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Research Article

Continued Fractions of Order Six and New Eisenstein Series Identities

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We prove two identities for Ramanujan's cubic continued fraction and a continued fraction of Ramanujan, which are analogues of Ramanujan's identities for the Rogers-Ramanujan continued fraction. We further derive Eisenstein series identities associated with Ramanujan's cubic continued fraction and Ramanujan's continued fraction of order six.

1. Introduction

Throughout this paper, we assume that $|q| < 1$ and for each positive integer n, we use the standard product notation

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right), \quad n \geq 1,$$
$$(a;q)_\infty := \prod_{j=0}^{\infty}\left(1-aq^j\right). \tag{1}$$

Srinivasa Ramanujan made some significant contributions to the theory of continued fraction expansions. The most beautiful continued fraction expansions can be found in Chapters 12 and 16 of his second notebook [1].

The celebrated Rogers-Ramanujan continued fraction is defined by [2]

$$R(q) := q^{1/5}\frac{f\left(-q,-q^4\right)}{f\left(-q^2,-q^3\right)} = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}, \tag{2}$$

where

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},$$
$$= 1+\sum_{n=1}^{\infty}(ab)^{n(n-1)/2}\left(a^n+b^n\right), \tag{3}$$
$$|ab| < 1,$$

is Ramanujan's general theta function.

Ramanujan eventually found several generalizations and ramifications of $R(q)$ which can be found in his notebooks [1] and "lost notebook" [3]. Recently, Liu [4] and Chan et al. [5] have established several new identities associated with the Rogers-Ramanujan continued fraction $R(q)$ including Eisenstein series identities involving $R(q)$.

The beautiful Ramanujan's cubic continued fraction $G(q)$, first introduced by Srinivasa Ramanujan in his second letter to Hardy [2, page xxvii], is defined by

$$G(q) := q^{1/3}\frac{f\left(-q,-q^5\right)}{f\left(-q^3,-q^3\right)}$$
$$= \cfrac{q^{1/3}}{1+\cfrac{q+q^2}{1+\cfrac{q^2+q^4}{1+\cfrac{q^3+q^6}{1+\cdots}}}}. \tag{4}$$

Adiga et al. [6], Bhargava et al. [7], Chan [8], and Vasuki et al. [9] have proved several elegant theorems for $G(q)$, many of which are analogues of well-known properties satisfied by the Rogers-Ramanujan continued fraction.

Recently, Vasuki et al. [10] have studied the following continued fraction of order six:

$$X(q) := q^{1/4}\frac{f(-q,-q^5)}{f(-q^2,-q^4)} = \frac{q^{1/4}(1-q^{-2})}{(1-q^{-3/2})+\dfrac{(q^{1/4}-q^{-1/4})(q^{-7/4}-q^{7/4})}{(1-q^{-3/2})(1+q^3)+\cdots}}. \quad (5)$$

The continued fraction (5) is a special case of a fascinating continued fraction identity recorded by Ramanujan in his second notebook [1], [11, page 24]. Furthermore, they have established modular relations between the continued fractions $X(q)$ and $X(q^n)$ for $n = 2, 3, 5, 7$, and 11.

In Section 3 of this paper, we establish two new identities associated with the continued fractions $G(q)$ and $X(q)$, using the quintuple product identity. In Section 4, we derive Eisenstein series identities associated with $G(q)$ and $X(q)$.

2. Definitions and Preliminary Results

In this section, we present some basic definitions and preliminary results. One of the most interesting special cases of $f(a,b)$ is [11, Entry 22]

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \quad (6)$$

Note that the Dedekind eta function $\eta(\tau) = q^{1/24}f(-q)$, where $q = e^{2\pi i\tau}$, Im $\tau > 0$. We need the following three lemmas to prove our main results.

Lemma 1 (see [11, Entry 30, page 46]). *One has*

$$f(a,b)+f(-a,-b) = 2f(a^3b, ab^3). \quad (7)$$

Lemma 2 (see [11, page 80]). *One has*

$$f\left(B^3q,\frac{q^5}{B^3}\right) - B^2 f\left(\frac{q}{B^3},B^3q^5\right) = f(-q^2)\frac{f(-B^2,-q^2/B^2)}{f(Bq,q/B)}. \quad (8)$$

Lemma 3 (see [12, Lemma 2(ii)]). *Let $m = [s/(s-r)]$, $l = m(s-r)-r$, $k = -m(s-r)+s$, and $h = mr-(m(m-1)(s-r))/2$, $0 \le r < s$. Here $[x]$ denotes the largest integer less than or equal to x. Then*

(i) $f(q^{-r},q^s) = q^{-h}f(q^l,q^k)$;

(ii) $f(-q^{-r},-q^s) = (-1)^m q^{-h}f(-q^l,-q^k)$.

3. Main Results

The Jacobi triple product identity states that

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}z^n = (q;q)_\infty (z;q)_\infty \left(\frac{q}{z};q\right)_\infty, \quad z \neq 0. \quad (9)$$

In Ramanujan's notation, the Jacobi triple product identity takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \quad (10)$$

The Jacobi triple product identity was first proved by Gauss [13]. Using (3), we have

$$f(-e^{2iz},-qe^{-2iz}) = 2ie^{iz}\sum_{n=0}^{\infty}(-1)^{n+1}q^{n(n+1)/2}\sin(2n+1)z. \quad (11)$$

Putting $a = -e^{2iz}$ and $b = -qe^{-2iz}$ in (10), we obtain

$$f(-e^{2iz},-qe^{-2iz}) = (1-e^{2iz})(qe^{2iz};q)_\infty(qe^{-2iz};q)_\infty(q;q)_\infty. \quad (12)$$

Putting $z = \pi/6$ and $z = 2\pi/6$, respectively, in (12), we obtain

$$f(-e^{i\pi/3},-qe^{-i\pi/3}) = (1-e^{i\pi/3})(qe^{i\pi/3};q)_\infty(qe^{-i\pi/3};q)_\infty(q;q)_\infty, \quad (13)$$

$$f(-e^{i2\pi/3},-qe^{-i2\pi/3}) = (1-e^{i2\pi/3})(qe^{i2\pi/3};q)_\infty(qe^{-i2\pi/3};q)_\infty(q;q)_\infty. \quad (14)$$

Multiplying (13) and (14) together and using the identities

$$(1-e^{i\pi/3})(1-e^{i2\pi/3}) = -i\sqrt{3},$$
$$(1-x)(1-xe^{2\pi i/6})(1-xe^{-2\pi i/6})(1-xe^{4\pi i/6}) \times (1-xe^{-4\pi i/6})(1-xe^{6\pi i/6}) = (1-x^6), \quad (15)$$

in the resulting equation and then after some simplifications, we obtain the following identity:

$$f(-e^{i\pi/3},-qe^{-i\pi/3})f(-e^{i2\pi/3},-qe^{-i2\pi/3}) = -i\sqrt{3}q^{-1/4}\frac{\eta^2(\tau)\eta(6\tau)}{\eta(2\tau)}. \quad (16)$$

Theorem 4. *Let $|q| < 1$, $\alpha = -1$, and $\beta = 1$. Then*

$$\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^{n/2}+q^n)} - \prod_{n=1}^{\infty}\frac{1}{(1+\beta q^{n/2}+q^n)} = 4q^{1/24}\frac{\eta(\tau)\eta^2(3\tau)}{\eta(\tau)\eta^2(3\tau/2)+\eta^2(\tau/2)\eta(3\tau)}G(q), \quad (17)$$

$$\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^{n/2}+q^n)} - \prod_{n=1}^{\infty}\frac{1}{(1+\beta q^{n/2}+q^n)} = 4q^{1/24}\frac{\eta(\tau)\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta^2(3\tau/2)+\eta^2(\tau/2)\eta(3\tau)}X(q). \quad (18)$$

Proof. We may rewrite (13) and (14) as follows:

$$f\left(-e^{i\pi/3},-qe^{-i\pi/3}\right) = \left(1-e^{i\pi/3}\right)q^{-1/24}\eta(\tau)\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right),$$
$$f\left(-e^{i2\pi/3},-qe^{-i2\pi/3}\right) = \left(1-e^{i2\pi/3}\right)q^{-1/24}\eta(\tau)\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right), \tag{19}$$

where

$$\alpha=-2\cos\frac{\pi}{3}=-1,\qquad \beta=-2\cos\frac{2\pi}{3}=1. \tag{20}$$

Then,

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)=\frac{q^{1/24}}{\eta(\tau)}\frac{f\left(-e^{i\pi/3},-qe^{-i\pi/3}\right)}{\left(1-e^{i\pi/3}\right)}, \tag{21}$$

$$\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)=\frac{q^{1/24}}{\eta(\tau)}\frac{f\left(-e^{i2\pi/3},-qe^{-i2\pi/3}\right)}{\left(1-e^{i2\pi/3}\right)}. \tag{22}$$

Multiplying the above two equations together and then using (16), in resulting identity, we find that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)\left(1+\beta q^n+q^{2n}\right)=q^{-1/6}\frac{\eta(6\tau)}{\eta(2\tau)}. \tag{23}$$

Subtracting (22) from (21), we obtain

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)-\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right) = \frac{q^{1/24}}{\eta(\tau)}\left(\frac{f\left(-e^{i\pi/3},-qe^{-i\pi/3}\right)}{\left(1-e^{i\pi/3}\right)}-\frac{f\left(-e^{i2\pi/3},-qe^{-i2\pi/3}\right)}{\left(1-e^{i2\pi/3}\right)}\right). \tag{24}$$

Using (11) in (24), we deduce that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)-\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right) = \frac{q^{-1/12}}{\eta(\tau)}\sum_{n=0}^{\infty}(-1)^nP(n)q^{(2n+1)^2/8}, \tag{25}$$

where

$$P(n)=\frac{2\sin(2n+1)(\pi/6)}{ie^{-i\pi/6}\left(1-e^{i\pi/3}\right)}-\frac{2\sin(2n+1)(2\pi/6)}{ie^{-i2\pi/6}\left(1-e^{i2\pi/3}\right)}. \tag{26}$$

Now, by direct computations, we find that

$$P(6m+0)=0,\qquad P(6m+1)=2,$$
$$P(6m+2)=2,\qquad P(6m+3)=-2, \tag{27}$$
$$P(6m+4)=-2,\qquad P(6m+5)=0.$$

Therefore,

$$\sum_{n=0}^{\infty}(-1)^nP(n)q^{(2n+1)^2/8} = 2\left\{-\sum_{m=0}^{\infty}q^{(12m+3)^2/8}+\sum_{m=0}^{\infty}q^{(12m+5)^2/8}+\sum_{m=0}^{\infty}q^{(12m+7)^2/8}-\sum_{m=0}^{\infty}q^{(12m+9)^2/8}\right\}. \tag{28}$$

In the right-hand side of the above equation, changing m to $m-1$ in the first two summations and also changing m to $-m$ in the last two summations, we obtain

$$\sum_{n=0}^{\infty}(-1)^nP(n)q^{(2n+1)^2/8} = 2\left\{\sum_{m=-\infty}^{\infty}q^{(12m-7)^2/8}-\sum_{m=-\infty}^{\infty}q^{(12m-9)^2/8}\right\}. \tag{29}$$

Now, using the definition of $f(a,b)$ in the right-hand side of the above equation, we find that

$$\sum_{n=0}^{\infty}(-1)^nP(n)q^{(2n+1)^2/8} = -2q^{9/8}\left\{f\left(q^9,q^{27}\right)-q^2f\left(q^3,q^{33}\right)\right\}. \tag{30}$$

In the quintuple product identity (8), replacing q by q^6 and then setting $B=q$, we find that

$$f\left(q^9,q^{27}\right)-q^2f\left(q^3,q^{33}\right)=\frac{f\left(-q^{12}\right)f\left(-q^2,-q^{10}\right)}{f\left(q^5,q^7\right)}. \tag{31}$$

Combining (25), (30), and (31), we find that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)-\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right) = \frac{-2q^{25/24}}{\eta(\tau)}\left\{\frac{f\left(-q^{12}\right)f\left(-q^2,-q^{10}\right)}{f\left(q^5,q^7\right)}\right\}. \tag{32}$$

Dividing both sides of (32) by $\prod_{n=1}^{\infty}(1+\alpha q^n+q^{2n})(1+\beta q^n+q^{2n})$ and then using (23), we obtain

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)^{-1}-\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)^{-1} = 2q^{29/24}\frac{\eta(2\tau)}{\eta(\tau)\eta(6\tau)}\left\{\frac{f\left(-q^{12}\right)f\left(-q^2,-q^{10}\right)}{f\left(q^5,q^7\right)}\right\}. \tag{33}$$

Replacing a by q and b by q^2 in (7), we find that

$$f\left(q,q^2\right)+f\left(-q,-q^2\right)=2f\left(q^5,q^7\right). \tag{34}$$

Now, using the above equation in the right-hand side of (33) and then changing q to $q^{1/2}$ throughout, we obtain (17). Equation (18) follows from the following identity:

$$G(q) = \frac{\eta(2\tau)\eta(6\tau)}{\eta^2(3\tau)} X(q). \tag{35}$$

This completes the proof of Theorem 4. □

4. Eisenstein Series Identities Associated with $G(q)$ and $X(q)$

In this section, we present four Eisenstein series identities associated with $G(q)$ and $X(q)$.

Theorem 5. *Let* $|q| < 1$. *Then*

$$\sum_{\substack{n=1\\ n\equiv 1(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} - \sum_{\substack{n=1\\ n\equiv 2(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} = \frac{\eta^3(18\tau)}{\eta(6\tau)}\left[\frac{1}{G(q^3)} - 1 - G(q^3)\right]. \tag{36}$$

Proof. Changing n to $-n$ in the second summation, of the left-hand side of Theorem 5, we have

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} - \sum_{\substack{n=1\\ n\equiv 2(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} \\
&= \sum_{\substack{n=1\\ n\equiv 1(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} - \sum_{\substack{n=-\infty\\ n\equiv 1(\bmod 3)}}^{-1} \frac{q^{-n} - q^{-3n} + q^{-5n}}{1-q^{-6n}} \\
&= \sum_{n=-\infty}^{\infty} \frac{q^{3n+1}}{1-q^{18n+6}} - \sum_{n=-\infty}^{\infty} \frac{q^{9n+3}}{1-q^{18n+6}} + \sum_{n=-\infty}^{\infty} \frac{q^{15n+5}}{1-q^{18n+6}}.
\end{aligned} \tag{37}$$

Using a corollary of Ramanujan's ${}_1\Psi_1$ summation formula [11, Entry 17, page 32]

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{(az, q/az, q, q; q)_\infty}{(a, q/a, z, q/z; q)_\infty}, \quad |q| < |z| < 1, \tag{38}$$

and Lemma 3 in (37), we find that

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} - \sum_{\substack{n=1\\ n\equiv 2(\bmod 3)}}^{\infty} \frac{q^n - q^{3n} + q^{5n}}{1-q^{6n}} \\
&= q\frac{(q^9, q^9, q^{18}, q^{18}; q^{18})_\infty}{(q^6, q^{12}, q^3, q^{15}; q^{18})_\infty} - q^3\frac{(q^{15}, q^3, q^{18}, q^{18}; q^{18})_\infty}{(q^6, q^{12}, q^9, q^9; q^{18})_\infty} \\
&\quad + q^5\frac{(q^{21}, q^{-3}, q^{18}, q^{18}; q^{18})_\infty}{(q^6, q^{12}, q^{15}, q^3; q^{18})_\infty} \\
&= \frac{(q^{18}; q^{18})_\infty^2}{(q^6, q^{12}; q^{18})_\infty}\left\{q\frac{f(-q^9, -q^9)}{f(-q^3, -q^{15})} - q^3\frac{f(-q^3, -q^{15})}{f(-q^9, -q^9)}\right. \\
&\quad \left. -q^2\frac{f(-q^3, -q^{15})}{f(-q^3, -q^{15})}\right\}.
\end{aligned} \tag{39}$$

Using (4) in (39) and after some simplifications, we obtain (36). □

Theorem 6. *Let* $|q| < 1$. *Then*

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1(\bmod 6)}}^{\infty} \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1-q^{6n}} \\
&\quad - \sum_{\substack{n=1\\ n\equiv 5(\bmod 6)}}^{\infty} \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1-q^{6n}} \\
&= \frac{\eta(12\tau)\eta(18\tau)\eta(36\tau)}{\eta(6\tau)} \\
&\quad \times \left[\frac{1}{X(q^6)} - \frac{\eta^2(18\tau)}{\eta(12\tau)\eta(36\tau)} - X(q^6)\right].
\end{aligned} \tag{40}$$

Proof. Using the identity

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{x^n}{1-yq^n} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x^n y^m q^{mn} = \sum_{n=0}^{\infty} \frac{y^n}{1-xq^n}, \quad |x|, |y| < 1,
\end{aligned} \tag{41}$$

the left-hand side of Theorem 6 can be written as

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1(\bmod 6)}}^{\infty} \frac{q^n - q^{2n} - q^{4n}}{1-q^{6n}} - \sum_{\substack{n=-\infty\\ n\equiv 1(\bmod 6)}}^{-1} \frac{-q^{-2n} - q^{-4n} + q^{-5n}}{1-q^{-6n}} \\
&= \sum_{n=-\infty}^{\infty} \frac{q^{6n+1}}{1-q^{36n+6}} - \sum_{n=-\infty}^{\infty} \frac{q^{12n+2}}{1-q^{36n+6}} - \sum_{n=-\infty}^{\infty} \frac{q^{24n+4}}{1-q^{36n+6}}.
\end{aligned} \tag{42}$$

Using (38) in (42), we obtain

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1(\bmod 6)}}^{\infty} \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1-q^{6n}} \\
&\quad - \sum_{\substack{n=1\\ n\equiv 5(\bmod 6)}}^{\infty} \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1-q^{6n}} \\
&= \frac{(q^{36};q^{36})_\infty^2}{(q^6,q^{30};q^{36})_\infty} \\
&\quad \left\{ q\frac{(q^{12},q^{24};q^{36})_\infty}{(q^6,q^{30};q^{36})_\infty} - q^2\frac{(q^{18},q^{18};q^{36})_\infty}{(q^{12},q^{24};q^{36})_\infty} \right. \\
&\quad \left. -q^4\frac{(q^6,q^{30};q^{36})_\infty}{(q^{12},q^{24};q^{36})_\infty} \right\}.
\end{aligned} \tag{43}$$

Using (5) in (43) and after some simplifications, we obtain (40). □

Differentiating both sides of (12) and then setting $z = 0$ yield

$$f'(-1,-q) = -2i(q;q)_\infty^3, \tag{44}$$

where f' denotes the partial derivative of f with respect to z.

Now we prove a lemma, which is useful to prove Eisenstein series identities associated with $G(q)$ and $X(q)$.

Lemma 7. *Consider the following:*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1-q^{6n}} \sin 2nz \\
&= \left(-e^{2iz} f'\left(-1,-q^6\right) f\left(-q^3,-q^3\right) f\left(-q,-q^5\right)\right. \\
&\quad \left.\times f\left(-e^{4iz},-q^6e^{-4iz}\right)\right) \\
&\quad \times \left(4q^2 f\left(-qe^{2iz},-q^5e^{-2iz}\right)\right. \\
&\qquad \left.\times f(-q^{-1}e^{2iz},-q^7e^{-2iz}) f(-q^2e^{2iz},-q^4e^{-2iz})\right)^{-1} \\
&\quad \times \frac{1}{f\left(-q^{-2}e^{2iz},-q^8e^{-2iz}\right)}.
\end{aligned} \tag{45}$$

Proof. For simplicity, we use $F(z,q)$ to denote the logarithmic derivative of $iq^{1/8}e^{-iz}f(-e^{2iz},-qe^{-2iz})$ with respect to z. To prove this lemma we need the following identity, which can be found in [14, Theorem 5], [15, Corollary 2]:

$$\begin{aligned}
&F(z_1,q) + F(z_2,q) + F(z_3,q) - F(z_1+z_2+z_3,q) \\
&= \left(-f'(-1,-q) f\left(-e^{2i(z_1+z_2)},-qe^{-2i(z_1+z_2)}\right)\right. \\
&\quad \left.\times f\left(-e^{2i(z_2+z_3)},-qe^{-2i(z_2+z_3)}\right)\right) \\
&\quad \times \left(f\left(-e^{2iz_1},-qe^{-2iz_1}\right)\right. \\
&\qquad \left.\times f(-e^{2iz_2},-qe^{-2iz_2}) f(-e^{2iz_3},-qe^{-2iz_3})\right)^{-1} \\
&\quad \times \frac{f\left(-e^{2i(z_1+z_3)},-qe^{-2i(z_1+z_3)}\right)}{f\left(-e^{2i(z_1+z_2+z_3)},-qe^{-2i(z_1+z_2+z_3)}\right)}.
\end{aligned} \tag{46}$$

As the proof of this lemma is similar to that of Lemma 1 in [4], we omit the details. □

Using (45), we derive the following Eisenstein series identity.

Theorem 8. *Let $|q| < 1$. Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n\left(q^n - q^{2n} - q^{4n} + q^{5n}\right)}{1-q^{6n}} \\
&= \frac{\eta^6(3\tau)}{\eta^2(\tau)} G(q) = \frac{\eta^4(3\tau)\,\eta(6\tau)\,\eta(2\tau)}{\eta^2(\tau)} X(q).
\end{aligned} \tag{47}$$

Proof. Dividing both sides of (45) by z and then letting $z \to 0$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n\left(q^n - q^{2n} - q^{4n} + q^{5n}\right)}{1-q^{6n}} \\
&= \left(\left(q^6;q^6\right)_\infty^6 f\left(-q^3,-q^3\right) f\left(-q^1,-q^5\right)\right) \\
&\quad \times \left(q^2 f\left(-q^1,-q^5\right) f\left(-q^{-1},-q^7\right)\right. \\
&\qquad \left.\times f(-q^2,-q^4) f(-q^{-2},-q^8)\right)^{-1}.
\end{aligned} \tag{48}$$

Using Lemma 3 in (48), we complete the proof of Theorem 8. □

We use $(\cdot/p)$ to denote the Legendre symbol modulo p. Setting $z = \pi/3$ in (45) and noting that $\sin(2n\pi/3) = (\sqrt{3}/2)(n/3)$, we find that

$$\begin{aligned}
&\frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right) \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1-q^{6n}} \\
&= \left(-e^{2i\pi/3} f'\left(-1,-q^6\right) f\left(-q^3,-q^3\right)\right. \\
&\quad \left.\times f\left(-q^1,-q^5\right) f\left(-e^{4i\pi/3},-q^6e^{-4i\pi/3}\right)\right) \\
&\quad \times \left(4q^2 f\left(-e^{2i((\pi/3)+\pi\tau)},-q^6e^{-2i((\pi/3)+\pi\tau)}\right)\right. \\
&\qquad \left.\times f(-e^{2i((\pi/3)-\pi\tau)},-q^6e^{-2i((\pi/3)-\pi\tau)})\right)^{-1} \\
&\quad \times (1)\left(f\left(-e^{2i((\pi/3)+2\pi\tau)},-q^6e^{-2i((\pi/3)+2\pi\tau)}\right)\right. \\
&\qquad \left.\times f(-e^{2i((\pi/3)-2\pi\tau)},-q^6e^{-2i((\pi/3)-2\pi\tau)})\right)^{-1}.
\end{aligned} \tag{49}$$

Recall the identity [16, Eq. (3. 1)]

$$\begin{aligned} & f\left(-e^{2i((\pi/3)-z)}, -qe^{-2i((\pi/3)-z)}\right) \\ & \quad \times f\left(-e^{2i((\pi/3)+z)}, -qe^{-2i((\pi/3)+z)}\right) \\ & = \frac{e^{-2iz}(q;q)_\infty^3}{e^{(-i2\pi)/3}(q^3;q^3)_\infty} \frac{f\left(-e^{6iz}, -q^3 e^{-6iz}\right)}{f\left(-e^{2iz}, -qe^{-2iz}\right)}. \end{aligned} \tag{50}$$

Using the above identity in (49), we obtain the following Eisenstein series identity.

Theorem 9. *Let* $|q| < 1$. *Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n - q^{2n} - q^{4n} + q^{5n}}{1 - q^{6n}} \\ & = \frac{\eta^2(3\tau)\,\eta(\tau)\,\eta(9\tau)\,\eta(18\tau)}{\eta^3(6\tau)} G(q) \\ & = \frac{\eta(\tau)\,\eta(2\tau)\,\eta(9\tau)\,\eta(18\tau)}{\eta^2(6\tau)} X(q). \end{aligned} \tag{51}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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