Hindawi Publishing Corporation
International Journal of Computational Mathematics
Volume 2014, Article ID 643230, 6 pages
http://dx.doi.org/10.1155/2014/643230

Research Article

Extended f-Vector Equilibrium Problem

Khushbu and Zubair Khan

Department of Mathematics, Integral University, Lucknow, Uttar Pradesh 226026, India

Correspondence should be addressed to Khushbu; khushboo_bbkk@yahoo.in

Received 25 June 2014; Revised 23 October 2014; Accepted 14 November 2014; Published 3 December 2014

Academic Editor: Sheung-Hung Poon

We introduce and study extended f-vector equilibrium problem. By using KKM-Fan Theorem as basic tool, we prove existence theorem in the setting of Hausdorff topological vector space and reflexive Banach space. Some examples are also given.

1. Introduction

Equilibrium problems have been extensively studied in recent years; the origin of this can be traced back to Blum and Oettli [1] and Noor and Oettli [2]. The equilibrium problem is a generalization of classical variational inequalities and provides us with a systematic framework to study a wide class of problems arising in finance, economics, operations research, and so forth. General equilibrium problems have been extended to the case of vector-valued bifunctions, known as vector equilibrium problems. Vector equilibrium problems have attracted increasing interest of many researchers and provide a unified model for several classes of problems, for example, vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems; see [1–4] and references therein. Many existence results for vector equilibrium problems have been established by several eminent researchers; see, for example, [5–13].

The generalized monotonicity plays an important role in the literature of equilibrium problems and variational inequalities. There are a substantial number of papers on existence results for solving equilibrium problems and variational inequalities based on different monotonicity notions such as monotonicity, pseudomonotonicity, and quasimonotonicity.

Let X and Y be two Hausdorff topological vector spaces, let K be a nonempty, closed, and convex subset of X, and let C be a pointed, closed, convex cone in Y with $\operatorname{int} C \neq \emptyset$. Given a vector-valued mapping $f : K \times K \to Y$, the vector equilibrium problem consists of finding $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{1}$$

Inspired by the concept of monotonicity, KKM-Fan Theorem, and the other work done in the direction of generalization of vector equilibrium problems (see [14–16]) we introduce and study extended f-vector equilibrium problem and prove some existence results in the setting of Hausdorff topological vector spaces and reflexive Banach spaces.

2. Preliminaries

The following definitions and concepts are needed to prove the results of this paper.

Definition 1. The Hausdorff topological vector space Y is said to be an ordered space denoted by (Y, C) if ordering relations are defined in Y by a pointed, closed, convex cone C of Y as follows:

$$\begin{gathered} \forall x, y \in Y, \quad y \leqq x \Longleftrightarrow x - y \in C, \\ \forall x, y \in Y, \quad y \leq x \Longleftrightarrow x - y \in C \setminus \{0\}, \\ \forall x, y \in Y, \quad y \nleq x \Longleftrightarrow x - y \notin C \setminus \{0\}. \end{gathered} \tag{2}$$

If the interior of C is $\operatorname{int} C \neq \emptyset$, then the weak ordering relations in Y are also defined as follows:

$$\begin{aligned} &\forall x, y \in Y, \quad y < x \Longleftrightarrow x - y \in \operatorname{int} C, \\ &\forall x, y \in Y, \quad y \nless x \Longleftrightarrow x - y \notin \operatorname{int} C. \end{aligned} \tag{3}$$

Throughout this paper, unless otherwise specified, we assume that (Y, C) is an ordered Hausdorff topological vector space with $\operatorname{int} C \neq \emptyset$.

Definition 2. Let K be a nonempty convex subset of a topological vector space X. A set-valued mapping $A : K \to 2^X$ is said to be KKM-mapping, if, for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of K, $\operatorname{Co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} A(x_i)$, where $\operatorname{Co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

The following KKM-Fan Theorem is important for us to prove the existence results of this paper.

Theorem 3 (KKM-Fan Theorem). *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $A : K \to 2^X$ be a KKM-mapping such that $A(x)$ is closed for all $x \in K$ and $A(x)$ is compact for at least one $x \in K$; then*

$$\bigcap_{x \in K} A(x) \neq \emptyset. \tag{4}$$

Lemma 4. *Let (Y, C) be an ordered topological vector space with a pointed, closed, and convex cone C. Then for all $x, y \in Y$, one has the following:*

(i) $y - x \in \operatorname{int} C$ *and* $y \notin \operatorname{int} C$ *imply* $x \notin \operatorname{int} C$;

(ii) $y - x \in C$ *and* $y \notin \operatorname{int} C$ *imply* $x \notin \operatorname{int} C$;

(iii) $y - x \in -\operatorname{int} C$ *and* $y \notin -\operatorname{int} C$ *imply* $x \notin -\operatorname{int} C$;

(iv) $y - x \in -C$ *and* $y \notin -\operatorname{int} C$ *imply* $x \notin -\operatorname{int} C$.

Let X and Y be the Hausdorff topological vector spaces and let K be a nonempty, closed, convex subset of X. Let C be a pointed, closed, convex cone in Y with $\operatorname{int} C \neq \emptyset$. Let $g : K \times X \to Y$ be a vector-valued mapping and let $f : K \to X$ be another mapping. We introduce the following extended f-vector equilibrium problem.

Find $x_0 \in K$ such that for all $z \in K$ and $\lambda \in (0, 1]$

$$g(\lambda x_0 + (1 - \lambda) z, f(y)) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{5}$$

If $\lambda = 1$ and $f(y) = y \in K$, the problem (5) reduces to the vector equilibrium problem of finding $x_0 \in K$ such that

$$g(x_0, y) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{6}$$

Problem (6) was introduced and studied by Xuan and Nhat [13].

In addition, if $Y = R$ and $C = R_+$, then problem (5) reduces to the equilibrium problem of finding $x_0 \in K$ such that

$$g(x_0, y) \geq 0, \quad \forall y \in K. \tag{7}$$

Problem (7) was introduced and studied by Blum and Oettli [1].

The fact that problem (5) is much more general than many existing equilibria, vector equilibrium problems, and so forth motivated us to study extended f-vector equilibrium problem given by (5).

Definition 5. Let X and Y be the Hausdorff topological vector spaces and let K be a nonempty, closed, convex subset of X and let C be a pointed, closed, convex cone in Y with $\operatorname{int} C \neq \emptyset$. Let $g : K \times X \to Y$ and $f : K \to X$ be mappings. Then, g is said to be

(i) f-monotone with respect to C, if and only if, for all $x, y, z \in K$, $\lambda \in (0, 1]$,

$$g(\lambda x + (1 - \lambda) z, f(y)) + g(\lambda y + (1 - \lambda) z, f(x)) \in -C; \tag{8}$$

(ii) f-hemicontinuous, if and only if, for all $x, y \in K$, $t \in [0, 1]$, the mapping $t \to g(ty + (1 - t)x, f(x))$ is continuous at 0^+;

(iii) f-pseudomonotone, if and only if, for all $x, y, z \in K$, $\lambda \in (0, 1]$,

$$\begin{gathered} g(\lambda x + (1 - \lambda) z, f(y)) \notin -\operatorname{int} C \\ \text{implies } g(\lambda y + (1 - \lambda) z, f(x)) \notin \operatorname{int} C; \end{gathered} \tag{9}$$

(iv) f-generally convex, if and only if, for all $x, y, z, \in K$, $\lambda \in (0, 1]$,

$$\begin{gathered} g(z, f(x)) \notin -\operatorname{int} C, \\ g(z, f(y)) \notin -\operatorname{int} C, \\ \text{imply } g(z, f(\lambda x + (1 - \lambda) y)) \notin -\operatorname{int} C. \end{gathered} \tag{10}$$

In support of Definition 5, we have the following examples.

Example 6. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, and $C = \{(x, y) : x \leq 0, y \leq 0\}$.

Let $g : K \times X \to Y$ and $f : K \to X$ be mappings such that

$$\begin{gathered} g(x, y) = (y, x^2), \quad \forall x, y \in K, \\ f(x) = x^2, \quad \forall x \in K. \end{gathered} \tag{11}$$

Then,

$$\begin{aligned} g(\lambda x &+ (1 - \lambda) z, f(y)) + g(\lambda y + (1 - \lambda) z, f(x)) \\ &= (f(y), (\lambda x + (1 - \lambda) z)^2) \\ &\quad + (f(x), (\lambda y + (1 - \lambda) z)^2) \\ &= (y^2, (\lambda x + (1 - \lambda) z)^2) \\ &\quad + (x^2, (\lambda y + (1 - \lambda) z)^2) \in -C; \end{aligned} \tag{12}$$

that is, $g(\lambda x + (1 - \lambda) z, f(y)) + g(\lambda y + (1 - \lambda) z, f(x)) \in -C$.

Hence, g is f-monotone with respect to C.

Example 7. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, and $C = \{(x, y) : x \leq 0, y \leq 0\}$. Let $F : [0, 1] \to Y$ be a mapping such that

$$F(t) = g(ty + (1 - t) x, f(x)), \quad \forall t \in [0, 1]. \tag{13}$$

Let $g : K \times X \to Y$ and $f : K \to X$ be mappings such that

$$g(x, y) = (y, x^2), \quad \forall x, y \in K, \qquad f(x) = x^2, \quad \forall x \in K. \tag{14}$$

Then,

$$\begin{aligned} F(t) &= g(ty + (1-t)x, f(x)) \\ &= (f(x), (ty + (1-t)x)^2) \\ &= (x^2, (ty + (1-t)x)^2) \end{aligned} \tag{15}$$

which implies that $t \to g(ty + (1-t)x, f(x))$ is continuous at 0^+. Hence, g is f-hemicontinuous.

Example 8. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, and $C = \{(x, y) : x \geq 0, y \geq 0\}$.

Let $g : K \times X \to Y$ and $f : K \to X$ be mappings such that

$$g(x, y) = (y, -x^2), \quad \forall x, y \in K, \qquad f(x) = mx, \quad \forall x \in K, \ m \text{ is constant}. \tag{16}$$

Then,

$$\begin{aligned} & g(\lambda x + (1-\lambda)z, f(y)) \\ &\quad = (f(y), -(\lambda x + (1-\lambda)z)^2) \\ &\quad = (my, -(\lambda x + (1-\lambda)z)^2) \notin -\operatorname{int} C \end{aligned} \tag{17}$$

implies $m > 0$, so it follows that

$$\begin{aligned} & g(\lambda y + (1-\lambda)z, f(x)) \\ &\quad = (f(x), -(\lambda y + (1-\lambda)z)^2) \\ &\quad = (mx, -(\lambda x + (1-\lambda)z)^2) \notin \operatorname{int} C. \end{aligned} \tag{18}$$

Hence, g is f-pseudomonotone with respect to C.

Example 9. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, and $C = \{(x, y) : x \geq 0, y \leq 0\}$.

Let $g : K \times X \to Y$ and $f : K \to X$ be mappings such that

$$g(x, y) = (y, x^2), \quad \forall x, y \in K, \qquad f(x) = mx, \quad \forall x \in K, \ m \text{ is constant}. \tag{19}$$

Then,

$$\begin{aligned} g(z, f(x)) &= (f(x), z^2) \\ &= (mx, z^2) \notin -\operatorname{int} C \end{aligned} \tag{20}$$

implies $m > 0$, and

$$\begin{aligned} g(z, f(y)) &= (f(y), z^2) \\ &= (my, z^2) \notin -\operatorname{int} C \end{aligned} \tag{21}$$

again implies $m > 0$, so it follows that

$$\begin{aligned} & g(z, f(\lambda x + (1-\lambda)y)) \\ &\quad = (f(\lambda x + (1-\lambda)y), z^2) \\ &\quad = (m(\lambda x + (1-\lambda)y), z^2) \\ &\quad = (\lambda mx + (1-\lambda)my, z^2) \notin -\operatorname{int} C. \end{aligned} \tag{22}$$

Hence, g is f-generally convex.

Definition 10. A mapping $\eta : K \times K \to X$ is said to be affine in the first argument if and only if, for all $x, y, z \in K$ and $t \in [0, 1]$,

$$\eta(tx + (1-t)y, z) = t\eta(x, z) + (1-t)\eta(y, z). \tag{23}$$

Similarly, one can define the affine property of η with respect to the second argument.

Definition 11. Let (Y, C) be an ordered topological vector space. A mapping $T : X \to Y$ is called C-convex if and only if, for each pair $x, y \in K$ and $\lambda \in (0, 1]$,

$$T(\lambda x + (1-\lambda)y) \leq_C \lambda T(x) + (1-\lambda)T(y). \tag{24}$$

3. Existence Results

We prove the following equivalence lemma which we need for the proof of our main results.

Lemma 12. *Let X be a Hausdorff topological vector space, let K be a closed, convex subset of X, and let (Y, C) be an ordered Hausdorff topological vector space with* $\operatorname{int} C \neq \emptyset$. *Let $g : K \times X \to Y$ be a vector-valued mapping which is C-convex in the second argument, f-monotone with respect to C, positive homogeneous in the second argument, and f-hemicontinuous and let $f : K \to X$ be a continuous, affine mapping such that, for $z \in K$, $\lambda \in (0, 1]$, $g(\lambda x + (1-\lambda)z, f(x)) = 0$ for all $x \in K$. Then for all $z \in K$ and $\lambda \in (0, 1]$, the following statements are equivalent. Find $x_0 \in K$ such that*

(i) $g(\lambda x_0 + (1-\lambda)z, f(y)) \notin -\operatorname{int} C, \forall y \in K$;

(ii) $g(\lambda y + (1-\lambda)z, f(x_0)) \notin \operatorname{int} C, \forall y \in K$.

Proof. (i)⇒(ii). For all $z \in K$ and $\lambda \in (0, 1]$, let x_0 be a solution of (i); then we have

$$g(\lambda x_0 + (1-\lambda)z, f(y)) \notin -\operatorname{int} C. \tag{25}$$

Since g is f-monotone with respect to C, we have

$$\begin{aligned} & g(\lambda x_0 + (1-\lambda)z, f(y)) \\ &\quad + g(\lambda y + (1-\lambda)z, f(x_0)) \in -C \\ &\Longrightarrow g(\lambda x_0 + (1-\lambda)z, f(y)) \\ &\quad \in -C - g(\lambda y + (1-\lambda)z, f(x_0)). \end{aligned} \tag{26}$$

Suppose to the contrary that (ii) is false. Then, there exists $y \in K$ such that

$$g(\lambda y + (1-\lambda) z, f(x_0)) \in \operatorname{int} C. \tag{27}$$

By (26), we obtain

$$g(\lambda x_0 + (1-\lambda) z, f(y)) \in -C - \operatorname{int} C \subset -\operatorname{int} C \tag{28}$$

which contradicts (i). Thus, (i) $\Rightarrow$(ii).

(ii)$\Rightarrow$(i). Conversely, suppose that (ii) holds. Let $y \in K$ be arbitrary and taking $y_\alpha = \alpha y + (1-\alpha)x_0$, $\alpha \in (0,1)$ so $y_\alpha \in K$ as K is convex.

Therefore,

$$\begin{aligned} & g(\lambda y_\alpha + (1-\lambda) z, f(x_0)) \notin \operatorname{int} C \\ & \quad \Longrightarrow (1-\alpha) g(\lambda y_\alpha + (1-\lambda) z, f(x_0)) \notin \operatorname{int} C. \end{aligned} \tag{29}$$

As $g(\lambda x + (1-\lambda)z, f(x)) = 0$, we have

$$\begin{aligned} & g(\lambda y_\alpha + (1-\lambda) z, f(y_\alpha)) = 0 \\ & \quad \Longrightarrow g(\lambda y_\alpha + (1-\lambda) z, f(\alpha y + (1-\alpha) x_0)) = 0. \end{aligned} \tag{30}$$

Since f is affine, we have

$$g(\lambda y_\alpha + (1-\lambda) z, \alpha f(y) + (1-\alpha) f(x_0)) = 0. \tag{31}$$

Since g is C-convex in the second argument, therefore we have

$$\begin{aligned} 0 &= g(\lambda y_\alpha + (1-\lambda) z, \alpha f(y) + (1-\alpha) f(x_0)) \\ &\leq_C \alpha g(\lambda y_\alpha + (1-\lambda) z, f(y)) \\ &\quad + (1-\alpha) g(\lambda y_\alpha + (1-\lambda) z, f(x_0)) \\ &\Longrightarrow (1-\alpha) g(\lambda y_\alpha + (1-\lambda) z, f(x_0)) \\ &\quad + \alpha g(\lambda y_\alpha + (1-\lambda) z, f(y)) \in C. \end{aligned} \tag{32}$$

By (29), as $(1-\alpha)g(\lambda y_\alpha + (1-\lambda)z, f(x_0)) \notin \operatorname{int} C$ and using Lemma 4, we have from (32)

$$\begin{aligned} & -\alpha g(\lambda y_\alpha + (1-\lambda) z, f(y)) \notin \operatorname{int} C \\ & \quad \Longrightarrow g(\lambda y_\alpha + (1-\lambda) z, f(y)) \notin -\operatorname{int} C. \end{aligned} \tag{33}$$

Since g is f-hemicontinuous, therefore, for $\alpha \to 0^+$, we have

$$g(\lambda x_0 + (1-\lambda) z, f(y)) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{34}$$

Thus, (ii)$\Rightarrow$(i). This completes the proof. □

Theorem 13. *Let X be a Hausdorff topological vector space and let K be a compact and convex subset of X and let (Y, C) be an ordered Hausdorff topological vector space with $\operatorname{int} C \neq \emptyset$. Let $g : K \times X \to Y$ be a vector-valued, affine mapping which is f-monotone with respect to C, positive homogeneous in the second argument, and f-hemicontinuous and let $f : K \to X$ be a continuous, affine mapping such that, for $z \in K$, $\lambda \in (0,1]$, $g(\lambda x + (1-\lambda)z, f(x)) = 0$ for all $x \in K$ and let the mapping $x \to g(\lambda y + (1-\lambda)x, f(x))$ be continuous. Then problem (5) admits a solution; that is, for all $z \in K$ and $\lambda \in (0,1]$, there exists $x_0 \in K$ such that*

$$g(\lambda x_0 + (1-\lambda) z, f(y)) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{35}$$

Proof. For $y \in K$, we define

$$\begin{aligned} M(y) &= \{x \in K : g(\lambda x + (1-\lambda) z, f(y)) \notin -\operatorname{int} C\}, \\ S(y) &= \{x \in K : g(\lambda y + (1-\lambda) z, f(x)) \notin \operatorname{int} C\}. \end{aligned} \tag{36}$$

Clearly $M(y) \neq \emptyset$, as $y \in M(y)$. We divide the proof into three steps.

Step 1. We claim that $M : K \to 2^K$ is a KKM-mapping. If M is not a KKM-mapping, then there exists $u \in \operatorname{Co}\{y_1, y_2, \ldots, y_n\}$ such that, for all $t_i \in [0,1]$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n t_i = 1$, we have

$$u = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n M(y_i). \tag{37}$$

Thus, we have

$$g(\lambda u + (1-\lambda) z, f(y_i)) \in -\operatorname{int} C. \tag{38}$$

Since g is affine in the second argument and $g(\lambda u + (1-\lambda)z, f(u)) = 0$, we have

$$\begin{aligned} & \sum_{i=1}^n t_i g(\lambda u + (1-\lambda) z, f(y_i)) \in -\operatorname{int} C \\ & \quad \Longrightarrow g\left(\lambda u + (1-\lambda) z, f\left(\sum_{i=1}^n t_i y_i\right)\right) \in -\operatorname{int} C \\ & \quad \Longrightarrow g(\lambda u + (1-\lambda) z, f(u)) \in -\operatorname{int} C. \end{aligned} \tag{39}$$

It follows that $0 \in -\operatorname{int} C$ which contradicts to the pointedness of $C(x)$ and hence M is a KKM-mapping.

Step 2. One has $\bigcap_{y \in K} M(y) = \bigcap_{y \in K} S(y)$ and S is also a KKM-mapping.

If $x \in M(y)$, then $g(\lambda x + (1-\lambda)z, f(y)) \notin -\operatorname{int} C$. By the f-monotonicity of g with respect to C, we have

$$\begin{aligned} & g(\lambda x + (1-\lambda) z, f(y)) \\ & \quad + g(\lambda y + (1-\lambda) z, f(x)) \in -C \\ & \quad \Longrightarrow g(\lambda x + (1-\lambda) z, f(y)) \in -C \\ & \quad - g(\lambda y + (1-\lambda) z, f(x)). \end{aligned} \tag{40}$$

Suppose that $x \notin S(y)$. Then, we have

$$g(\lambda y + (1-\lambda) z, f(x)) \in \operatorname{int} C. \tag{41}$$

It follows from (40) that

$$g(\lambda x + (1-\lambda) z, f(y)) \in -C - \operatorname{int} C \subset -\operatorname{int} C \tag{42}$$

which contradicts the fact that $x \in M(y)$. Therefore, $x \in S(y)$; that is, $M(y) \subset S(y)$. Then,

$$\bigcap_{y \in k} M(y) \subset \bigcap_{y \in K} S(y). \tag{43}$$

On the other hand, suppose that $x \in \bigcap_{y\in K} S(y)$. We have

$$g(\lambda y + (1-\lambda) z, f(x)) \notin \operatorname{int} C, \quad \forall y \in K. \tag{44}$$

By Lemma 12, we have

$$g(\lambda x + (1-\lambda) z, f(y)) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{45}$$

That is, $x \in \bigcap_{y\in K} M(y)$. Hence,

$$\bigcap_{y\in k} M(y) \supset \bigcap_{y\in K} S(y). \tag{46}$$

So,

$$\bigcap_{y\in k} M(y) = \bigcap_{y\in K} S(y). \tag{47}$$

Also $\bigcap_{y\in K} S(y) \neq \emptyset$, since $y \in S(y)$. From above, we know that $M(y) \subset S(y)$ and, by Step 1, we know that M is a KKM-mapping. Thus, S is also a KKM-mapping.

Step 3. For all $y \in K$, $S(y)$ is closed.

Let $\{x_n\}$ be a sequence in $S(y)$ such that $\{x_n\}$ converges to $x \in K$. Then,

$$g(\lambda y + (1-\lambda) z, f(x_n)) \notin \operatorname{int} C \quad \forall n. \tag{48}$$

Since the mapping $x \to g(\lambda y+(1-\lambda)z, f(x))$ is continuous, we have

$$\begin{aligned} &g(\lambda y + (1-\lambda) z, f(x_n)) \\ &\quad \longrightarrow g(\lambda y + (1-\lambda) z, f(x)) \notin \operatorname{int} C. \end{aligned} \tag{49}$$

We conclude that $x \in S(y)$; that is, $S(y)$ is a closed subset of a compact set K and hence compact.

By KKM-Theorem 3, $\bigcap_{y\in K} S(y) \neq \emptyset$ and also $\bigcap_{y\in K} M(y) \neq \emptyset$. Hence, there exists $x_0 \in \bigcap_{y\in k} M(y) = \bigcap_{y\in K} S(y)$; that is, there exists $x_0 \in K$ such that

$$g(\lambda x_0 + (1-\lambda) z, f(y)) \notin -\operatorname{int} C \quad \forall y, z \in K,\ \lambda \in (0,1]. \tag{50}$$

Thus, x_0 is a solution of problem (5). □

In support of Theorem 13, we give the following example.

Example 14. Let $= Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $C = \{x : x \geq 0\}$.

Let $g : K \times X \to Y$ and $f : K \to X$ be mappings such that

$$\begin{gathered} g(x, y) = xy, \quad \forall x, y \in K, \\ f(x) = -mx \quad \forall x \in K,\ m > 0. \end{gathered} \tag{51}$$

Then,

(i) for any $x, y, z \in K$,

$$\begin{aligned} &g(\lambda x + (1-\lambda) z, f(y)) + g(\lambda y + (1-\lambda) z, f(x)) \\ &\quad = (\lambda x + (1-\lambda) z)(-my) \\ &\qquad + (\lambda y + (1-\lambda) z)(-mx) \\ &\quad = -2\lambda mxy - (1-\lambda)(myz + mzx) \\ &\quad = -m(2\lambda xy + (1-\lambda)(yz + zx)) \in -C \quad \text{for } m > 0; \end{aligned} \tag{52}$$

that is, g is f-monotone with respect to C;

(ii) for any $r > 0$,

$$g(x, ry) = xry = r(xy) = rg(x, y); \tag{53}$$

that is, g is positive homogeneous in the second argument;

(iii) let $F : [0,1] \to Y$ be a mapping such that $F(t) = g(ty + (1-t)x, f(x))$, $\forall t \in [0,1]$; then,

$$F(t) = g(ty + (1-t)x, f(x)) = (ty + (1-t)x)(-mx) \tag{54}$$

which is a continuous mapping; that is, $t \to g(ty + (1-t)x, f(x))$ is continuous at 0^+. Hence, g is f-hemicontinuous.

(iv) Let $G : K \to Y$ be a mapping such that

$$G(x) = g(\lambda y + (1-\lambda) x, f(x)), \quad \forall x \in K; \tag{55}$$

then,

$$G(x) = g(\lambda y + (1-\lambda) x, f(x)) = (\lambda y + (1-\lambda) x)(-mx) \tag{56}$$

which implies that $x \to g(\lambda y + (1-\lambda)x, f(x))$ is continuous mapping.

Hence, all the conditions of Theorem 13 are satisfied.
In addition,

$$\begin{aligned} &g(\lambda x + (1-\lambda) z, f(y)) \\ &\quad = (\lambda x + (1-\lambda) z)(-my) \\ &\quad = -m(\lambda x + (1-\lambda) z) y \notin \operatorname{int} C \quad \text{for } m > 0. \end{aligned} \tag{57}$$

Thus, it follows that x is a solution of problem (5) for all $z \in K$ and $\lambda \in (0,1]$.

Corollary 15. *Let K be a compact and convex subset of X and let (Y, C) be an ordered topological vector space with $\operatorname{int} C \neq \emptyset$. Let $g : K \times X \to Y$ be a vector-valued mapping which is f-pseudomonotone with respect to C and let $f : K \to X$ be a continuous, affine mapping such that, for $z \in K$, $\lambda \in (0,1]$, $g(\lambda x + (1-\lambda)z, f(x)) = 0$ for all $x \in K$. Let the mapping $x \to g(\lambda x+(1-\lambda)z, f(y))$ be continuous. Then, problem (5) is solvable.*

Proof. By Step 1 of Theorem 13, it follows that M is a KKM-mapping. Also it follows from f-pseudomonotonicity of g that $M(y) \subset S(y)$; thus S is also a KKM-mapping. By Step 3 of Theorem 13, the conclusion follows. □

Theorem 16. *Let X be a reflexive Banach space; let (Y, C) be an ordered topological vector space with $\operatorname{int} C \neq \emptyset$. Let K be a nonempty, bounded, and convex subset of X. Let $g : K \times X \to Y$ be a vector-valued mapping which is f-monotone with respect to C, positive homogeneous in the second argument, f-hemicontinuous, and f-generally convex on K. Let $f : K \to X$*

be a continuous, affine mapping such that, for $z \in K$, $\lambda \in (0,1]$, $g(\lambda x + (1-\lambda)z, f(x)) = 0$ for all $x \in K$. Then, problem (5) admits a solution, that is, for all $z \in K$ and $\lambda \in (0,1]$, there exists $x_0 \in K$ such that

$$g(\lambda x_0 + (1-\lambda)z, f(y)) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{58}$$

Proof. For each $y \in K$, let

$$M(y) = \{x \in K : g(\lambda x + (1-\lambda)z, f(y)) \notin -\operatorname{int} C\},$$
$$S(y) = \{x \in K : g(\lambda y + (1-\lambda)z, f(x)) \notin \operatorname{int} C\} \tag{59}$$

for all $z \in K$ and $\lambda \in (0,1]$.

From the proof of Theorem 13, we know that $S(y)$ is closed and S is a KKM-mapping. We also know that

$$\bigcap_{y\in k} M(y) = \bigcap_{y\in K} S(y). \tag{60}$$

Since K is a bounded, closed, and convex subset of a reflexive Banach space X, therefore, K is weakly compact.

Now, we show that $S(y)$ is convex. Suppose that $y_1, y_2 \in S(y)$ and $t_1, t_2 \geq 0$ with $t_1 + t_2 = 1$.

Then,

$$g(\lambda y + (1-\lambda)z, f(y_i)) \notin \operatorname{int} C, \quad i = 1, 2. \tag{61}$$

Since g is f-generally convex, we have

$$g(\lambda y + (1-\lambda)z, f(t_1 y_1 + t_2 y_2)) \notin \operatorname{int} C; \tag{62}$$

that is, $t_1 y_1 + t_2 y_2 \in S(y)$, which implies that $S(y)$ is convex. Since $S(y)$ is closed and convex, $S(y)$ is weakly closed.

As S is a KKM-mapping, $S(y)$ is weakly closed subset of K; therefore $S(y)$ is weakly compact. By KKM-Theorem 3, there exists $x_0 \in K$ such that $x_0 \in \bigcap_{y\in k} M(y) = \bigcap_{y\in K} S(y) \neq \emptyset$. That is, there exists $x_0 \in K$ such that

$$g(\lambda x_0 + (1-\lambda)z, f(y)) \notin -\operatorname{int} C,$$
$$\forall y, z \in K, \quad \lambda \in (0,1]. \tag{63}$$

Hence, problem (5) is solvable. □

4. Conclusion

In this paper, some existence results for extended f-vector equilibrium problem are proved in the setting of Hausdorff topological vector spaces and reflexive Banach spaces. The concept of monotonicity plays an important role in obtaining existence results. The results of this paper can be viewed as generalizations of some known equilibrium problems as explained by (6) and (7).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[2] M. A. Noor and W. Oettli, "On general nonlinear complementarity problems and quasi-equilibria," *Le Matematiche*, vol. 49, no. 2, pp. 313–331, 1994.

[3] G.-Y. Chen and B. D. Craven, "A vector variational inequality and optimization over an efficient set," *Zeitschrift für Operations Research*, vol. 34, no. 1, pp. 1–12, 1990.

[4] G. Y. Chen and B. D. Craven, "Existence and continuity of solutions for vector optimization," *Journal of Optimization Theory and Applications*, vol. 81, no. 3, pp. 459–468, 1994.

[5] Q. H. Ansari and J.-C. Yao, "An existence result for the generalized vector equilibrium problem," *Applied Mathematics Letters*, vol. 12, no. 8, pp. 53–56, 1999.

[6] Q. H. Ansari, S. Schaible, and J. C. Yao, "System of vector equilibrium problems and its applications," *Journal of Optimization Theory and Applications*, vol. 107, no. 3, pp. 547–557, 2000.

[7] Q. H. Ansari and J. C. Yao, "On vector quasi—equilibrium problems," in *Equilibrium Problems and Variational Models*, vol. 68 of *Nonconvex Optimization and Its Applications*, pp. 1–18, 2003.

[8] Q. H. Ansari and F. Flores-Bazán, "Generalized vector quasi-equilibrium problems with applications," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 1, pp. 246–256, 2003.

[9] M. Bianchi, N. Hadjisavvas, and S. Schaible, "Vector equilibrium problems with generalized monotone bifunctions," *Journal of Optimization Theory and Applications*, vol. 92, no. 3, pp. 527–542, 1997.

[10] A. P. Farajzadeh and B. S. Lee, "On dual vector equilibrium problems," *Applied Mathematics Letters*, vol. 25, no. 6, pp. 974–979, 2012.

[11] A. Khaliq, "Implicit vector quasi-equilibrium problems with applications to variational inequalities," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 63, no. 5–7, pp. e1823–e1831, 2005.

[12] J. Li and N.-J. Huang, "Implicit vector equilibrium problems via nonlinear scalarisation," *Bulletin of the Australian Mathematical Society*, vol. 72, no. 1, pp. 161–172, 2005.

[13] N. Xuan and P. Nhat, "On the existence of equilibrium points of vector functions," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 141–156, 1998.

[14] R. Ahmad and M. Akram, "Extended vector equilibrium problem," *British Journal of Mathematics & Computer Science*, vol. 4, pp. 546–556, 2013.

[15] A. Farajzadeh, "On Vector Equilibrium Problems," nsc10.cankaya.edu.tr.

[16] M. Rahaman, A. Kılıçman, and R. Ahmad, "Extended mixed vector equilibrium problems," *Abstract and Applied Analysis*, vol. 2014, Article ID 376759, 6 pages, 2014.