaces $\mathbb{V}_{k}, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_{2}(\mathbb{R}) \bigcap C_{0}(\mathbb{R})$ are constructed. The multiresolution analysis of $L_{2}(\mathbb{R})$ is then carried out in this section in terms of nested sequences of vector subspaces $\mathbb{V}_{k}, k \in \mathbb{Z}$.

## 2. Construction of a CHFIF

In this section, a brief introduction on the construction of CHFIF is given. A Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) is constructed as the graph of the attractor of a suitably defined Iterated Function System (IFS).

Given an interpolation data $\left\{\left(x_{i}, y_{i}\right) \in \mathbb{R}^{2}: i=0,1, \ldots\right.$, $N\}$, where $0=x_{0}<x_{1}<\cdots<x_{N}=1$, a CHFIF is constructed as follows. Consider a generalized interpolation data $\left\{\left(x_{i}, y_{i}, z_{i}\right) \in \mathbb{R}^{3}: i=0,1, \ldots, N\right\}$, where $z_{i}$ are real numbers. We denote the interval $\left[x_{0}, x_{N}\right]$ by $I$ and the intervals $\left[x_{n-1}, x_{n}\right]$ by $I_{n}$ for $n=1,2, \ldots, N$. Define the functions $L_{n}: I \rightarrow I_{n}$ and $F_{n}: I \times \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ by

$$
\begin{gather*}
L_{n}(x)=a_{n} x+b_{n}  \tag{1}\\
F_{n}(x, y, z)=\left(\alpha_{n} y+\beta_{n} z+p_{n}(x), \gamma_{n} z+q_{n}(x)\right) \tag{2}
\end{gather*}
$$

where $a_{n}=\left(x_{n}-x_{n-1}\right) /\left(x_{N}-x_{0}\right), b_{n}=\left(x_{N} x_{n-1}-x_{0} x_{n}\right) /\left(x_{N}-\right.$ $x_{0}$ ), and the functions $F_{n}$ satisfy the join-up conditions:

$$
\begin{equation*}
F_{n}\left(x_{0}, y_{0}, z_{0}\right)=\left(y_{n-1}, z_{n-1}\right), \quad F_{n}\left(x_{N}, y_{N}, z_{N}\right)=\left(y_{n}, z_{n}\right) . \tag{3}
\end{equation*}
$$

In (2), the variables $\alpha_{n}, \gamma_{n}$ are free variables, $\beta_{n}$ are constrained variables such that $\left|\alpha_{n}\right|<1,\left|\gamma_{n}\right|<1,\left|\beta_{n}\right|+\left|\gamma_{n}\right|<1$, and $p_{n}(x)$, $q_{n}(x)$ are linear polynomials given by

$$
\begin{equation*}
p_{n}(x)=c_{n} x+d_{n}, \quad q_{n}(x)=e_{n} x+h_{n} . \tag{4}
\end{equation*}
$$

The desired IFS for construction of a CHFIF for the generalized interpolation data $\left\{\left(x_{i}, y_{i}, z_{i}\right) \in \mathbb{R}^{3}: i=0,1, \ldots, N\right\}$ is now defined as

$$
\begin{equation*}
\left\{\mathbb{R}^{3}, \omega_{n}: n=1,2, \ldots N\right\} \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{n}(x, y, z)=\left(L_{n}(x), F_{n}(x, y, z)\right) \tag{6}
\end{equation*}
$$

The following theorem gives the existence of an attractor of the IFS defined by (5) associated with the generalized interpolation data.

Theorem 1 (see [19]). Let $\left\{\mathbb{R}^{3}, \omega_{n}: n=1, \ldots N\right\}$ be the IFS defined by (5) associated with the generalized data $\left\{\left(x_{i}, y_{i}, z_{i}\right) \in \mathbb{R}^{3}: i=0,1, \ldots, N\right\}$. Let $\alpha_{n}, \beta_{n}$ and $\gamma_{n}$ in the definition of $\omega_{n}$ satisfy $\left|\alpha_{n}\right|<1,\left|\gamma_{n}\right|<1$ and $\left|\beta_{n}\right|+\left|\gamma_{n}\right|<1$
for $n=1, \ldots, N$. Then there exists a metric $\tau$ on $\mathbb{R}^{3}$, equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to $\tau$. In particular, there exists a unique nonempty compact set $G \subseteq \mathbb{R}^{3}$ such that $G=\bigcup_{n=1}^{N} \omega_{n}(G)$.

The following theorem is instrumental for precise definition of a CHFIS.

Theorem 2 (see [19]). Let $G$ be the attractor of the IFS for the given interpolation data. Then $G$ is graph of a continuous function $f: I \rightarrow \mathbb{R}^{2}$ such that $f\left(x_{i}\right)=\left(y_{i}, z_{i}\right)$ for $i=$ $0,1, \ldots, N$, that is, $G=\{(x, f(x)): x \in I$ and $f(x)=$ $(y(x), z(x))\}$.

Suppose $f(x)$ is written component-wise as $f(x)=$ $\left(f_{1}(x), f_{2}(x)\right)$. Then the Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) is defined as follows.

Definition 3. The Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) for the given interpolation data $\left\{\left(x_{i}, y_{i}\right): i=0,1, \ldots, N\right\}$ is defined as the first component $f_{1}(x)$ of the function $f(x)$.

It is easily seen that the graph of CHFIF $f_{1}(x)$ is the projection of the graph of $f(x)$ on $\mathbb{R}^{2}$.

## 3. Auxiliary Results

In order to develop the multiresolution analysis of $L_{2}(\mathbb{R})$ based on CHFIF, the space of CHFIF is introduced in this section. Further, a few auxiliary results and a result on the dimension of the vector space of CHFIF are found here.

Let

$$
\begin{equation*}
t_{n}=\left(p_{n}, q_{n}\right) \tag{7}
\end{equation*}
$$

and $t=\left(t_{1}, t_{2}, \ldots, t_{N}\right)$, where $p_{n}$ and $q_{n}$ are polynomials of degree at most 1 given by (4). Then, $T=\left\{t=\left(t_{1}, \ldots, t_{N}\right)\right.$ : $\left.t_{i}=\left(p_{i}, q_{i}\right), p_{i}, q_{i} \in \mathscr{P}_{1}, i=1,2, \ldots, N\right\}$ is a vector space with usual pointwise addition and scalar multiplication, where $\mathscr{P}_{1}$ is the class of polynomials of degree at most 1 . It is easily seen that on $\mathbb{B}\left(I, \mathbb{R}^{2}\right)$, the set of bounded functions from $I$ to $\mathbb{R}^{2}$ with respect to maximum metric $d^{*}(f, g)=\max _{x \in I}\left\{\mid f_{1}(x)-\right.$ $g_{1}(x)\left|,\left|f_{2}(x)-g_{2}(x)\right|\right\}$, the function $\Phi_{t}$ defined by

$$
\begin{equation*}
\left(\Phi_{t} f\right)(x)=F_{n}\left(L_{n}^{-1}(x), f\left(L_{n}^{-1}(x)\right)\right) \tag{8}
\end{equation*}
$$

for $x \in I_{n}, n=1,2, \ldots, N$, is a contraction map. Therefore, by Banach contraction mapping theorem, $\Phi_{t}$ has a unique fixed point $f_{t} \in \mathbb{B}\left(I, \mathbb{R}^{2}\right)$. By join-up conditions (3), it follows that $f_{t} \in \mathbb{C}\left(I, \mathbb{R}^{2}\right)$, the set of continuous functions from $I$ to $\mathbb{R}^{2}$. The following proposition gives the existence of a linear isomorphism between the vector space $T$ and the vector subspace $\mathscr{A}$ of $\mathbb{C}\left(I, \mathbb{R}^{2}\right)$ defined by $\mathscr{A}=\left\{f=\left(f_{1}, f_{2}\right) \in\right.$ $\mathbb{C}\left(I, \mathbb{R}^{2}\right): f_{2} \circ L_{n}-\gamma_{n} f_{2} \in \mathscr{P}_{1}$ and $f_{1} \circ L_{n}-\alpha_{n} f_{1}-\beta_{n} f_{2} \in$ $\mathscr{P}_{1}$ for all $\left.n=1,2, \ldots, N\right\}$.

Proposition 4. The mapping $\Theta: T \rightarrow \mathscr{A} \subset \mathbb{C}\left(I, \mathbb{R}^{2}\right)$ defined by $\Theta(t)=f_{t}$ is a linear isomorphism.

Proof. The assertion of the proposition is proved by establishing
(i) $\left(a f_{t}+f_{s}\right)_{i}(x)=a f_{t, i}(x)+f_{s, i}(x), i=1,2$, where $f_{t}$ and $a f_{t}+f_{s}$ are written component-wise as $f_{t}=\left(f_{t, 1}, f_{t, 2}\right)$ and $a f_{t}+f_{s}=\left(\left(a f_{t}+f_{s}\right)_{1},\left(a f_{t}+f_{s}\right)_{2}\right),(\mathrm{ii})\left(a f_{t}+f_{s}\right)=f_{a t+s}$, (iii) $\Theta$ is onto and (iv) $\Theta$ is one-one.

The identity (i) follows by equating the components of left and right hand side in the identity $\left(a f_{t}+f_{s}\right)(x)=a f_{t}(x)+$ $f_{s}(x)$.
(ii) $\left(a f_{t}+f_{s}\right)=f_{a t+s}:$ By the definition of $\Phi_{t}$ :

$$
\begin{align*}
& \left(\Phi_{a t+s}\left(a f_{t}+f_{s}\right)\right)(x) \\
& =F_{n}\left(L_{n}^{-1}(x),\left(a f_{t}+f_{s}\right)\left(L_{n}^{-1}(x)\right)\right) \\
& =\left(\alpha_{n}\left(a f_{t}+f_{s}\right)_{1}\left(L_{n}^{-1}(x)\right)+\beta_{n}\left(a f_{t}+f_{s}\right)_{2}\left(L_{n}^{-1}(x)\right)\right. \\
& \quad+\left(a p_{n}+\widehat{p}_{n}\right)\left(L_{n}^{-1}(x)\right), \gamma_{n}\left(a f_{t}+f_{s}\right)_{2}\left(L_{n}^{-1}(x)\right) \\
& \left.\quad+\left(a q_{n}+\widehat{q}_{n}\right)\left(L_{n}^{-1}(x)\right)\right) . \tag{9}
\end{align*}
$$

Using identity (i), it follows that

$$
\begin{align*}
& \left(\Phi_{a t+s}\left(a f_{t}+f_{s}\right)\right)(x) \\
& =\left(\alpha_{n}\left(a f_{t, 1}\left(L_{n}^{-1}(x)\right)+f_{s, 1}\left(L_{n}^{-1}(x)\right)\right)\right. \\
& \quad+\beta_{n}\left(a f_{t, 2}\left(L_{n}^{-1}(x)\right)+f_{s, 2}\left(L_{n}^{-1}(x)\right)\right) \\
& \quad+\left(a p_{n}\left(L_{n}^{-1}(x)\right)+\widehat{p}_{n}\left(L_{n}^{-1}(x)\right)\right)  \tag{10}\\
& \quad \gamma_{n}\left(a f_{t, 2}\left(L_{n}^{-1}(x)\right)+f_{s, 2}\left(L_{n}^{-1}(x)\right)\right) \\
& \left.\quad+\left(a q_{n}\left(L_{n}^{-1}(x)\right)+\widehat{q}_{n}\left(L_{n}^{-1}(x)\right)\right)\right)
\end{align*}
$$

The above equation gives the following on simplification:

$$
\begin{align*}
& \left(\Phi_{a t+s}\left(a f_{t}+f_{s}\right)\right)(x) \\
& =a\left(\alpha_{n} f_{t, 1}\left(L_{n}^{-1}(x)\right)+\beta_{n} f_{t, 2}\left(L_{n}^{-1}(x)\right)+p_{n}\left(L_{n}^{-1}(x)\right),\right. \\
& \left.\quad \gamma_{n} f_{t, 2}\left(L_{n}^{-1}(x)\right)+q_{n}\left(L_{n}^{-1}(x)\right)\right) \\
& +\left(\alpha_{n} f_{s, 1}\left(L_{n}^{-1}(x)\right)+\beta_{n} f_{s, 2}\left(L_{n}^{-1}(x)\right)+\widehat{p}_{n}\left(L_{n}^{-1}(x)\right),\right. \\
& \left.\quad \gamma_{n} f_{s, 2}\left(L_{n}^{-1}(x)\right)+\widehat{q}_{n}\left(L_{n}^{-1}(x)\right)\right) \\
& =a f_{t}(x)+f_{s}(x) . \tag{11}
\end{align*}
$$

Therefore, $a f_{t}+f_{s}$ is a fixed point of $\Phi_{a t+s}$ for all $a \in \mathbb{R}$ and $t, s \in T$. By uniqueness of fixed point of $\Phi_{a t+s}$, it follows that $\left(a f_{t}+f_{s}\right)=f_{a t+s}$.
(iii) $\Theta$ is onto: Let $f=\left(f_{1}, f_{2}\right) \in \mathscr{A} \subset \mathbb{C}\left(I, \mathbb{R}^{2}\right)$. Define $q_{n}(f)=f_{2} \circ L_{n}-\gamma_{n} f_{2}$ and $p_{n}(f)=f_{1} \circ L_{n}-\alpha_{n} f_{1}-\beta_{n} f_{2}$ for $n=1, \ldots, N$. Suppose $t(f)=\left(t_{1}(f), t_{2}(f), \ldots, t_{N}(f)\right)$, where $t_{n}(f)=\left(p_{n}(f), q_{n}(f)\right)$. Then $t(f) \in T$ whenever $f \in \mathscr{A}$. Also $f_{t(f)}=f$.
(iv) $\Theta$ is one-one: Let $f_{t}(x)=(0,0)$ for all values of $x \in I$. Then, $f_{t}(x)=(0,0) \Leftrightarrow \Phi_{t}\left(f_{t}\right)(x)=(0,0) \Leftrightarrow$ $F_{n}\left(L_{n}^{-1}(x), f_{t}\left(L_{n}^{-1}(x)\right)\right)=(0,0) \Leftrightarrow\left(p_{n}, q_{n}\right)=(0,0)$ for every $n \Leftrightarrow t=(0, \ldots, 0)$.

To introduce the space of CHFIFs, let the set $\mathcal{S}_{0}$ consisting of functions $f: I \rightarrow \mathbb{R}^{2}$ be defined as $\mathcal{S}_{0}=\{f: f=$ $\left(f_{1}, f_{2}\right), f_{1}$ is a CHFIF passing through $\left\{\left(x_{i}, y_{i}\right) \in \mathbb{R}^{2}: i=\right.$ $0,1, \ldots, N\}$, and $f_{2}$ is an AFIF passing through $\left\{\left(x_{i}, z_{i}\right) \in\right.$ $\left.\left.\mathbb{R}^{2}: i=0,1, \ldots, N\right\}\right\}$. Then, $\mathcal{S}_{0}$ is a vector space, with usual pointwise addition and scalar multiplication. The desired space of CHFIFs is now defined as follows.

Definition 5. Let $\mathcal{S}_{0}^{1}$ be the set of functions $f_{1}: I \rightarrow \mathbb{R}$ that are first components of functions $f \in \mathcal{S}_{0}$. The space of CHFIFs is the set $\mathcal{S}_{0}^{1}$ together with the maximum metric $d^{*}(f, g)=\max _{x \in I}|f(x)-g(x)|$.

It is easily seen that the space of CHFIFs $\mathcal{S}_{0}^{1}$ is also a vector space with pointwise addition and scalar multiplication. The following proposition gives the dimension of $\mathcal{S}_{0}^{1}$.

Proposition 6. The dimension of space of CHFIFs is $2 N$.
Proof. Consider the operator $\Phi_{t} f=\left(\Phi_{t, 1} f_{1}, \Phi_{t, 2} f_{2}\right)$. The operators $\Phi_{t, i}: \mathbb{B}(I, \mathbb{R}) \rightarrow \mathbb{B}(I, \mathbb{R}), i=1,2$, where $\mathbb{B}(I, \mathbb{R})$ is the set of bounded functions, satisfy

$$
\begin{equation*}
\Phi_{t, 1} f_{1}(x)=\alpha_{n} f_{1}\left(L_{n}^{-1}(x)\right)+\beta_{n} f_{2}\left(L_{n}^{-1}(x)\right)+p_{n}\left(L_{n}^{-1}(x)\right) \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
\Phi_{t, 2} f_{2}(x)=\gamma_{n} f_{2}\left(L_{n}^{-1}(x)\right)+q_{n}\left(L_{n}^{-1}(x)\right) \tag{13}
\end{equation*}
$$

for $x \in\left[x_{n-1}, x_{n}\right]$. By Proposition 4 and (13), it follows that $f_{2}$ is completely determined by $f_{2}(i / N)$ for $i=0,1, \ldots, N$. Further, it follows by (12) that $f_{1}$ depends on $f_{2}$. Then, for $f=$ $\left(f_{1}, f_{2}\right) \in \mathcal{S}_{0}$, the function $f_{1}$ is the unique CHFIF passing through $\left(i / N, y_{i}\right)$, while the function $f_{2}$ is the unique AFIF passing through $\left(i / N, z_{i}\right)$. Hence,

$$
\begin{equation*}
\text { dimension of } \mathcal{S}_{0}=2(N+1) \tag{14}
\end{equation*}
$$

Now, consider the projection map $P: \mathcal{S}_{0} \rightarrow \mathcal{S}_{0}^{1}$. Then, kernel of $P \equiv\left\{f \in \mathcal{S}_{0}\right.$ such that $\left.P(f)=0\right\}$ is a proper subset of $\mathcal{S}_{0}$ and consists of elements of the forms $(0,0)$ and $\left(0, f_{2}\right)$. For the element $\left(0, f_{2}\right) \in \operatorname{Ker} P$, it is observed that $\beta_{n} f_{2}\left(L_{n}^{-1}(x)\right)+p_{n}\left(L_{n}^{-1}(x)\right)=0$ for $x \in I_{n}$. Hence, for all $x \in I$, it is seen that $f_{2}(x)=\left(-1 / \beta_{n}\right) p_{n}(x)$. With $x=$ $x_{0}$, it follows that $c_{i}=\left(\beta_{i} / \beta_{1}\right) c_{1}$ and $d_{i}=\left(\beta_{i} / \beta_{1}\right) d_{1}, i=$ $2, \ldots, N$. Consequently, if $\left(0, f_{2}\right) \in \operatorname{Ker} P$ then $f_{2}$ is a linear polynomial. So, dimension of $\operatorname{Ker} P=2$. Therefore, by RankNullity Theorem, dimension of $\mathcal{S}_{0}^{1}=$ dimension of $\mathcal{S}_{0}-$ dimension of $\operatorname{Ker} P=2(N+1)-2=2 N$.

Remark 7. By Proposition 4 and (14), it follows that the map $\theta: \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \rightarrow \mathcal{S}_{0}$ defined by $\theta(y, z)=f$ is a linear isomorphism, where $f=\left(f_{1}, f_{2}\right) \in \mathcal{S}_{0}, f_{1}$ is the unique CHFIF passing through the points $\left(x_{i}, y_{i}\right)$, and $f_{2}$ is the unique AFIF passing through the points $\left(x_{i}, z_{i}\right)$,
$y=\left(y_{0}, y_{1}, \ldots, y_{N}\right)$, and $z=\left(z_{0}, z_{1}, \ldots, z_{N}\right)$. Thus, $\mathcal{\delta}_{0}$ is linearly isomorphic to $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. Consider the metric space $\left(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}, d_{\mathbb{R}^{2(N+1)}}\right)$, where $d_{\mathbb{R}^{2(N+1)}}$ is given by $d_{\mathbb{R}^{2(N+1)}}(y \times z, \bar{y} \times \bar{z})=\max _{0 \leq i \leq N}\left(\left|y_{i}-\bar{y}_{i}\right|,\left|z_{i}-\bar{z}_{i}\right|\right), y=$ $\left(y_{0}, y_{1}, \ldots, y_{N}\right), z=\left(z_{0}, z_{1}, \ldots, z_{N}\right), \bar{y}=\left(\bar{y}_{0}, \bar{y}_{1}, \ldots, \bar{y}_{N}\right)$, and $\bar{z}=\left(\bar{z}_{0}, \bar{z}_{1}, \ldots, \bar{z}_{N}\right)$. Then, with the restriction of metric $d^{*}$ on the set $\mathcal{S}_{0}$, it is observed by (8) that the maps $\theta$ and $\theta^{-1}$ are continuous. Hence $\mathcal{S}_{0}$ is closed and complete subspace of $L_{2}(\mathbb{R})$.

Remark 8. Let $\left\{f_{n, 1}\right\}$ be a sequence in $\mathcal{S}_{0}^{1}$ such that $\lim _{n \rightarrow \infty} f_{n, 1}=f_{1}^{*}$ and $\left\{f_{n}=\left(f_{n, 1}, f_{n, 2}\right)\right\}$ be a convergent sequence in $\mathcal{S}_{0}$, where the functions $f_{n, 2}$ are AFIFs. Since $\mathcal{S}_{0}$ is closed, $\lim _{n \rightarrow \infty} f_{n}=f^{*} \equiv\left(f_{1}^{*}, f_{2}^{*}\right) \in \mathcal{S}_{0}$. Thus, $f_{1}^{*} \in \mathcal{S}_{0}^{1}$ and, consequently, $\mathcal{\delta}_{0}^{1}$ is closed subspace of $L_{2}(\mathbb{R})$.

## 4. Multiresolution Analysis Based on CHFIF

In this section, the multiresolution analysis of $L_{2}(\mathbb{R})$ is generated by using a finite set of CHFIFs. For this purpose, the sets $\mathbb{V}_{k}, k \in \mathbb{Z}$, consisting of a collection of CHFIFs, are defined and it is shown that these form a nested sequence. The multiresolution analysis of $L_{2}(\mathbb{R})$ is then generated by constructing Riesz bases of vector subspaces $\mathbb{V}_{k}$ consisting of certain orthogonal functions in $L_{2}(\mathbb{R})$.

To introduce certain sets of CHFIFs needed for multiresolution of $L_{2}(\mathbb{R})$, let $L_{2}\left(\mathbb{R}, \mathbb{R}^{2}\right)$ be a collection of functions $f: \mathbb{R} \rightarrow \mathbb{R}^{2}$ such that $f=\left(f_{1}, f_{2}\right)$ and $f_{1}, f_{2} \in L_{2}(\mathbb{R})$ and $C_{0}\left(\mathbb{R}, \mathbb{R}^{2}\right)$ be a collection of functions $f: \mathbb{R} \rightarrow \mathbb{R}^{2}$ such that $f=\left(f_{1}, f_{2}\right)$ and $f_{1}, f_{2} \in C_{0}(\mathbb{R})$, the set of all real valued continuous functions defined on $\mathbb{R}$ which vanish at infinity. Define the set $\widetilde{\mathbb{V}}_{0}$ as

$$
\begin{equation*}
\widetilde{\mathbb{V}}_{0}=\widetilde{S}_{0} \bigcap L_{2}\left(\mathbb{R}, \mathbb{R}^{2}\right) \bigcap C_{0}\left(\mathbb{R}, \mathbb{R}^{2}\right) \tag{15}
\end{equation*}
$$

where $\widetilde{S}_{0}=\left\{f: f=\left(f_{1}, f_{2}\right),\left.f_{1}\right|_{[i-1, i)}\right.$ is a CHFIFand $\left.f_{2}\right|_{[i-1, i)}$ is an AFIF, $\left.i \in \mathbb{Z}\right\}$.

That the set $\widetilde{\mathbb{V}}_{0}$ is not empty is easily seen by considering a function $f=\left(f_{1}, f_{2}\right) \in \mathcal{S}_{0}$, with $f\left(x_{0}\right)=(0,0)=f\left(x_{N}\right)$ and $f(x)=(0,0)$ for $x \notin I$, which obviously belongs to $\widetilde{\mathbb{V}}_{0}$. Let, for $k \in \mathbb{Z}$,

$$
\begin{equation*}
\widetilde{\mathbb{V}}_{k}=\left\{f: f\left(N^{-k} \cdot\right) \in \widetilde{\mathbb{V}}_{0}\right\} . \tag{16}
\end{equation*}
$$

The sets $\widetilde{\mathbb{V}}_{0}$ and $\widetilde{\mathbb{V}}_{k}$ are seen to be closed sets as follows. Let $\left\{f_{n}\right\}$ be a sequence in $\widetilde{\mathbb{V}}_{0}$ such that $\lim _{n \rightarrow \infty} f_{n}=f^{*}=\left(f_{1}^{*}, f_{2}^{*}\right)$. Now, $\left.\lim _{n \rightarrow \infty} f_{n}\right|_{[i-1, i)}=\left.f^{*}\right|_{[i-1, i)}=\left(\left.f_{1}^{*}\right|_{[i-1, i)},\left.f_{2}^{*}\right|_{[i-1, i)}\right)$. By Remarks 7 and 8 , it is observed that $\left.f_{1}^{*}\right|_{[i-1, i)}$ is a CHFIF and $\left.f_{2}^{*}\right|_{[i-1, i)}$ is an AFIF, $i \in \mathbb{Z}$. Thus, $f^{*} \in \widetilde{S}_{0}$, which implies that $\widetilde{S}_{0}$ is a closed set. Consequently, $\widetilde{\mathbb{V}}_{0}$ and $\widetilde{\mathbb{V}}_{k}, k \in \mathbb{Z}$ are closed sets. Now, for $k \in \mathbb{Z} \backslash 0$, define

$$
\begin{gather*}
\mathbb{V}_{0}=\left\{f_{1}: f_{1}\right. \text { is the first component of some } \\
\left.f=\left(f_{1}, f_{2}\right) \in \widetilde{\mathbb{V}}_{0}\right\},  \tag{17}\\
\mathbb{V}_{k}=\left\{f_{1}: f_{1}\left(N^{-k} \cdot\right) \in \mathbb{V}_{0}\right\} .
\end{gather*}
$$

It follows from Proposition 4 that the sets $\mathbb{V}_{k}$, with $L_{2}$-norm, are vector subspaces of $L_{2}(\mathbb{R})$. The following proposition shows that these subspaces $\mathbb{V}_{k}$ form a nested sequence.

Proposition 9. The subspaces $\mathbb{V}_{k}, k \in \mathbb{Z}$, form a nested sequence $\cdots \supseteq \mathbb{V}_{-1} \supseteq \mathbb{V}_{0} \supseteq \mathbb{V}_{1} \supseteq \cdots$.

Proof. To show that $\mathbb{V}_{k} \supseteq \mathbb{V}_{k+1}$ for all $k \in \mathbb{Z}$, it suffices to prove the inclusion relation for $k=0$. Let $f \in \mathbb{V}_{1}$. Then, $\left.f\right|_{[0, N)}=$ $\left.g_{1}\right|_{[0, N)}$ for some $g=\left(g_{1}, g_{2}\right) \in \widetilde{\mathbb{V}}_{1}$. If $G=\operatorname{graph}\left(\left.g\right|_{[0, N]}\right)$ then, $G=\bigcup_{i=1}^{N} w_{i}(G)$ implies, for $j \in\{1, \ldots, N\}, w_{j}(G)=$ $\bigcup_{i=1}^{N} w_{j} \circ w_{i} \circ w_{j}^{-1}\left(w_{j}(G)\right)$, where $w_{i}(G)=\left(L_{i}(x), F_{i}(x, y, z)\right)$ for all $(x, y, z) \in G, i=1, \ldots N$. Expressing $w_{i}$ and $w_{j}$ 。 $w_{i} \circ w_{j}^{-1}$ in matrix form as $w_{i}(x, y, z)=A_{i}(x, y, z)+B_{i}$ and $w_{j} \circ w_{i} \circ w_{j}^{-1}(x, y, z)=A_{i, j}(x, y, z)+B_{i, j}$, it is observed that nonzero entries in matrices $A_{i}$ and $A_{i, j}$ occur at the same places. Consequently, $w_{j}(G)$ is graph of $\left.g\right|_{[j-1, j)}$, so that $g \in \widetilde{\mathbb{V}}_{0}$. It therefore follows that $\left.g_{1}\right|_{[j-1, j)}$ is a CHFIF on the interval $[j-1, j)$. Thus, the function $\left.f\right|_{[j-1, j)}=\left.g_{1}\right|_{[j-1, j)}$ is a CHFIF on the interval $[j-1, j)$ and consequently, $f \in \mathbb{V}_{0}$.

In order to generate a multiresolution analysis of $L_{2}(\mathbb{R})$ using CHFIFs, the inner product on the vector space $\mathbb{V}_{k}, k \in \mathbb{Z}$, is defined by $\left\langle f_{1}, \widehat{f}_{1}\right\rangle=\int_{\mathbb{R}} f_{1}(x) \widehat{f}_{1}(x) d x$. Using $f_{1}\left(L_{n}(x)\right)=\alpha_{n} f_{1}(x)+\beta_{n} f_{2}(x)+p_{n}(x)$ and $\widehat{f}_{1}\left(L_{n}(x)\right)=$ $\widehat{\alpha}_{n} \widehat{f}_{1}(x)+\widehat{\beta}_{n} \widehat{f}_{2}(x)+\widehat{p}_{n}(x)$, it is observed that, for $f_{1}, \widehat{f}_{1} \in \mathbb{V}_{0}$,

$$
\begin{align*}
& \left\langle f_{1}, \widehat{f}_{1}\right\rangle \\
& =\left(\sum _ { n = 1 } ^ { N } a _ { n } \left(\alpha_{n} \widehat{\beta}_{n}\left\langle f_{1}, \widehat{f}_{2}\right\rangle+\beta_{n} \widehat{\alpha}_{n}\left\langle f_{2}, \widehat{f}_{1}\right\rangle+\beta_{n} \widehat{\beta}_{n}\left\langle f_{2}, \widehat{f}_{2}\right\rangle\right.\right. \\
& \quad+\alpha_{n}\left\langle f_{1}, \widehat{p}_{n}\right\rangle+\widehat{\alpha}_{n}\left\langle\widehat{f}_{1}, p_{n}\right\rangle+\beta_{n}\left\langle f_{2}, \widehat{p}_{n}\right\rangle \\
& \left.\left.\quad+\widehat{\beta}_{n}\left\langle\widehat{f}_{2}, p_{n}\right\rangle+\left\langle p_{n}, \widehat{p}_{n}\right\rangle\right)\right) \\
& \quad \times\left(1-\sum_{n=1}^{N} a_{n} \alpha_{n} \widehat{\alpha}_{n}\right)^{-1}, \tag{18}
\end{align*}
$$

where $a_{n}, \alpha_{n}$ and $\beta_{n}, p_{n} ; \widehat{\alpha}_{n}$ and $\widehat{\beta}_{n}, \widehat{p}_{n}$, are given by (1), (2), and (4), respectively, for the interpolation data $\left\{\left(x_{i}, y_{i}, z_{i}\right)\right.$ : $i=0,1, \ldots, N\}$ and $\left\{\left(x_{i}, \widehat{y}_{i}, \widehat{z}_{i}\right): i=0,1, \ldots, N\right\}$. Using (18), the set of orthogonal functions that forms the Riesz basis of set $\mathbb{V}_{0}$ is now constructed as follows.

Let the free variables $\alpha_{j}, \gamma_{j}$ and constrained variables $\beta_{j}$, $j=1, \ldots, N, N>1$, in the construction of CHFIF be chosen such that $\alpha_{j}+\beta_{j} \neq \gamma_{j}$ for at least one $j$. Consider, the points $y_{i}$ and $z_{i} \in R^{N+1}, i=0, \ldots, N$, given by

$$
\begin{gather*}
y_{0}=\left(1, r_{1}, \ldots, r_{N-1}, 0\right), \quad y_{N}=\left(0, s_{1}, \ldots, s_{N-1}, 1\right), \\
y_{i}=(0, \ldots, 1, \ldots, 0), \quad i=1, \ldots, N-1, \\
y_{N+1+i}=\left(0, u_{i, 1}, \ldots, u_{i, N-1}, 0\right), \quad i=0, \ldots, N ; \tag{19}
\end{gather*}
$$

$$
\begin{array}{r}
z_{i}=(0, \ldots, 0), \quad z_{N+1+i}=(0, \ldots, 1, \ldots, 0) \\
i=0, \ldots, N \tag{20}
\end{array}
$$

and a set of $2(N+1)$ functions $\tilde{f}_{i}=\left(\tilde{f}_{i, 1}, \tilde{f}_{i, 2}\right) \in \mathcal{S}_{0}$, $i=0, \ldots, 2 N+1$, where the CHFIF $\tilde{f}_{i, 1}$ passes through the points $\left(x_{k}, y_{i_{k}}\right), k=0, \ldots, N+1, y_{i_{k}}$ being the $k$ th component of $y_{i}$ and AFIF $\tilde{f}_{i, 2}$ passes through the points $\left(x_{k}, z_{i_{k}}\right), k=0, \ldots, N+1, z_{i_{k}}$ being the $k$ th component of $z_{i}$. Let the function $\widetilde{f}_{i}^{*}: \mathbb{R} \rightarrow \mathbb{R}^{2}, i=0,1, \ldots 2 N+1$, be the extension of the function $\tilde{f}_{i}: I \rightarrow \mathbb{R}^{2}$ such that $\tilde{f}_{i}^{*}(x)=\tilde{f}_{i}(x)$ for $x \in I$ and $\tilde{f}_{i}^{*}(x)=(0,0)$ for $x \notin I$.

For ensuring the orthogonality of the functions $\tilde{f}_{i, 1}^{*}$ with respect to the inner product in $L_{2}(\mathbb{R})$, let the values of $r_{i}, s_{i}$ and $u_{i, j}, i, j=1, \ldots, N-1$, in (19) be chosen such that

$$
\begin{gather*}
\left\langle\tilde{f}_{i, 1}, \tilde{f}_{0,1}\right\rangle=0, \quad\left\langle\tilde{f}_{i, 1}, \tilde{f}_{N, 1}\right\rangle=0,  \tag{21}\\
\left\langle\tilde{f}_{N+1+j, 1}, \tilde{f}_{i, 1}\right\rangle=0 .
\end{gather*}
$$

Let, for $i=1,2, \ldots, N-1$,

$$
\begin{equation*}
\zeta_{i}=\left\langle\tilde{f}_{N+1+i, 1}, \tilde{f}_{0,1}\right\rangle, \quad \eta_{i}=\left\langle\tilde{f}_{N+1+i, 1}, \tilde{f}_{N, 1}\right\rangle \tag{22}
\end{equation*}
$$

The free variables $\alpha_{j}, \gamma_{j}$ and constrained variables $\beta_{j}, j=$ $1,2, \ldots, N$, in (2) are $3 N$ variables and $\zeta_{i}=\eta_{i}=0, i=$ $1, \ldots, N-1$ is a system of $2 N-2$ equations. Suppose there exist no $\alpha_{j}, \gamma_{j}$ and $\beta_{j}, j=1, \ldots, N$, in $(-1,1)$ such that $\zeta_{i}=\eta_{i}=0$, $i=1, \ldots, N-1$; then dimension of $S_{0}^{1}<2 N$, which is a contradiction. Hence, there exists at least one set of $\alpha_{j}, \gamma_{j}$ and $\beta_{j}, j=1, \ldots, N$, in $(-1,1)$ such that $\zeta_{i}=\eta_{i}=0$, $i=1, \ldots, N-1$. The free variables $\alpha_{j}, \gamma_{j}$ and constrained variables $\beta_{j}, j=1,2, \ldots, N$, in (2) are chosen such that, for $i=1,2, \ldots, N-1, \zeta_{i}=0$, and $\eta_{i}=0$.

It is easily seen that the functions $\widetilde{f}_{i}^{*}, i=0, \ldots, 2 N+$ $1, \tilde{f}_{i, 1}^{*}, i=0, \ldots, N$ and the functions $\tilde{f}_{j, 2}^{*}, j=N+$ $1, \ldots, 2 N+1$, are linearly independent. Now, by (8), $\widetilde{f}_{j, 1}^{*}(x)=$ $\alpha_{n} \widetilde{f}_{j, 1}^{*}\left(L_{n}^{-1}(x)\right)+\beta_{n} \widetilde{f}_{j, 2}^{*}\left(L_{n}^{-1}(x)\right)+p_{n, j}\left(L_{n}^{-1}(x)\right)$ and $\widetilde{f}_{j, 2}^{*}(x)=$ $\gamma_{n} \widetilde{f}_{j, 2}^{*}\left(L_{n}^{-1}(x)\right)+q_{n, j}\left(L_{n}^{-1}(x)\right), j=0,1, \ldots, 2 N+1$, where $p_{n, j}$ and $q_{n, j}$ are linear polynomials. By (20), the functions $\widetilde{f}_{j, 2}^{*}, j=N+2, \ldots, 2 N$, are nonlinear polynomials. Hence, $\sum_{k=1}^{N-1} a_{k} \widetilde{f}_{N+1+k, 1}^{*}(x)-\alpha_{n} \sum_{k=1}^{N-1} a_{k} \widetilde{f}_{N+1+k, 1}^{*}\left(L_{n}^{-1}(x)\right)=0$ if and only if $a_{k}=0$, which implies $\widetilde{f}_{N+1+k, 1}^{*}, k=1, \ldots, N-1$, are linearly independent. The linear independence of $\tilde{f}_{k, 1}^{*}$, $\widetilde{f}_{N+1+k, 1}^{*}, k=1, \ldots, N-1$ together with (21) now ensures the same number of orthogonal functions by applying the GramSchmidt process.

Let $\left\{\phi_{i, 1}\right\}_{i=1}^{2 N-1} \subset \mathbb{V}_{0}, i \neq N$, be a sequence of orthogonal functions obtained from the sequence $\left\{\widetilde{f}_{i, 1}^{*}\right\}_{i=1}^{2 N}, i \neq N, N+1$, by the Gram-Schmidt process. Set

$$
\phi_{N, 1}= \begin{cases}\tilde{f}_{N, 1}^{*}(x) & x \in[0,1)  \tag{23}\\ \tilde{f}_{0,1}^{*}(x-1) & x \in[1,2) \\ 0 & \text { otherwise }\end{cases}
$$

It is easily seen by Proposition 6 that none of the functions $\phi_{i, 1}, i=1,2, \ldots, 2 N-1$, are identically zero. Further, by (21) and (22), it follows that $\left\{\phi_{i, 1}: i=1,2, \ldots, 2 N-1\right\}$ is an orthogonal set. This is the set that leads to the generation of multiresolution analysis of $L_{2}(\mathbb{R})$ in the following theorem.

Theorem 10. Let free variables $\alpha_{j}, \gamma_{j}$ and constrained variables $\beta_{j}, j=1, \ldots, N, N>1$, in the construction of CHFIF be chosen such that $\alpha_{j}+\beta_{j} \neq \gamma_{j}$ for at least one $j$ and let $\zeta_{i}, \eta_{i}$ given by (22) be such that $\zeta_{i}=0, \eta_{i}=0, i=1, \ldots, N-1$. Then,

$$
\begin{equation*}
\mathbb{V}_{0}=\operatorname{clos}_{L^{2}} \operatorname{span}\left\{\phi_{i, 1}(\cdot-l): i=1, \ldots, 2 N-1, l \in \mathbb{Z}\right\} \tag{24}
\end{equation*}
$$

where $\phi_{i, 1} \in \mathbb{V}_{0}$. Also, the set $\left\{\phi_{i, 1}\right\}_{i=1}^{2 N-1}$ generates a continuous, compactly supported multiresolution analysis of $L_{2}(\mathbb{R})$.

Proof. It is obvious that functions $\phi_{i, 1}, i=1, \ldots, 2 N-1$, are compactly supported and are elements of $\mathbb{V}_{0}$. Now, $f \in \mathbb{V}_{0}$ implies $\left.f\right|_{[i-1, i)}=\left.g_{1}\right|_{[i-1, i)}$ is a CHFIF for some $g=\left(g_{1}, g_{2}\right) \in$ $\widetilde{\mathbb{V}}_{0}$. Since every $g=\left(g_{1}, g_{2}\right) \in \widetilde{\mathbb{V}}_{0}$ is determined by $g_{1}(i / N)$ and $g_{2}(i / N), i \in \mathbb{Z}$, the function $g$ has a unique expansion in terms of the functions $\widetilde{f}_{i}^{*}=\left(\widetilde{f}_{i, 1}^{*}, \widetilde{f}_{i, 2}^{*}\right), i=0, \ldots 2 N+1$, and their integer translates. Hence, the function $f=g_{1} \in \mathbb{V}_{0}$ has a unique expansion in terms of the functions $\widetilde{f}_{i, 1}^{*}, i=1, \ldots N-$ $1, N+2, \ldots, 2 N-2, \phi_{N, 1}$ and their integer translates. Thus, CHFIF $f$ has a unique expansion in terms of the functions $\phi_{i, 1}, i=1, \ldots, 2 N-1$, and their integer translates, that is, $f=\sum_{k}\left(\sum_{i=1}^{2 N-1} K_{k, i} \phi_{i, 1}(x-k)\right)$ where $K_{k, i}=\int_{\mathbb{R}} f(x) \phi_{i, 1}(x-$ $k) d x$. Since $f \in \mathbb{V}_{0}$ is arbitrary, $\mathbb{V}_{0}=\operatorname{span}\left\{\phi_{i, 1}(\cdot-l), i=\right.$ $1, \ldots, 2 N-1, l \in \mathbb{Z}\}$. Let $\left\{f_{n, 1}\right\}$ be a sequence in $\mathbb{V}_{0}$ such that $\lim _{n \rightarrow \infty} f_{n, 1}=f_{1}^{*}$ and let $\left\{f_{n}=\left(f_{n, 1}, f_{n, 2}\right)\right\}$ be a convergent sequence in $\mathbb{V}_{0}$, where $f_{n, 2}$ are AFIFs. Since $\widetilde{\mathbb{V}}_{0}$ is closed, $\lim _{n \rightarrow \infty} f_{n}=f^{*}=\left(f_{1}^{*}, f_{2}^{*}\right) \in \widetilde{\mathbb{V}}_{0}$ which gives that $\left.f_{1}^{*}\right|_{[i-1, i)}, i \in \mathbb{Z}$ is a CHFIF. Hence, $f_{1}^{*} \in \mathbb{V}_{0}$. It therefore follows that $\mathbb{V}_{0}$ is closed and $\mathbb{V}_{0}=\operatorname{clos}_{L^{2}} \operatorname{span}\left\{\phi_{i, 1}(\cdot-l), i=\right.$ $1, \ldots, 2 N-1, l \in \mathbb{Z}\}$.

Now, the following steps show that the set $\left\{\phi_{i, 1}\right\}_{i=1}^{2 N-1}$ indeed generates a continuous, compactly supported multiresolution analysis of $L_{2}(\mathbb{R})$.
(a) By Proposition 9, it follows that $\cdots \supseteq \mathbb{V}_{-1} \supseteq \mathbb{V}_{0} \supseteq \mathbb{V}_{1} \supseteq$
(b) To prove that $\bigcap_{k \in \mathbb{Z}} \mathbb{V}_{k}=\{0\}$, let $I_{n}=[n, n+1], n \in \mathbb{Z}$, and $U_{0}=\left\{f_{\chi_{I_{0}}}: f \in \mathbb{V}_{0}\right\}$ where $f_{\chi_{I_{0}}}(x)=f(x)$ if $x \in I_{0}$ and $f_{\chi_{I_{0}}}(x)=0$ if $x \notin I_{0}$. Since the space $U_{0}$ is finite dimensional over $\mathbb{R}$, the norms $\|\cdot\|_{\infty}$ and $\|\cdot\|_{L^{2}}$ restricted to $U_{0}$ are equivalent. Hence, there exists a positive constant $c$ such that $\|f\|_{\infty} \leq c\|f\|_{L^{2}}$ for all $f \in U_{0}$. By the property of translation invariance, it is observed that $\left\|f_{\chi_{I_{n}}}\right\|_{\infty} \leq c\left\|f_{\chi_{I_{n}}}\right\|_{L^{2}}$ for any $f \in$ $U_{0}$. Thus, $\|f\|_{\infty} \leq \sup _{n}\left\|f_{\chi_{I_{n}}}\right\|_{\infty} \leq c \sum_{n \in \mathbb{Z}}\left\|f_{\chi_{L_{n}}}\right\|_{L^{2}}=c\|f\|_{L^{2}}$ for any $f \in \mathbb{V}_{0}$. It therefore follows by the definition of $\mathbb{V}_{k}$ that $\|f\|_{\infty} \leq c N^{k / 2}\|f\|_{L^{2}}$ for all $f \in \mathbb{V}_{k}$. Consequently, if $f \in \bigcap_{k \in \mathbb{Z}} \mathbb{V}_{k}$, then $\|f\|_{\infty}=0$ which implies $f=0$.
(c) For showing that $\operatorname{clos}_{L_{2}} \bigcup_{m \in \mathbb{Z}} \mathbb{V}_{m}=L^{2}(\mathbb{R})$, let $f=$ $\left(f_{1}, f_{2}\right) \in \widetilde{\mathbb{V}}_{0}$, where $f_{1}$ is a CHFIF passing through $\left(x_{k}, 1\right)$ and $f_{2}$ is an AFIF passing through $\left(x_{k}, z_{k}\right)$. For all $x \in \mathbb{R}$, by (19),

$$
\begin{array}{r}
f_{1}=\sum_{k}\left(\sum_{i=1}^{N-1} C_{k, i} \tilde{f}_{i, 1}^{*}(x-k)+\phi_{N}(x-k)\right.  \tag{25}\\
\left.+\sum_{i=1}^{N-1} D_{k, i} \tilde{f}_{N+1+i, 1}^{*}(x-k)\right)
\end{array}
$$

$$
\frac{1}{\left\|\phi_{N, 1}\right\|^{2}}\left(\begin{array}{cc}
\left(\int_{I}\left|f_{0,1}(x)\right|^{2} d x\right)^{1 / 2} & \left(\int_{I}\left|f_{0,1}(x)\right|\left|f_{N, 1}(x)\right| d x\right)^{1 / 2} \\
\left(\int_{I}\left|f_{0,1}(x)\right|\left|f_{N, 1}(x)\right| d x\right)^{1 / 2} & \left(\int_{I}\left|f_{N, 1}(x)\right|^{2} d x\right)^{1 / 2}
\end{array}\right)
$$

Since the functions $f_{0,1}$ and $f_{N, 1}$ are linearly independent, the determinant of the matrix is positive which implies $\tau>0$. Taking $A=\sqrt{\tau}\left\|\phi_{N, 1}\right\|_{L_{2}}$ and $B=\sqrt{3}\left\|\phi_{N, 1}\right\|_{L_{2}}$, it is seen that, for every $c=\left\{c_{i}\right\} \in l^{2}, A\|c\|_{l^{2}} \leq\left\|\sum c_{i} \phi_{N, 1}(\cdot-i)\right\|_{L^{2}} \leq B\|c\|_{l^{2}}$. Further, the functions $\phi_{i, 1}, i=1, \ldots, 2 N-1, i \neq N$, and their integer translates are mutually orthogonal. Therefore, the functions $\phi_{i, 1}, i=1, \ldots, 2 N-1$, and their integer translates form a Riesz basis for $\mathbb{V}_{0}$.

Remark 11. By Theorem 10, it follows that the set $\left\{\hat{\phi}_{i, 1}: i=\right.$ $1,2, \ldots, 2 N-1\} \subset \mathbb{V}_{0}$, where $\widehat{\phi}_{i, 1}(x)=\phi_{i, 1}(x) /\left\|\phi_{i, 1}\right\|_{L^{2}}, i=$ $1,2, \ldots, 2 N-1$, actually generates a continuous, compactly supported multiresolution analysis of $L_{2}(\mathbb{R})$ by orthonormal functions.

## 5. Conclusions

In this paper, multiresolution analysis arising from Coalescence Hidden-variable Fractal Interpolation Functions is developed, since CHFIF based wavelets generally more satisfactorily preserve the features of the functions simulating natural objects or outcome of scientific experiments that are partly self-affine and partly non-self-affine compared to AFIF based wavelets. The availability of a larger set of free variables and constrained variables with CHFIF in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_{2}(\mathbb{R})$ than that provided by multiresolution analysis based only on affine FIFs. In our approach, the vector space of CHFIFs is introduced, its dimension is determined, and Riesz bases of vector subspaces $\mathbb{V}_{k}, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_{2}(\mathbb{R}) \bigcap C_{0}(\mathbb{R})$ are constructed.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.
where

$$
\begin{array}{r}
C_{k, i}=\left(1-r_{i}-s_{i}-\sum_{j=1}^{N-1} u_{j, i} z_{j}\right), \quad D_{k, i}=z_{i},  \tag{26}\\
i=1, \ldots, N-1 .
\end{array}
$$

Now, since $\phi_{i, 1}$ are continuous and compactly supported, by using (b) and Proposition 3.1 of [11], it follows that $\bigcup_{k \in \mathbb{Z}} \mathbb{V}_{k}$ is dense in $L_{2}(\mathbb{R})$.
(d) For proving that the functions $\phi_{i, 1}, i=1, \ldots, 2 N-1$, and their integer translates form a Riesz basis for $\mathbb{V}_{0}$, let $\tau$ be the smallest eigenvalue of the matrix

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