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Research Article

Some Regularity Criteria for the 3D Boussinesq Equations in the Class $L^2(0,T;\dot{B}^{-1}_{\infty,\infty})$

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We consider the three-dimensional Boussinesq equations and obtain some regularity criteria via the velocity gradient (or the vorticity, or the deformation tensor) and the temperature gradient.

1. Introduction

Consider the following three-dimensional (3D) Boussinesq equations:

$$\begin{gathered}\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\,\mathbf{u}-\Delta\mathbf{u}+\nabla\pi=\theta\mathbf{e}_3,\\ \theta_t+(\mathbf{u}\cdot\nabla)\,\theta-\Delta\theta=0,\\ \nabla\cdot\mathbf{u}=0,\\ \mathbf{u}\,(x,0)=\mathbf{u}_0,\qquad \theta\,(x,0)=\theta_0.\end{gathered}\tag{1}$$

Here, $\mathbf{u} = (u_1(x,t),u_2(x,t),u_3(x,t))$ is the fluid velocity, $\pi=\pi(x,t)$ is a scalar pressure, and $\theta=\theta(x,t)\geq 0$ is the temperature, while $\mathbf{u}_0$ and θ_0 are the prescribed initial velocity and temperature, respectively.

When $\theta=0$, (1) reduces to the incompressible Navier-Stokes equations. The regularity of its weak solutions and the existence of global strong solutions are challenging open problems; see [1–3]. Initialed by [4, 5], there have been a lot of literatures devoted to finding sufficient conditions to ensure the smoothness of the solutions; see [6–18] and so forth. Since the convective terms are similar in Navier-Stokes equations and Boussinesq equations, the authors also consider the regularity conditions for (1); see [19–23] and so forth.

Motivated by [7], we will consider the regularity criteria for (1) and extend the result of [7] to the case of Boussinesq equations.

Before stating the precise result, let us recall the weak formulation of (1).

Definition 1. Let $\mathbf{u}_0\in L^2(\mathbb{R}^3)$, $\theta_0\in L^1\cap L^\infty(\mathbb{R}^3)$, and $T>0$. A measurable pair $(\mathbf{u},\theta)$ is said to be a weak solution of (1) in $(0,T)$, provided that

(1) $(\mathbf{u},\theta)\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$, $\theta\in L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^3))$;

(2) $(1)_{1,2,3}$ are satisfied in the sense of distributions;

(3) the energy inequality

$$\|(\mathbf{u},\theta)\|_{L^2}^2+2\int_0^t\|\nabla(\mathbf{u},\theta)\|_{L^2}^2\mathrm{d}s\leq\int_0^t\int_{\mathbb{R}^3}\theta u_3\,\mathrm{d}x\,\mathrm{d}s,\tag{2}$$

for all $0\leq t\leq T$.

Now, our main result reads the following.

Theorem 2. *Let $\mathbf{u}_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot\mathbf{u}_0=0$ in the sense of distributions, $\theta_0\in L^1\cap L^\infty(\mathbb{R}^3)$. Suppose that $(\mathbf{u},\theta)$ is a weak solution of (1) in $[0,T)$, and*

$$(\nabla\mathbf{u},\nabla\theta)\in L^2\left(0,T;\dot{B}^{-1}_{\infty,\infty}\left(\mathbb{R}^3\right)\right);\tag{3}$$

then the solution $(\mathbf{u},\theta)\in C^\infty((0,T)\times\mathbb{R}^3)$.

Due to the divergence-free condition of the fluid velocity $\mathbf{u}$, we have

$$\Delta u_i=\sum_{k=1}^3\partial_k\left(\partial_k u_i-\partial_i u_k\right),\qquad \Delta u_i=\sum_{k=1}^3\partial_k\left(\partial_k u_i+\partial_i u_k\right).\tag{4}$$

Thus,

$$\partial_j u_i = -\sum_{k=1}^{3} \mathscr{R}_j \mathscr{R}_k \left(\partial_k u_i - \partial_i u_k\right), \\ \partial_j u_i = -\sum_{k=1}^{3} \mathscr{R}_j \mathscr{R}_k \left(\partial_k u_i + \partial_i u_k\right). \tag{5}$$

Here, $\mathscr{R}_j = \partial_j / \sqrt{-\Delta}$ is the Riesz transformation.

Using (5), we can deduce easily from Theorem 2 the following.

Corollary 3. *Let* $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ *with* $\nabla \cdot \mathbf{u}_0 = 0$ *in the sense of distributions,* $\theta_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$. *Suppose that* $(\mathbf{u}, \theta)$ *is a weak solution of* (1) *in* $[0, T)$, *and*

$$(\nabla \times \mathbf{u}, \nabla\theta) \in L^2\left(0, T; \dot{B}^{-1}_{\infty,\infty}\left(\mathbb{R}^3\right)\right), \tag{6}$$

or

$$(\mathrm{Def}\, \mathbf{u}, \nabla\theta) \in L^2\left(0, T; \dot{B}^{-1}_{\infty,\infty}\left(\mathbb{R}^3\right)\right); \tag{7}$$

then the solution $(\mathbf{u}, \theta) \in C^\infty((0,T) \times \mathbb{R}^3)$. *Here,* $\nabla \times \mathbf{u} = \begin{vmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$ *is the vorticity and* $(\mathrm{Def}\ \mathbf{u})_{ij} = \partial_i u_j + \partial_j u_i$ *is the deformation tensor (the symmetric tensor of the rate of strain).*

The rest of this paper is organized as follows. In Section 2, we recall the definition of Besov spaces and the related interpolation inequalities. Section 3 is devoted to proving Theorem 2.

2. Preliminaries

We first introduce the Littlewood-Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathscr{F} f = \widehat{f}$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix\cdot\xi} \mathrm{d}x. \tag{8}$$

Let us choose a nonnegative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \le \widehat{\varphi}(\xi) \le 1, \qquad \widehat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1, \\ 0, & \text{if } |\xi| \ge 2, \end{cases} \tag{9}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi\left(\frac{x}{2}\right), \\ \varphi_j(x) = 2^{3j}\varphi\left(2^j x\right), \\ \psi_j(x) = 2^{3j}\psi\left(2^j x\right), \quad j \in \mathbb{Z}. \tag{10}$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and Δ_j are, respectively, defined by

$$S_j f = \varphi_j * f, \qquad \Delta_j f = \psi_j * f. \tag{11}$$

Observe that $\Delta_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \longrightarrow 0, \quad \text{as } j \longrightarrow -\infty; \tag{12}$$

$$S_j f \longrightarrow f, \quad \text{as } j \longrightarrow \infty, \tag{13}$$

in the L^2 sense. By telescoping the series, we have the following Littlewood-Paley decomposition:

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f, \tag{14}$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the L^2 sense.

Let $s \in \mathbb{R}$; $p, q \in [1, \infty]$; the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by the full dyadic decomposition such as

$$\dot{B}^s_{p,q} = \left\{ f \in \mathscr{Z}'\left(\mathbb{R}^3\right); \|f\|_{\dot{B}^s_{p,q}} = \left\| \left\{ 2^{js} \left\|\Delta_j f\right\|_{L^p} \right\}_{j=-\infty}^{\infty} \right\|_{\ell^q} < \infty \right\}, \tag{15}$$

where $\mathscr{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathscr{Z}\left(\mathbb{R}^3\right) = \left\{ f \in \mathcal{S}\left(\mathbb{R}^3\right); D^\alpha \widehat{f}(0) = 0,\ \forall \alpha \in \mathbb{N}^3 \right\}. \tag{16}$$

The following interpolation inequality will be needed in Section 3:

$$\|f\|_{L^4} \le C \|f\|^{1/2}_{\dot{H}^1} \|f\|^{1/2}_{\dot{B}^{-1}_{\infty,\infty}}, \quad \forall f \in \dot{H}^1\left(\mathbb{R}^3\right) \cap \dot{B}^{-1}_{\infty,\infty}\left(\mathbb{R}^3\right). \tag{17}$$

See [24] for the proof.

3. Proof of Theorem 2

In this section, we will prove Theorem 2.

Multiplying $(1)_1$ by $-\Delta\mathbf{u}$ and $(1)_2$ by $-\Delta\theta$, integrating over $\mathbb{R}^3$, and summing up, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} &\|\nabla(\mathbf{u},\theta)\|^2_{L^2} + \|\Delta(\mathbf{u},\theta)\|^2_{L^2} \\ &= \int_{\mathbb{R}^3} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\Delta\mathbf{u}\mathrm{d}x - \int_{\mathbb{R}^3} \theta\Delta u_3 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} [(\mathbf{u}\cdot\nabla)\theta]\cdot\Delta\theta\mathrm{d}x \\ &= -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_j \partial_j u_i \partial_k u_i \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} \nabla\theta\cdot\nabla u_3 \mathrm{d}x - \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_j \partial_j\theta \partial_k\theta \mathrm{d}x \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{18}$$

Invoking Hölder inequality, (17), and Young inequality, we may bound I_1 as

$$\begin{aligned} I_1 &\leq \|\nabla \mathbf{u}\|_{L^4}^2 \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C\|\nabla \mathbf{u}\|_{\dot{B}^{-1}_{\infty,\infty}} \|\Delta \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C\|\nabla \mathbf{u}\|^2_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla \mathbf{u}\|^2_{L^2} + \frac{1}{2}\|\Delta \mathbf{u}\|^2_{L^2}. \end{aligned} \tag{19}$$

For I_2, we use Hölder inequality to dominate as

$$I_2 \leq \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla\theta|^2 + |\nabla \mathbf{u}|^2\right) dx = \frac{1}{2}\left(\|\nabla\theta\|^2_{L^2} + \|\nabla \mathbf{u}\|^2_{L^2}\right). \tag{20}$$

Finally, I_3 can be estimated similarly as I_1,

$$\begin{aligned} I_3 &\leq \|\nabla \theta\|_{L^4}^2 \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C\|\nabla \theta\|_{\dot{B}^{-1}_{\infty,\infty}} \|\Delta \theta\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C\|\nabla \theta\|^2_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla \mathbf{u}\|^2_{L^2} + \frac{1}{2}\|\Delta \theta\|^2_{L^2}. \end{aligned} \tag{21}$$

Substituting (19), (20), and (21) into (18), we gather

$$\begin{aligned} &\frac{d}{dt}\|\nabla(\mathbf{u},\theta)\|^2_{L^2} + \|\Delta(\mathbf{u},\theta)\|^2_{L^2} \\ &\quad \leq C\left(\|\nabla(\mathbf{u},\theta)\|^2_{\dot{B}^{-1}_{\infty,\infty}} + 1\right)\|\nabla(\mathbf{u},\theta)\|^2_{L^2}. \end{aligned} \tag{22}$$

Gronwall inequality together with (3) then implies that

$$(\mathbf{u},\theta) \in L^\infty\left(0,T;H^1\left(\mathbb{R}^3\right)\right) \cap L^2\left(0,T;H^2\left(\mathbb{R}^3\right)\right). \tag{23}$$

Then, we may use standard energy method to drive high-order derivative bounds, which would imply $(\mathbf{u},\theta) \in C^\infty((0,T)\times\mathbb{R}^3)$ by Sobolev embedding theorems, as desired.

The proof of Theorem 2 is completed.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] E. Hopf, "Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen," *Mathematische Nachrichten*, vol. 4, pp. 213–231, 1951.

[2] P. G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman & Hall/CRC Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.

[3] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace," *Acta Mathematica*, vol. 63, no. 1, pp. 193–248, 1934.

[4] J. Serrin, "On the interior regularity of weak solutions of the Navier-Stokes equations," *Archive for Rational Mechanics and Analysis*, vol. 9, pp. 187–191, 1962.

[5] J. Serrin, "The initial value problem for the Navier-Stokes equations," in *Nonlinear Problems*, pp. 69–98, University of Wisconsin Press, Madison, Wis, USA, 1963.

[6] H. B. da Veiga, "A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$," *Chinese Annals of Mathematics B*, vol. 16, no. 4, pp. 407–412, 1995.

[7] S. Gala, "A remark on the blow-up criterion of strong solutions to the Navier-Stokes equations," *Applied Mathematics and Computation*, vol. 217, no. 22, pp. 9488–9491, 2011.

[8] X. W. He and S. Gala, "Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the pressure in the class $L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3602–3607, 2011.

[9] J. Neustupa, A. Novotný, and P. Penel, "An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity," in *Topics in Mathematical Fluid Mechanics*, vol. 10 of *Quaderni di Matematica*, pp. 163–183, 2002.

[10] Z. J. Zhang, "A logarithmically improved regularity criterion for the 3D Boussinesq equations via the pressure," *Acta Applicandae Mathematicae*, 2013.

[11] Z. J. Zhang, "A remark on the regularity criterion for the 3D Navier-Stokes equations involving the gradient of one velocity component," *Journal of Mathematical Analysis and Applications*, 2014.

[12] Z. J. Zhang, "A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component," *Communications on Pure and Applied Analysis*, vol. 12, no. 1, pp. 117–124, 2013.

[13] Z. J. Zhang, P. Li, and D. X. Zhong, "Navier-Stokes equations with regularity in two entries of the velocity gradient tensor," *Applied Mathematics and Computation*, vol. 228, pp. 546–551, 2014.

[14] Z. J. Zhang, Z. A. Yao, P. Li, C. C. Guo, and M. Lu, "Two new regularity criteria for the 3D Navier-Stokes equations via two entries of the velocity gradient tensor," *Acta Applicandae Mathematicae*, vol. 123, pp. 43–52, 2013.

[15] Z. J. Zhang, D. X. Zhong, and L. Hu, "A new regularity criterion for the 3D Navier-Stokes equations via two entries of the velocity gradient tensor," *Acta Applicandae Mathematicae*, vol. 129, no. 1, pp. 175–181, 2014.

[16] Y. Zhou, "A new regularity criterion for weak solutions to the Navier-Stokes equations," *Journal de Mathématiques Pures et Appliquées*, vol. 84, no. 11, pp. 1496–1514, 2005.

[17] Y. Zhou and M. Pokorný, "On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component," *Journal of Mathematical Physics*, vol. 50, no. 12, Article ID 123514, 11 pages, 2009.

[18] Y. Zhou and M. Pokorný, "On the regularity of the solutions of the Navier-Stokes equations via one velocity component," *Nonlinearity*, vol. 23, no. 5, pp. 1097–1107, 2010.

[19] J. S. Fan and Y. Zhou, "A note on regularity criterion for the 3D Boussinesq system with partial viscosity," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 802–805, 2009.

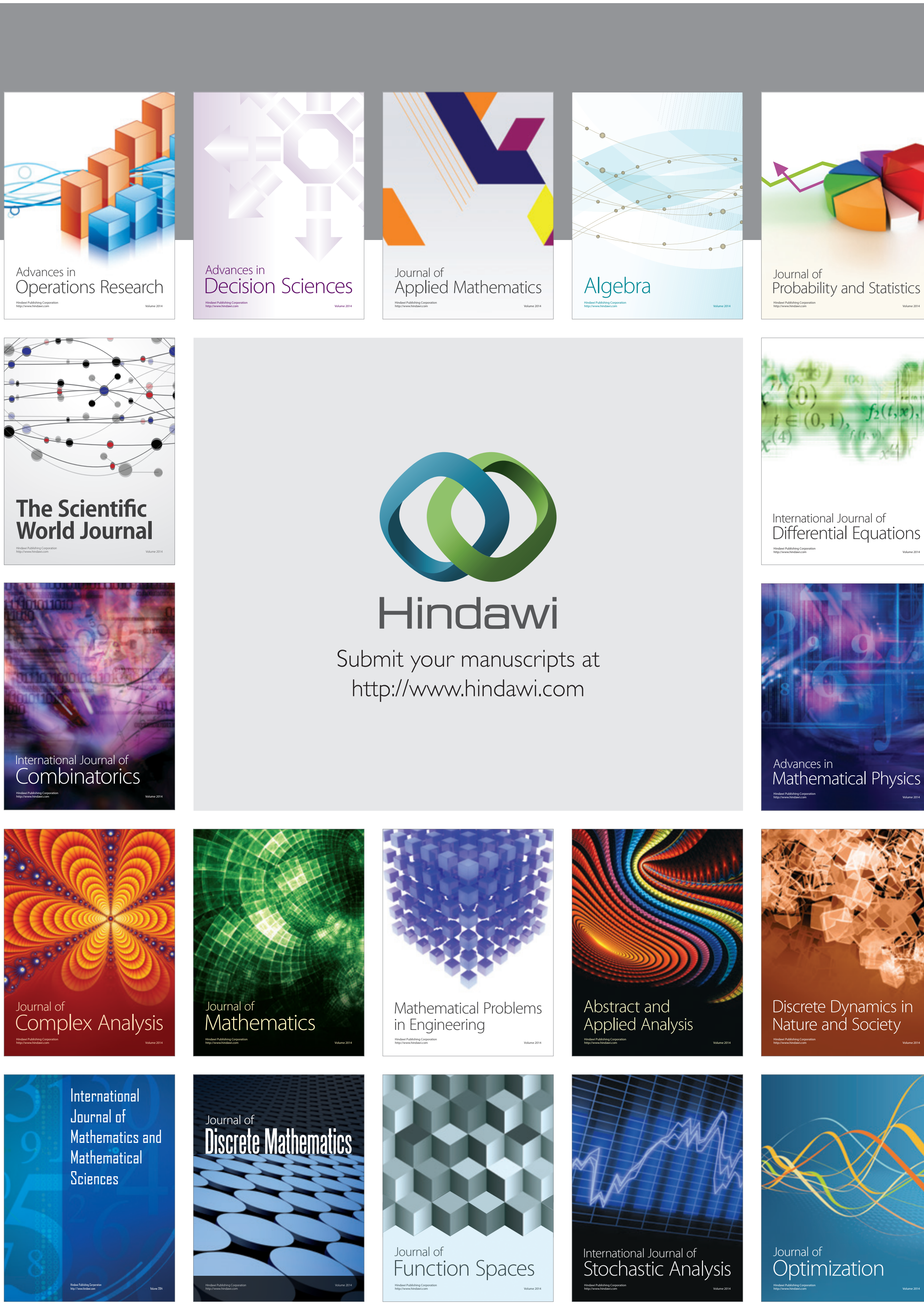
[20] N. Ishimura and H. Morimoto, "Remarks on the blow-up criterion for the 3-D Boussinesq equations," *Mathematical Models & Methods in Applied Sciences*, vol. 9, no. 9, pp. 1323–1332, 1999.

[21] H. Qiu, Y. Du, and Z. Yao, "Blow-up criteria for 3D Boussinesq equations in the multiplier space," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 1820–1824, 2011.

[22] H. Qiu, Y. Du, and Z. Yao, "Serrin-type blow-up criteria for 3D Boussinesq equations," *Applicable Analysis*, vol. 89, no. 10, pp. 1603–1613, 2010.

[23] Z. Zhang, "A remark on the regularity criterion for the 3D Boussinesq equations involving the pressure gradient," *Abstract and Applied Analysis*, vol. 2014, Article ID 510924, 4 pages, 2014.

[24] Y. Meyer, P. Gerard, and F. Oru, "Inégalités de Sobolev précisées," in *Séminaire sur les Équations aux Dérivées Partielles, 1996-1997*, Exp. No. 4, p. 8, École Polytechnique, Palaiseau, France, 1997.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