Hindawi Publishing Corporation
Chinese Journal of Mathematics
Volume 2014, Article ID 676470, 11 pages
http://dx.doi.org/10.1155/2014/676470

*Research Article*

# New Oscillation Criteria for Third-Order Nonlinear Mixed Neutral Difference Equations

**Elmetwally Mohammed Elabbasy,[1] Magdy Yoseph Barsom,[1] and Faisal Saleh AL-dheleai[2]**

[1] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[2] *Department of Mathematics, Faculty of Education and Languages, Amran University, Amran, Yemen*

Correspondence should be addressed to Faisal Saleh AL-dheleai; faisalsaleh69@yahoo.com

Received 8 September 2013; Accepted 14 November 2013; Published 10 February 2014

Academic Editors: C. Bai, Y. Fu, W. Klingenberg, and P. K. Sahoo

Copyright © 2014 Elmetwally Mohammed Elabbasy et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Some new oscillation criteria are established for a third-order nonlinear mixed neutral difference equation. Our results improve and extend some known results in the literature. Several examples are given to illustrate the importance of the results.

## 1. Introduction

By a Riccati transformation technique, we present some new oscillation criteria for the nonlinear difference equation of the form

$$\begin{aligned}\Delta\left(a(n)\Delta\left(b(n)\Delta\left(x(n)+p_1(n)x(n-\sigma_1)\right.\right.\right.\\ \left.\left.\left.+p_2(n)x(n+\sigma_2)\right)\right)\right)\\ +q_1(n)x(n-\sigma_3)+q_2(n)x(n+\sigma_4)=0,\\ n\geq n_0,\end{aligned}\tag{1}$$

where $n_0 \in N$ is a fixed integer and $\Delta$ denotes the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$ and $\Delta^i x(n) = \Delta(\Delta^{i-1}x(n))$. Throughout this paper, we will assume the following hypotheses:

($A_1$) $a(n), b(n) > 0$ for $n \in N(n_0)$,

($A_2$) $p_i(n)$ is positive, $0 \leq p_i(n) \leq p_i$, where $p_i$ are constants for $i = 1, 2$,

($A_3$) $q_i(n) > 0$ on $N(n_0)$ for $i = 1, 2$,

($A_4$) $\sigma_i \geq 0$ are constants for $i = 1, 2, 3, 4$.

We set $z(n) := x(n) + p_1(n)x(n-\sigma_1) + p_2(n)x(n+\sigma_2)$. By a solution of (1), we mean a nontrivial sequence $\{x(n)\}$ defined on $N(n_0)$, which satisfies (1) for all $n \geq n_0$. A solution $\{x(n)\}$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation (1) is called oscillatory if all its solutions are oscillatory. In recent years, there has been an increasing interest in the study of the oscillatory behavior of solutions of difference equations; we refer to the books [1–5] and the papers [6–8] and the references cited therein.

In [9], via comparison with first-order oscillatory difference equations, Agarwal et al. proved that several theorems provided sufficient conditions for oscillation of all solutions of the third-order difference equation of the form

$$\begin{aligned}\Delta\left(\frac{1}{a_2(n)}\left(\Delta\left(\frac{1}{a_1(n)}\right)(\Delta x(n))^{\alpha_1}\right)^{\alpha_2}\right)\\ \pm q(n)f\left(x\left(g(n)\right)\right)=0,\end{aligned}\tag{2}$$

depending on the condition

$$\sum^{\infty}\left(a_i(n)\right)^{1/\alpha_i}=\infty,\quad i=1,2.\tag{3}$$

In [10], by a Riccati transformation technique, Schmeidel studied the oscillatory and asymptotic behavior of solutions of the third-order difference equation

$$\Delta\left(a(n)\Delta\left(b(n)\left(\Delta y(n)\right)\right)\right)+p(n)f\left(y(n+l)\right)=0,\tag{4}$$

using the condition

$$\sum_{n=1}^{\infty} a^{-1}(n) = \infty, \qquad \sum_{n=1}^{\infty} b^{-1}(n) = \infty. \tag{5}$$

In [11], by the generalized Riccati transformation technique, Saker et al. established some new oscillation criteria for a certain class of third-order nonlinear delay difference equation of the form

$$\Delta\left(c(n)\,\Delta(d(n))\,(\Delta x(n))^{\gamma}\right) + q(n)\,f(x(n-\sigma)) = 0. \tag{6}$$

In [12], by a Riccati transformation technique, Selvaraj et al. established some sufficient conditions for oscillation of all solutions of the third-order nonlinear difference equation of the form

$$\Delta\left(\frac{1}{a(n)}\Delta^2 y(n)\right) + p(n)\,f\left(y(\sigma(n))\right) = 0, \tag{7}$$

using the condition

$$\sum_{n=n_0}^{\infty} a(n) = \infty. \tag{8}$$

In [13], by means of a Riccati transformation technique, Thandapani et al. discussed the oscillatory behavior of the solutions of the difference equation of the form

$$\begin{aligned}&\Delta\left(a(n)\left(\Delta^2\left(x(n) + p(n)\,x(n-\delta)\right)\right)^{\alpha}\right)\\&\quad + q(n)\,x^{\alpha}(n-\tau) = 0,\end{aligned} \tag{9}$$

using the condition

$$\sum_{n=n_0}^{\infty} (a(n))^{-1/\alpha} = \infty. \tag{10}$$

In [14], by using condition (5) and a Riccati transformation technique, Aktas et al. obtained some new oscillation criteria of third-order nonlinear delay difference equation of the form

$$\begin{aligned}&\Delta\left(a(n)\,\Delta(b(n))\,(\Delta x(n))\right) + p(n)\,\Delta x(n+1)\\&\quad + q(n)\,f(x(n-\sigma)) = 0.\end{aligned} \tag{11}$$

In [15], using condition (8), Selvaraj et al. considered nonlinear third-order difference equations of the form

$$\Delta^2\left(\frac{1}{a(n)}\Delta x(n)\right) + q(n)\,f(x(\sigma(n))) = 0, \tag{12}$$

and they studied the oscillatory behavior of solutions of (12).

In [16], Saker investigated the third-order difference equation

$$\Delta\left(a(n)\left(\Delta(b(n))\,(\Delta x(n))\right)^{\gamma}\right) + q(n)\,f\left(x\left(g(n)\right)\right) = 0, \tag{13}$$

using the condition

$$\sum_{n=n_0}^{\infty} a^{-\gamma}(n) = \infty, \qquad \sum_{n=n_0}^{\infty} b^{-1}(n) = \infty. \tag{14}$$

In [17], under condition (5), Selvaraj et al. discussed the oscillatory behavior of neutral delay difference equation of the form

$$\begin{aligned}&\Delta\left(a(n)\,\Delta\left(b(n)\,\Delta\left(x(n) + c(n)\,x(n-k)\right)\right)\right)\\&\quad + q(n+1)\,f\left(x(n+1-l)\right) = 0.\end{aligned} \tag{15}$$

In [18], by comparison with some first difference equations whose oscillatory characters are known and by means of a Riccati transformation technique, Elabbasy et al. obtained several new sufficient conditions for the oscillation of all solutions of the nonlinear difference equation with Delay of the form

$$\begin{aligned}&\Delta\left(a(n)\left(\Delta\left(b(n)\,(\Delta x(n))^{\alpha_1}\right)\right)^{\alpha_2}\right)\\&\quad + \sum_{i=1}^{n} q_i(n)\,f\left(x\left(g_i(n)\right)\right) = 0.\end{aligned} \tag{16}$$

In this paper, the details of the proofs of results for nonoscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions. The paper is organized as follows. In Section 2, we will state and prove the main oscillation theorems and in Section 3 will we provide some examples to illustrate the main results.

## 2. Main Results

In this section, we establish some new oscillation criteria for (1) under the following condition

$$\sum_{n=n_0}^{\infty} a^{-1}(n) = \sum_{n=n_0}^{\infty} b^{-1}(n) = \infty. \tag{17}$$

In the following results, we will use the following notations:

$$\begin{aligned}&Q_1(n) := \min\left\{q_1(n), q_1(n-\sigma_1), q_1(n+\sigma_2)\right\},\\&Q_2(n) := \min\left\{q_2(n), q_2(n-\sigma_1), q_2(n+\sigma_2)\right\},\\&Q(n) = Q_1(n) + Q_2(n), \qquad \delta(n) := \sum_{v=n}^{\infty}\frac{1}{a(v)},\\&\varphi_1(n) := \frac{\rho(n)}{\rho^2(n+1)\,b(n-\sigma_3)}\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s),\\&\varphi_2(n) := \frac{\rho(n)}{\rho^2(n+1)\,b(n-\sigma_1)}\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s),\\&\vartheta(m,n) := \left(\frac{\Delta\rho(n)}{\rho(n+1)} - \frac{h(m,n)}{\sqrt{H(m,n)}}\right).\end{aligned} \tag{18}$$

We assume that there exists a double sequence $\{H(m,n) \mid m \geq n \geq 0\}$ and $h(m,n)$ such that

(i) $H(m,m) = 0$ for $m \geq 0$,

(ii) $H(m,n) > 0$ for $m > n > 0$,

(iii) $\Delta_2 H(m,n) = H(m,n+1) - H(m,n) \le 0$ for $m > n \ge 0$,

(iv) $h(m,n) = -\Delta_2 H(m,n)/\sqrt{H(m,n)}$.

We begin with these lemmas, which will be used in obtaining our main results. The proof of the following lemmas are similar to that of Lemmas 2.1, 2.2, and 2.3, respectively, in [18], and hence the details are omitted.

**Lemma 1.** *Let $\{x(n)\}$ be an eventually positive solution of* (1) *and suppose that $\{z(n)\}$ satisfies $\Delta z(n) > 0$, $\Delta(b(n)\Delta z(n)) > 0$, and $\Delta(a(n)\Delta(b(n)\Delta z(n))) \le 0$, for all $n \ge N$. Then there exists $n \ge n_1 \ge n_2$ such that*

$$\Delta z(n) \ge b^{-1}(n)\left(a(n)\Delta\left(b(n)\Delta z(n)\right)\right)\sum_{s=n_2}^{n-1} a^{-1}(s). \tag{19}$$

**Lemma 2.** *Assume that* (17) *holds. Let $\{x(n)\}$ be an eventually positive solution of* (1). *Then for sufficiently large $n$, there are only two possible cases:*

(I) $\Delta z(n) > 0$, $\Delta(b(n)(\Delta z(n))) > 0$,

*or*

(II) $\Delta z(n) < 0$, $\Delta(b(n)(\Delta z(n))) > 0$.

**Lemma 3.** *Assume that* (17) *holds. Let $\{x(n)\}$ be an eventually positive solution of* (1) *and suppose that (II) of Lemma 2 holds. If*

$$\sum_{v=n_0}^{\infty}\left(b^{-1}(v)\left(\sum_{u=n_0}^{v-1} a^{-1}(u) \times\left(\sum_{s=n_2}^{u-1}\left(q_1(s)+q_2(s)\right)\right)\right)\right) = \infty, \tag{20}$$

*then $x(n) \to 0$ as $n \to \infty$.*

Next, we state and prove the main theorems.

**Theorem 4.** *Assume that* (20) *holds. Further, assume that $\sigma_1 \le \sigma_3$ and there exists a positive nondecreasing sequence $\{\rho(n)\}$, such that*

$$\sum_{s=n_0}^{n-1}\left(\rho(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{(\Delta\rho(s))^2 b(s-\sigma_3)}{\rho(s)\sum_{u=n_2}^{s-\sigma_3-1} a^{-1}(u)}\right) = \infty. \tag{21}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty} x(n) = 0$.

*Proof.* Assume that (1) has a nonoscillatory solution, say $x(n) > 0$, $x(n-\sigma_1) > 0$, $x(n+\sigma_2) > 0$, $x(n-\sigma_3) > 0$, and $x(n+\sigma_4) > 0$ for all $n \ge n_0$. From (1), we see that $z(n) > x(n) > 0$ and

$$\begin{aligned}&\Delta\left(a(n)\Delta\left(b(n)\Delta z(n)\right)\right)\\&\quad = -q_1(n)x(n-\sigma_3) - q_2(n)x(n+\sigma_4) \le 0.\end{aligned} \tag{22}$$

Then, $a(n)\Delta(b(n)\Delta z(n))$ is nonincreasing sequence and thus $\Delta z(n)$ and $\Delta(b(n)\Delta z(n))$ are eventually of one sign. By Lemma 2, there exist two possible cases (I) and (II). Assume that (I) holds. From (1) and the definition of $z(n)$, we have

$$\begin{aligned}&\Delta\left(a(n)\Delta\left(b(n)\Delta z(n)\right)\right)\\&\quad + q_1(n)x(n-\sigma_3) + q_2(n)x(n+\sigma_4)\\&\quad + p_1\Delta\left(a(n-\sigma_1)\Delta\left(b(n-\sigma_1)\Delta z(n-\sigma_1)\right)\right)\\&\quad + p_1 q_1(n-\sigma_1)x(n-\sigma_1-\sigma_3)\\&\quad + p_1 q_2(n-\sigma_1)x(n+\sigma_4-\sigma_1)\\&\quad + p_2\Delta\left(a(n+\sigma_2)\Delta\left(b(n+\sigma_2)\Delta z(n+\sigma_2)\right)\right)\\&\quad + p_2 q_1(n+\sigma_2)x(n+\sigma_2-\sigma_3)\\&\quad + p_2 q_2(n+\sigma_2)x(n+\sigma_2+\sigma_4) = 0.\end{aligned} \tag{23}$$

Thus,

$$\begin{aligned}&\Delta\left(a(n)\Delta\left(b(n)\Delta z(n)\right)\right)\\&\quad + p_1\Delta\left(a(n-\sigma_1)\Delta\left(b(n-\sigma_1)\Delta z(n-\sigma_1)\right)\right)\\&\quad + p_2\Delta\left(a(n+\sigma_2)\Delta\left(b(n+\sigma_2)\Delta z(n+\sigma_2)\right)\right)\\&\quad + Q_1(n)z(n-\sigma_3) + Q_2(n)z(n+\sigma_4) \le 0.\end{aligned} \tag{24}$$

It follows from $\Delta z(n) > 0$ that $z(n+\sigma_4) \ge z(n-\sigma_3)$. Thus, by (24), we obtain

$$\begin{aligned}&\Delta\left(a(n)\Delta\left(b(n)\Delta z(n)\right)\right)\\&\quad + p_1\Delta\left(a(n-\sigma_1)\Delta\left(b(n-\sigma_1)\Delta z(n-\sigma_1)\right)\right)\\&\quad + p_2\Delta\left(a(n+\sigma_2)\Delta\left(b(n+\sigma_2)\Delta z(n+\sigma_2)\right)\right)\\&\quad + Q(n)z(n-\sigma_3) \le 0.\end{aligned} \tag{25}$$

Define a Riccati substitution

$$\omega_1(n) := \rho(n)\frac{a(n)\Delta\left(b(n)\Delta z(n)\right)}{z(n-\sigma_3)}. \tag{26}$$

Then $\omega_1(n) > 0$. From (26), we have

$$\begin{aligned}&\Delta\omega_1(n)\\&\quad = \Delta\rho(n)\frac{a(n+1)\left(\Delta\left(b(n+1)\Delta z(n+1)\right)\right)}{z(n+1-\sigma_3)}\\&\quad + \rho(n)\frac{\Delta\left(a(n)\left(\Delta\left(b(n)\Delta z(n)\right)\right)\right)}{z(n-\sigma_3)}\\&\quad - \rho(n)\frac{a(n+1)\Delta\left(b(n+1)\Delta z(n+1)\right)\Delta\left(z(n-\sigma_3)\right)}{z(n-\sigma_3)z(n+1-\sigma_3)}.\end{aligned} \tag{27}$$

From Lemma 1, $\Delta(a(n)(\Delta(b(n)\Delta z(n)))) \le 0$, and $n-\sigma_3 < n$, we get

$$\begin{aligned}
&\Delta\left(z\left(n-\sigma_3\right)\right)\\
&\ge b^{-1}\left(n-\sigma_3\right)\left(a\left(n-\sigma_3\right)\left(\Delta\left(b\left(n-\sigma_3\right)\Delta z\left(n-\sigma_3\right)\right)\right)\right)\\
&\quad\times\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s)\\
&\ge b^{-1}\left(n-\sigma_3\right)\left(a(n)\left(\Delta\left(b(n)\Delta z(n)\right)\right)\right)\\
&\quad\times\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s)\\
&\ge b^{-1}\left(n-\sigma_3\right)\left(a(n+1)\left(\Delta\left(b(n+1)\Delta z(n+1)\right)\right)\right)\\
&\quad\times\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s).
\end{aligned}\tag{28}$$

From (27) and (28), we obtain

$$\begin{aligned}
\Delta\omega_1(n) \le{}& \frac{\Delta\rho(n)}{\rho(n+1)}\omega_1(n+1)\\
&+\rho(n)\frac{\Delta\left(a(n)\Delta\left(b(n)\Delta z(n)\right)\right)}{z\left(n-\sigma_3\right)}\\
&-\frac{\rho(n)}{\rho^2(n+1)b\left(n-\sigma_3\right)}\left(\omega_1(n+1)\right)^2\\
&\times\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s).
\end{aligned}\tag{29}$$

Next, define another sequence $\omega_2(n)$ by

$$\omega_2(n) := \rho(n)\frac{a\left(n-\sigma_1\right)\Delta\left(b\left(n-\sigma_1\right)\Delta z\left(n-\sigma_1\right)\right)}{z\left(n-\sigma_3\right)}.\tag{30}$$

Then $\omega_2(n) > 0$. From (30), we have

$$\begin{aligned}
&\Delta\omega_2(n)\\
&=\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1)\\
&\quad+\rho(n)\frac{\Delta\left(a\left(n-\sigma_1\right)\Delta\left(b\left(n-\sigma_1\right)\Delta z\left(n-\sigma_1\right)\right)\right)}{z\left(n-\sigma_3\right)}\\
&\quad-\rho(n)\left(a\left(n+1-\sigma_1\right)\left(\Delta\left(b\left(n+1-\sigma_1\right)\Delta z\left(n+1-\sigma_1\right)\right)\right)\right.\\
&\qquad\times\Delta\left(z\left(n-\sigma_3\right)\right)\\
&\qquad\left.\times\left(z\left(n+1-\sigma_1\right)z\left(n-\sigma_3\right)\right)^{-1}\right).
\end{aligned}\tag{31}$$

From Lemma 1, and $\sigma_3 \ge \sigma_1$, we get

$$\begin{aligned}
&\Delta\left(z\left(n-\sigma_3\right)\right)\\
&\ge b^{-1}\left(n-\sigma_3\right)\left(a\left(n-\sigma_3\right)\left(\Delta\left(b\left(n-\sigma_3\right)\Delta z\left(n-\sigma_3\right)\right)\right)\right)\\
&\quad\times\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s)\\
&\ge\left(a\left(n-\sigma_1\right)\left(\Delta\left(b\left(n-\sigma_1\right)\Delta z\left(n-\sigma_1\right)\right)\right)\right)\\
&\quad\times b^{-1}\left(n-\sigma_3\right)\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s)\\
&\ge\left(a\left(n+1-\sigma_1\right)\left(\Delta\left(b\left(n+1-\sigma_1\right)\Delta z\left(n+1-\sigma_1\right)\right)\right)\right)\\
&\quad\times b^{-1}\left(n-\sigma_3\right)\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s).
\end{aligned}\tag{32}$$

Then from (30), (31), and the above inequality, we have

$$\begin{aligned}
&\Delta\omega_2(n)\\
&\le\rho(n)\frac{\Delta\left(a\left(n-\sigma_1\right)\Delta\left(b\left(n-\sigma_1\right)\Delta z\left(n-\sigma_1\right)\right)\right)}{z\left(n-\sigma_3\right)}\\
&\quad+\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1)\\
&\quad-\frac{\rho(n)}{\rho^2(n+1)b\left(n-\sigma_3\right)}\left(\omega_2(n+1)\right)^2\sum_{s=n_2}^{n-\sigma_3-1} a^{-1}(s).
\end{aligned}\tag{33}$$

Similarly, we define another sequence $\omega_3(n)$ by

$$\omega_3(n) := \rho(n)\frac{a\left(n+\sigma_2\right)\Delta\left(b\left(n+\sigma_2\right)\Delta z\left(n+\sigma_2\right)\right)}{z\left(n-\sigma_3\right)}.\tag{34}$$

Then $\omega_3(n) > 0$. From (34), we have

$$\begin{aligned}
&\Delta\omega_3(n)\\
&=\frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1)\\
&\quad+\rho(n)\frac{\Delta\left(a\left(n+\sigma_2\right)\Delta\left(b\left(n+\sigma_2\right)\Delta z\left(n+\sigma_2\right)\right)\right)}{z\left(n-\sigma_3\right)}\\
&\quad-\rho(n)\left(a\left(n+1+\sigma_2\right)\right.\\
&\qquad\times\left(\Delta\left(b\left(n+1+\sigma_2\right)\Delta z\left(n+1+\sigma_2\right)\right)\right)\\
&\qquad\times\Delta\left(z\left(n-\sigma_3\right)\right)\\
&\qquad\left.\times\left(z\left(n+1-\sigma_3\right)z\left(n-\sigma_3\right)\right)^{-1}\right).
\end{aligned}\tag{35}$$

From Lemma 1 and $n-\sigma_3 < n+\sigma_2$, we get

$$\begin{aligned}
&\Delta z(n-\sigma_3)\\
&\quad\geq b^{-1}(n-\sigma_3)\\
&\qquad\times\left(a(n-\sigma_3)\left(\Delta\left(b(n-\sigma_3)\Delta z(n-\sigma_3)\right)\right)\right)\\
&\qquad\times\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)\\
&\quad\geq\left(a(n+\sigma_2)\left(\Delta\left(b(n+\sigma_2)\Delta z(n+\sigma_2)\right)\right)\right)\\
&\qquad\times b^{-1}(n-\sigma_3)\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)\\
&\quad\geq\left(a(n+1+\sigma_2)\right.\\
&\qquad\left.\times\left(\Delta\left(b(n+1+\sigma_2)\Delta z(n+1+\sigma_2)\right)\right)\right)\\
&\qquad\times b^{-1}(n-\sigma_3)\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s).
\end{aligned}\tag{36}$$

Then from (34), (35), and the above inequality, we have

$$\begin{aligned}
&\Delta\omega_3(n)\\
&\quad\leq\frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1)\\
&\qquad+\rho(n)\frac{\Delta\left(a(n+\sigma_2)\Delta\left(b(n+\sigma_2)\Delta z(n+\sigma_2)\right)\right)}{z(n-\sigma_3)}\\
&\qquad-\frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_3)}\left(\omega_3(n+1)\right)^2\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s).
\end{aligned}\tag{37}$$

From (25), (29), (33), and (37), we obtain

$$\begin{aligned}
&\Delta\omega_1(n)+p_1\Delta\omega_2(n)+p_2\Delta\omega_3(n)\\
&\quad\leq-\rho(n)Q(n)+\frac{\Delta\rho(n)}{\rho(n+1)}\omega_1(n+1)\\
&\qquad-\frac{\rho(n)\left(\omega_1(n+1)\right)^2}{\rho^2(n+1)b(n-\sigma_3)}\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)\\
&\qquad+p_1\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1)\\
&\qquad-p_1\frac{\rho(n)\left(\omega_2(n+1)\right)^2}{\rho^2(n+1)b(n-\sigma_3)}\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)\\
&\qquad+p_2\frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1)\\
&\qquad-p_2\frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_3)}\\
&\qquad\times\left(\omega_3(n+1)\right)^2\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s).
\end{aligned}\tag{38}$$

Using (38) and the inequality

$$Bu-Au^2\leq\frac{B^2}{4A},\quad A>0,\tag{39}$$

we have

$$\begin{aligned}
&\Delta\omega_1(n)+p_1\Delta\omega_2(n)+p_2\Delta\omega_3(n)\\
&\quad\leq-\rho(n)Q(n)+\frac{1}{4}\frac{\left(\Delta\rho(n)\right)^2b(n-\sigma_3)}{\rho(n)\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)}\\
&\qquad+\frac{p_1}{4}\frac{\left(\Delta\rho(n)\right)^2b(n-\sigma_3)}{\rho(n)\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)}\\
&\qquad+\frac{p_2}{4}\frac{\left(\Delta\rho(n)\right)^2b(n-\sigma_3)}{\rho(n)\sum_{s=n_2}^{n-\sigma_3-1}a^{-1}(s)}.
\end{aligned}\tag{40}$$

Summing the last inequality from $n_3$ to $n-1$, we obtain

$$\begin{aligned}
&\sum_{s=n_3}^{n-1}\left(\rho(s)Q(s)\right.\\
&\qquad\left.-\frac{(1+p_1+p_2)}{4}\frac{\left(\Delta\rho(s)\right)^2b(s-\sigma_3)}{\rho(s)\sum_{u=n_2}^{s-\sigma_3-1}a^{-1}(u)}\right)\\
&\quad\leq\omega_1(n_3)+p_1\omega_2(n_3)+p_2\omega_3(n_3),
\end{aligned}\tag{41}$$

which yields

$$\begin{aligned}
&\sum_{s=n_3}^{n-1}\left(\rho(s)Q(s)\right.\\
&\qquad\left.-\frac{(1+p_1+p_2)}{4}\frac{\left(\Delta\rho(s)\right)^2b(s-\sigma_3)}{\rho(s)\sum_{u=n_2}^{s-\sigma_3-1}a^{-1}(u)}\right)\leq c_1,
\end{aligned}\tag{42}$$

where $c_1>0$ is a finite constant. But this contradicts (21). Next we assume that (II) holds. We are then back to the proof of Lemma 3 to show that $\lim_{n\to\infty}x(n)=0$. The proof is complete. □

**Corollary 5.** *Assume that all the assumptions of Theorem 4 hold, except that condition* (21) *is replaced by*

$$\begin{aligned}
&\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}\rho(s)Q(s)=\infty,\\
&\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}\frac{\left(\Delta\rho(s)\right)^2b(s-\sigma_3)}{\rho(s)\sum_{u=n_2}^{s-\sigma_3-1}a^{-1}(u)}<\infty.
\end{aligned}\tag{43}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty}x(n)=0$.

*Remark 6.* Note that, from Theorem 4, we can obtain different conditions for oscillation of all solutions of (1) by different choices of $\{\rho(n)\}$. Let $\rho(n)=1, \rho(n)=n$ and $\rho(n)=n^\lambda$, $n\geq n_0$, and $\lambda>1$ is a constant. By Theorem 4, we have the following results.

**Corollary 7.** *Assume that all the assumptions of Theorem 4 hold, except that condition* (21) *is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}Q(s)=\infty. \tag{44}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty}x(n)=0$.

**Corollary 8.** *Assume that all the assumptions of Theorem 4 hold, except that condition* (21) *is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}\left(sQ(s)-\frac{(1+p_1+p_2)}{4}\times\frac{b(s-\sigma_3)}{s\sum_{u=n_2}^{s-\sigma_3-1}a^{-1}(u)}\right)=\infty. \tag{45}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty}x(n)=0$.

**Corollary 9.** *Assume that all the assumptions of Theorem 4 hold, except that condition* (21) *is replaced by*

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left(l^{\lambda}Q(l)-\frac{(1+p_1+p_2)}{4}\times\frac{\left((l+1)^{\lambda}-l^{\lambda}\right)^2 b(l-\sigma_3)}{l^{\lambda}\sum_{u=n_2}^{l-\sigma_3-1}a^{-1}(u)}\right)=\infty. \tag{46}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty}x(n)=0$.

**Theorem 10.** *Assume that* (20) *holds. Let* $\sigma_1\le\sigma_3$ *and* $\{\rho(n)\}$ *be a positive sequence. Furthermore, one assumes that there exists a double sequence* $\{H(m,n)\mid m\ge n\ge 0\}$. *If*

$$\lim_{m\to\infty}\sup\frac{1}{H(m,0)}\times\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)Q(n)-(1+p_1+p_2)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)}\right)=\infty, \tag{47}$$

*then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty}x(n)=0$.

*Proof.* Proceeding as in the proof of Theorem 4, we assume that (1) has a nonoscillatory solution, say $x(n)>0$, $x(n-\sigma_1)>0$, $x(n+\sigma_2)>0$, $x(n-\sigma_3)>0$, and $x(n+\sigma_4)>0$ for all $n\ge n_0$. By Lemma 2, there are two possible cases. If (I) holds, from the proof of Theorem 4, we find that (38) holds for all $n\ge n_1$. From (38), we have

$$\begin{aligned}\rho(n)Q(n)\le{}&-\Delta\omega_1(n)-p_1\Delta\omega_2(n)\\&-p_2\Delta\omega_3(n)+\frac{\Delta\rho(n)}{\rho(n+1)}\omega_1(n+1)\\&-\varphi_1(n)\left(\omega_1(n+1)\right)^2\\&+p_1\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1)\\&-p_1\varphi_1(n)\left(\omega_2(n+1)\right)^2\\&+p_2\frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1)\\&-p_2\varphi_1(n)\left(\omega_3(n+1)\right)^2.\end{aligned} \tag{48}$$

Therefore, we have

$$\begin{aligned}&\sum_{n=k}^{m-1}H(m,n)\rho(n)Q(n)\\&\quad\le-\sum_{n=k}^{m-1}H(m,n)\Delta\omega_1(n)\\&\qquad-p_1\sum_{n=k}^{m-1}H(m,n)\Delta\omega_2(n)\\&\qquad-p_2\sum_{n=k}^{m-1}H(m,n)\Delta\omega_3(n)\\&\qquad+\sum_{n=k}^{m-1}H(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}\omega_1(n+1)\\&\qquad-\sum_{n=k}^{m-1}H(m,n)\varphi_1(n)\left(\omega_1(n+1)\right)^2\\&\qquad+p_1\sum_{n=k}^{m-1}H(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1)\\&\qquad-p_1\sum_{n=k}^{m-1}H(m,n)\varphi_1(n)\left(\omega_2(n+1)\right)^2\\&\qquad+p_2\sum_{n=k}^{m-1}\frac{H(m,n)\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1)\\&\qquad-p_2\sum_{n=k}^{m-1}H(m,n)\varphi_1(n)\left(\omega_3(n+1)\right)^2,\end{aligned} \tag{49}$$

which yields after summing by parts

$$\begin{aligned}
&\sum_{n=k}^{m-1} H(m,n)\rho(n)Q(n) \\
&\quad \le H(m,k)\omega_1(k) + \sum_{n=k}^{m-1} \vartheta(m,n)H(m,n)\omega_1(n+1) \\
&\qquad - \sum_{n=k}^{m-1} H(m,n)\varphi_1(n)\left(\omega_1(n+1)\right)^2 \\
&\qquad + p_1 H(m,k)\omega_2(k) \\
&\qquad + p_1 \sum_{n=k}^{m-1} \vartheta(m,n)H(m,n)\omega_2(n+1) \\
&\qquad - p_1 \sum_{n=k}^{m-1} H(m,n)\varphi_1(n)\left(\omega_2(n+1)\right)^2 \\
&\qquad + p_2 H(m,k)\omega_3(k) \\
&\qquad + p_2 \sum_{n=k}^{m-1} \vartheta(m,n)H(m,n)\omega_3(n+1) \\
&\qquad - p_2 \sum_{n=k}^{m-1} H(m,n)\varphi_1(n)\left(\omega_3(n+1)\right)^2.
\end{aligned} \tag{50}$$

By completing the square, we have

$$\begin{aligned}
&\sum_{n=k}^{m-1} H(m,n)\rho(n)Q(n) \\
&\quad \le H(m,k)\omega_1(k) \\
&\qquad + \sum_{n=k}^{m-1} \frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \\
&\qquad - \sum_{n=k}^{m-1} \left[ \sqrt{H(m,n)\varphi_1(n)}\,\omega_1(n+1) \right. \\
&\qquad\qquad \left. - \frac{\vartheta(m,n)}{2}\sqrt{\frac{H(m,n)}{\varphi_1(n)}} \right]^2 \\
&\qquad + p_1 H(m,k)\omega_2(k) \\
&\qquad + p_1 \sum_{n=k}^{m-1} \frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \\
&\qquad - p_1 \sum_{n=k}^{m-1} \left[ \sqrt{H(m,n)\varphi_1(n)}\,\omega_2(n+1) \right. \\
&\qquad\qquad \left. - \frac{\vartheta(m,n)}{2}\sqrt{\frac{H(m,n)}{\varphi_1(n)}} \right]^2 \\
&\qquad + p_2 H(m,k)\omega_3(k) + p_2 \sum_{n=k}^{m-1} \frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \\
&\qquad - p_2 \sum_{n=k}^{m-1} \left[ \sqrt{H(m,n)\varphi_1(n)}\,\omega_3(n+1) \right. \\
&\qquad\qquad \left. - \frac{\vartheta(m,n)}{2}\sqrt{\frac{H(m,n)}{\varphi_1(n)}} \right]^2.
\end{aligned} \tag{51}$$

Then,

$$\begin{aligned}
&\sum_{n=k}^{m-1} \left( H(m,n)\rho(n)Q(n) \right. \\
&\qquad \left. - (1+p_1+p_2)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \right) \\
&\quad \le H(m,k)\omega_1(k) + p_1 H(m,k)\omega_2(k) \\
&\qquad + p_2 H(m,k)\omega_3(k),
\end{aligned} \tag{52}$$

which implies

$$\begin{aligned}
&\sum_{n=k}^{m-1} \left( H(m,n)\rho(n)Q(n) \right. \\
&\qquad \left. - (1+p_1+p_2)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \right) \\
&\quad \le H(m,0)\left|\omega_1(k)\right| + p_1 H(m,0)\left|\omega_2(k)\right| \\
&\qquad + p_2 H(m,0)\left|\omega_3(k)\right|.
\end{aligned} \tag{53}$$

Hence,

$$\begin{aligned}
&\sum_{n=0}^{m-1} \left( H(m,n)\rho(n)Q(n) \right. \\
&\qquad \left. - (1+p_1+p_2)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \right) \\
&\quad \le H(m,0)\left\{ \sum_{n=0}^{k-1} \left|\rho(n)Q(n)\right| + \left|\omega_1(k)\right| \right. \\
&\qquad \left. + p_1\left|\omega_2(k)\right| + p_2\left|\omega_3(k)\right| \right\}.
\end{aligned} \tag{54}$$

Hence,

$$\begin{aligned}
&\lim_{m\to\infty} \sup \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \left( H(m,n)\rho(n)Q(n) \right. \\
&\qquad \left. - (1+p_1+p_2)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi_1(n)} \right)
\end{aligned}$$

$$\leq \left\{ \sum_{n=0}^{k-1} \left|\rho(n) Q(n)\right| + \left|\omega_1(k)\right| + p_1 \left|\omega_2(k)\right| + p_2 \left|\omega_3(k)\right| \right\} < \infty, \tag{55}$$

which is contrary to (47). If (II) holds, then we are back to the proof of Lemma 3 to show that $\lim_{n\to\infty} x(n) = 0$. This completes the proof of Theorem 10. □

**Corollary 11.** *Assume that all the assumptions of Theorem 10 hold, except that condition* (47) *is replaced by*

$$\begin{gathered} \lim_{n\to\infty} \sup \sum_{s=n_0}^{n-1} H(m,s)\, \rho(s)\, Q(s) = \infty, \\ \lim_{n\to\infty} \sup \sum_{s=n_0}^{n-1} \frac{\vartheta^2(m,s)\, H(m,s)}{4\varphi_1(s)} < \infty. \end{gathered} \tag{56}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty} x(n) = 0$.

**Theorem 12.** *Assume that* (20) *holds. Further, assume that* $\sigma_3 \leq \sigma_1$ *and there exists a positive nondecreasing sequence* $\{\rho(n)\}$, *such that*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n-1} \left( \rho(s) Q(s) - \frac{(1+p_1+p_2)}{4} \frac{(\Delta\rho(s))^2 b(s-\sigma_1)}{\rho(s) \sum_{u=n_2}^{s-\sigma_1-1} a^{-1}(u)} \right) = \infty. \tag{57}$$

*Then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty} x(n) = 0$.

*Proof.* Assume that (1) has a nonoscillatory solution, say $x(n) > 0$, $x(n-\sigma_1) > 0$, $x(n+\sigma_2) > 0$, $x(n-\sigma_3) > 0$ and $x(n+\sigma_4) > 0$, for all $n \geq n_0$. Proceeding as in the proof of Theorem 4, we get (24). By Lemma 2, there exist two possible cases (I) and (II). Assume that (I) holds. Then, we obtain (25).

From $\Delta z(n) > 0$, $\sigma_3 \leq \sigma_1$, we obtain

$$\begin{aligned} &\Delta(a(n)\Delta(b(n)\Delta z(n))) \\ &\quad + p_1 \Delta(a(n-\sigma_1)\Delta(b(n-\sigma_1)\Delta z(n-\sigma_1))) \\ &\quad + p_2 \Delta(a(n+\sigma_2)\Delta(b(n+\sigma_2)\Delta z(n+\sigma_2))) \\ &\quad + Q(n) z(n-\sigma_1) \leq 0. \end{aligned} \tag{58}$$

Using the Riccati transformation

$$\omega_1(n) := \rho(n) \frac{a(n)\Delta(b(n)\Delta z(n))}{z(n-\sigma_1)}. \tag{59}$$

Then, $\omega_1(n) > 0$. From (59), we have

$$\begin{aligned} &\Delta\omega_1(n) \\ &= \Delta\rho(n) \frac{a(n+1)(\Delta(b(n+1)\Delta z(n+1)))}{z(n+1-\sigma_1)} \\ &\quad + \rho(n) \frac{\Delta(a(n)(\Delta(b(n)\Delta z(n))))}{z(n-\sigma_1)} \\ &\quad - \rho(n) \frac{a(n+1)(\Delta(b(n+1)\Delta z(n+1)))\Delta(z(n-\sigma_1))}{z(n-\sigma_1) z(n+1-\sigma_1)}. \end{aligned} \tag{60}$$

From Lemma 1, $\Delta(a(n)(\Delta(b(n)\Delta z(n)))) \leq 0$, and $n-\sigma_1 < n$, we get

$$\begin{aligned} &\Delta z(n-\sigma_1) \\ &\quad \geq b^{-1}(n-\sigma_1)(a(n+1)(\Delta(b(n+1)\Delta z(n+1)))) \\ &\quad \times \sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s). \end{aligned} \tag{61}$$

From (60) and (61), we obtain

$$\begin{aligned} &\Delta\omega_1(n) \\ &\quad \leq \frac{\Delta\rho(n)}{\rho(n+1)} \omega_1(n+1) \\ &\quad + \rho(n) \frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{z(n-\sigma_1)} \\ &\quad - \frac{\rho(n)(\omega_1(n+1))^2}{\rho^2(n+1) b(n-\sigma_1)} \sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s). \end{aligned} \tag{62}$$

Next, define another sequence $\omega_2(n)$ by

$$\omega_2(n) := \rho(n) \frac{a(n-\sigma_1)\Delta(b(n-\sigma_1)\Delta z(n-\sigma_1))}{z(n-\sigma_1)}. \tag{63}$$

Then, $\omega_2(n) > 0$. From (63), we have

$$\begin{aligned} &\Delta\omega_2(n) \\ &= \frac{\Delta\rho(n)}{\rho(n+1)} \omega_2(n+1) \\ &\quad + \rho(n) \frac{\Delta(a(n-\sigma_1)\Delta(b(n-\sigma_1)\Delta z(n-\sigma_1)))}{z(n-\sigma_1)} \\ &\quad - \rho(n) \Big( a(n+1-\sigma_1) \\ &\qquad \times (\Delta(b(n+1-\sigma_1)\Delta z(n+1-\sigma_1))) \\ &\qquad \times \Delta(z(n-\sigma_1)) \\ &\qquad \times (z(n+1-\sigma_1) z(n-\sigma_1))^{-1} \Big). \end{aligned} \tag{64}$$

From Lemma 1, we get

$$\begin{aligned}\Delta(z(n-\sigma_1)) &\geq (a(n+1-\sigma_1) \\ &\quad \times (\Delta(b(n+1-\sigma_1)\Delta z(n+1-\sigma_1)))) \\ &\quad \times b^{-1}(n-\sigma_1)\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s).\end{aligned} \tag{65}$$

Then from (63), (64), and the above inequality, we have

$$\begin{aligned}\Delta\omega_2(n) &\leq \rho(n)\frac{\Delta(a(n-\sigma_1)\Delta(b(n-\sigma_1)\Delta z(n-\sigma_1)))}{z(n-\sigma_1)} \\ &\quad + \frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1) - \frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_1)} \\ &\quad \times (\omega_2(n+1))^2\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s).\end{aligned} \tag{66}$$

Similarly, we define another sequence $\omega_3(n)$ by

$$\omega_3(n) := \rho(n)\frac{a(n+\sigma_2)\Delta(b(n+\sigma_2)\Delta z(n+\sigma_2))}{z(n-\sigma_1)}. \tag{67}$$

Then, $\omega_3(n) > 0$. From (67), we have

$$\begin{aligned}\Delta\omega_3(n) &= \frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1) \\ &\quad + \rho(n)\frac{\Delta(a(n+\sigma_2)\Delta(b(n+\sigma_2)\Delta z(n+\sigma_2)))}{z(n-\sigma_1)} \\ &\quad - \rho(n)\big(a(n+1+\sigma_2) \\ &\qquad \times (\Delta(b(n+1+\sigma_2)\Delta z(n+1+\sigma_2))) \\ &\qquad \times \Delta(z(n-\sigma_1)) \\ &\qquad \times (z(n+1-\sigma_1)z(n-\sigma_1))^{-1}\big).\end{aligned} \tag{68}$$

From Lemma 1 and $n-\sigma_1 < n+\sigma_2$, we get

$$\begin{aligned}\Delta z(n-\sigma_1) &\geq (a(n+1+\sigma_2) \\ &\quad \times (\Delta(b(n+1+\sigma_2)\Delta z(n+1+\sigma_2)))) \\ &\quad \times b^{-1}(n-\sigma_1)\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s).\end{aligned} \tag{69}$$

Then from (67), (68), and the above inequality, we have

$$\begin{aligned}\Delta\omega_3(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1) \\ &\quad + \rho(n)\frac{\Delta(a(n+\sigma_2)\Delta(b(n+\sigma_2)\Delta z(n+\sigma_2)))}{z(n-\sigma_1)} \\ &\quad - \frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_1)}(\omega_3(n+1))^2 \\ &\quad \times \sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s).\end{aligned} \tag{70}$$

From (58), (62), (66) and (70), we obtain

$$\begin{aligned}&\Delta\omega_1(n) + p_1\Delta\omega_2(n) + p_2\Delta\omega_3(n) \\ &\quad \leq -\rho(n)Q(n) + \frac{\Delta\rho(n)}{\rho(n+1)}\omega_1(n+1) \\ &\qquad - \frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_1)}(\omega_1(n+1))^2 \\ &\qquad \times \sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s) + p_1\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1) \\ &\qquad - p_1\frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_1)}(\omega_2(n+1))^2 \\ &\qquad \times \sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s) + p_2\frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1) \\ &\qquad - p_2\frac{\rho(n)}{\rho^2(n+1)b(n-\sigma_1)}(\omega_3(n+1))^2 \\ &\qquad \times \sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s).\end{aligned} \tag{71}$$

Using (39) and (71), we have

$$\begin{aligned}&\Delta\omega_1(n) + p_1\Delta\omega_2(n) + p_2\Delta\omega_3(n) \\ &\quad \leq -\rho(n)Q(n) + \frac{1}{4}\frac{(\Delta\rho(n))^2 b(n-\sigma_1)}{\rho(n)\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s)} \\ &\qquad + \frac{p_1}{4}\frac{(\Delta\rho(n))^2 b(n-\sigma_1)}{\rho(n)\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s)} \\ &\qquad + \frac{p_2}{4}\frac{(\Delta\rho(n))^2 b(n-\sigma_1)}{\rho(n)\sum_{s=n_2}^{n-\sigma_1-1} a^{-1}(s)}.\end{aligned} \tag{72}$$

Summing the last inequality from $n_3$ to $n-1$, we obtain

$$\sum_{s=n_3}^{n-1}\left(\rho(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{(\Delta\rho(s))^2 b(s-\sigma_1)}{\rho(s)\sum_{u=n_2}^{s-\sigma_1-1}a^{-1}(u)}\right) \le \omega_1(n_3)+p_1\omega_2(n_3)+p_2\omega_3(n_3), \tag{73}$$

which yields

$$\sum_{s=n_3}^{n-1}\left(\rho(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{(\Delta\rho(s))^2 b(s-\sigma_1)}{\rho(s)\sum_{u=n_2}^{s-\sigma_1-1}a^{-1}(u)}\right) \le c_1, \tag{74}$$

where $c_1 > 0$ is a finite constant. But this contradicts (57). Next, we assume that (II) holds. We are then back to the proof of Lemma 3 to show that $\lim_{n\to\infty}x(n) = 0$. The proof is complete. □

**Theorem 13.** *Assume that* (20) *holds. Let* $\sigma_3 \le \sigma_1$ *and* $\{\rho(n)\}$ *be a positive sequence. Furthermore, assume that there exists a double sequence* $\{H(m,n) \mid m \ge n \ge 0\}$. *If*

$$\lim_{m\to\infty}\sup\frac{1}{H(m,0)} \times \sum_{n=0}^{m-1}\left(H(m,n)\rho(n)Q(n) - (1+p_1+p_2)\frac{\vartheta^2(m,n)H(m,n)}{4\varphi_2(n)}\right) = \infty, \tag{75}$$

*then every solution of* (1) *is oscillatory or* $\lim_{n\to\infty}x(n) = 0$.

*Proof.* Proceeding as in the proof of Theorem 12, we assume that (1) has a nonoscillatory solution, say $x(n) > 0$, $x(n-\sigma_1) > 0$, $x(n+\sigma_2) > 0$, $x(n-\sigma_3) > 0$, and $x(n+\sigma_4) > 0$ for all $n \ge n_0$. By Lemma 2, there are two possible cases. If (I) holds, from the proof of Theorem 12, we find that (71) holds for all $n \ge n_1$. From (71), we have

$$\begin{aligned}\rho(n)Q(n) &\le -\Delta\omega_1(n) - p_1\Delta\omega_2(n) - p_2\Delta\omega_3(n)\\ &\quad + \frac{\Delta\rho(n)}{\rho(n+1)}\omega_1(n+1) - \varphi_2(n)(\omega_1(n+1))^2\\ &\quad + p_1\frac{\Delta\rho(n)}{\rho(n+1)}\omega_2(n+1) - p_1\varphi_2(n)(\omega_2(n+1))^2\\ &\quad + p_2\frac{\Delta\rho(n)}{\rho(n+1)}\omega_3(n+1) - p_2\varphi_2(n)(\omega_3(n+1))^2.\end{aligned} \tag{76}$$

The remainder of the proof is similar to that of the proof of Theorem 10 and hence the details are omitted. □

## 3. Examples

In this section, we will show the applications of our oscillation criteria by three examples. We will see that the equations in the example are oscillatory or tend to zero based on the results in Section 2.

*Example 1.* Consider the following nonlinear neutral equation:

$$\begin{aligned}&\Delta(a(n)\Delta(b(n)\Delta(x(n)+p_1(n)x(n-1)\\ &\qquad +p_2(n)x(n+\sigma_2))))\\ &\quad + \frac{1}{n}x(n-\sigma_3) + \frac{1}{n}x(n+\sigma_4) = 0,\end{aligned} \tag{77}$$

where $\sigma_1 = 1$, $q_i(n) = 1/n$ for $i = 1, 2$. Assume that (17) holds. If we take $n_0 = 1$ and $\rho(n) = 1$, thus Corollary 7, asserts that every solution of (77) is oscillatory or tends to zero.

*Example 2.* Consider the following nonlinear neutral equation

$$\begin{aligned}&\Delta(n\Delta(n\Delta(x(n)+p_1(n)x(n-\sigma_1)\\ &\qquad +p_2(n)x(n+\sigma_2))))\\ &\quad + \frac{\beta}{n}x(n-\sigma_3) + \frac{\gamma}{n}x(n+\sigma_4) = 0,\end{aligned} \tag{78}$$

where $\sigma_3 \ge \sigma_1$, $p_1+p_2 \le 3$, and $\beta$ and $\gamma$ are positive constants. We see that (17) holds. If we take $n_0 = 1$, $\rho(n) = 1$, then by Theorem 4, every solution of (78) is oscillatory or tend to zero provided that $\beta+\gamma > 1$.

*Example 3.* Consider the following nonlinear neutral equation:

$$\begin{aligned}&\Delta\left(\frac{1}{n}\Delta\left(\frac{1}{n}\Delta(x(n)+p_1(n)x(n-2)\right.\right.\\ &\qquad \left.\left. +p_2(n)x(n+1))\right)\right)\\ &\quad + \frac{\beta}{n^2}x(n-\sigma_3) + \frac{\gamma}{n^2}x(n+\sigma_4) = 0,\end{aligned} \tag{79}$$

where $\sigma_1 \ge \sigma_3$, $p_1+p_2 \le 3$, $\beta$ and $\gamma$ are positive constants. We see that (17) holds. If we take $n_0 = 1$, $\rho(n) = 1$ then by Theorem 12, every solution of (79) is oscillatory or tends to zero provided that $\beta+\gamma > 1$.

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

## Acknowledgment

The authors would like to thank the anonymous referees very much for valuable suggestions, corrections, and comments, which resulted in a great improvement of the original paper.

## References


[1] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing, New York, NY, USA, 2005.

[2] S. Elayadi, *An Introduction to Difference Equations*, Springer, New York, NY, USA, 3rd edition, 2005.

[3] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, UK, 1991.

[4] T. H. Hilderbrandt, *Introduction to the Theory of Integration*, Academic Press, New York, NY, USA, 1963.

[5] W. G. Kelly and A. C. Peterson, *Difference Equations, An Introduction with Applications*, Academic Press, New York, NY, USA, 1991.

[6] B. Baculikova and J. Dzurina, "Oscillation of third-order neutral differential equations," *Mathematical and Computer Modelling*, vol. 52, pp. 215–226, 2010.

[7] B. Baculíková, E. M. Elabbasy, S. H. Saker, and J. Džurina, "Oscillation criteria for third-order nonlinear differential equations," *Mathematica Slovaca*, vol. 58, no. 2, pp. 201–220, 2008.

[8] C. G. Philos, "On the existence of nonoscillatory solutions tending to zero at $\infty$ for differential equations with positive delays," *Archiv der Mathematik*, vol. 36, no. 1, pp. 168–178, 1981.

[9] R. P. Agarwal, S. R. Grace, and D. O'Regan, "On the oscillation of certain third-order difference equations," *Advances in Difference Equations*, vol. 3, pp. 345–367, 2005.

[10] E. Schmeidel, "Oscillatory and asymptotically zero solutions of third order difference equations with quasi differences," *Opuscula Mathematica*, vol. 26, pp. 361–369, 2006.

[11] S. H. Saker, J. O. Alzabut, and A. Mukheimer, "On the oscillatory behavior for a certain class of third order nonlinear delay difference equations," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 67, pp. 1–16, 2010.

[12] B. Selvaraj and I. M. A. Jaffer, "On the oscillation of the solution to third order nonlinear difference equations," *The Journal of Mathematics and Computer Science*, vol. 7, pp. 873–876, 2010.

[13] E. Thandapani, M. Vijaya, and T. Li, "On the oscillation of third order half-linear neutral type difference equations," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 76, pp. 1–13, 2011.

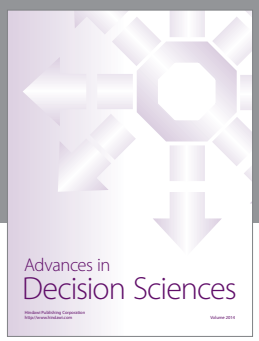
[14] M. F. Aktas, A. Tiryaki, and A. Zafer, "Oscillation of the third-order nonlinear delay difference equations," *Turkish Journal of Mathematics*, vol. 36, pp. 422–436, 2012.

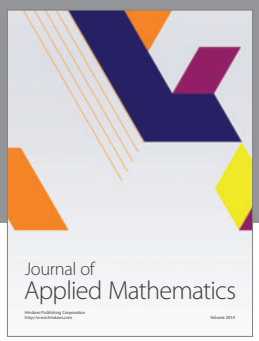
[15] B. Selvaraj, P. Mohankumar, and A. Balasubramanian, "Oscillatory solutions of certain third order non-linear difference equations," *International Journal of Nonlinear Science*, vol. 14, pp. 216–219, 2012.

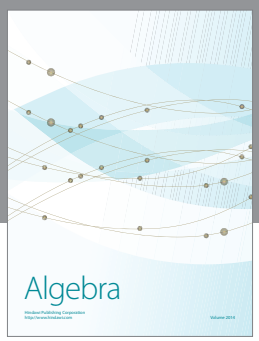
[16] S. H. Saker, "Oscillation of a certain class of third order nonlinear difference equations," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 35, pp. 651–669, 2012.

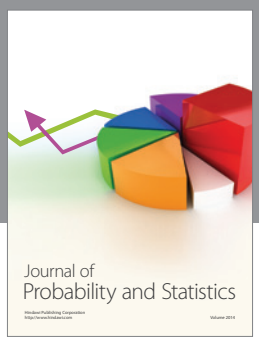
[17] B. Selvaraj, P. Mohankumar, and V. Ananthan, "Oscillatory and non oscillatory behavior of neutral delay difference equations," *International Journal of Nonlinear Science*, vol. 13, pp. 472–474, 2012.

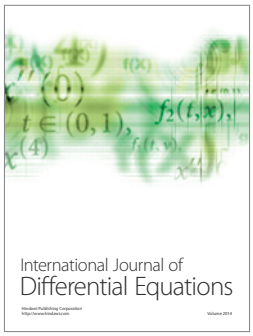
[18] E. M. Elabbasy, M. Y. Barsom, and F. S. AL-dheleai, "Oscillation properties of third order nonlinear delay difference equations," *European International Journal of Science and Technology*, vol. 2, pp. 97–116, 2013.

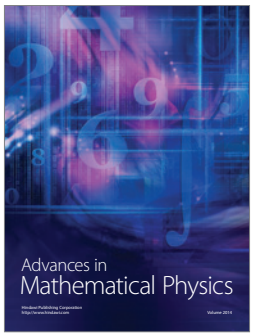
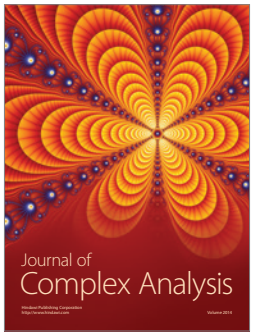
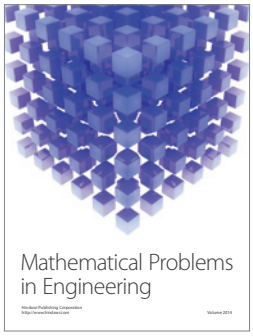
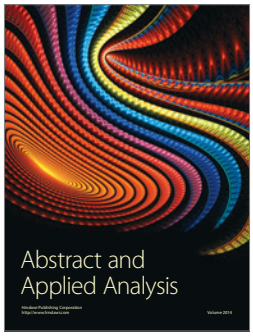
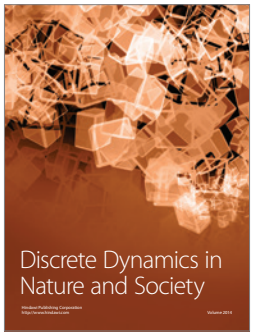
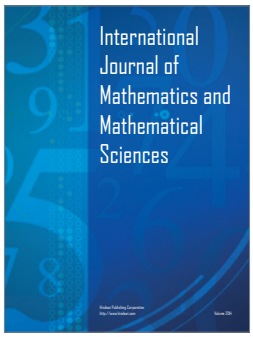
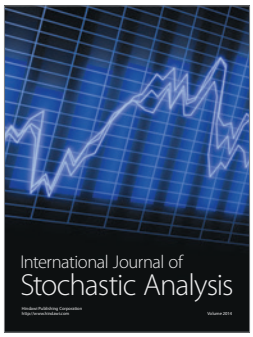










Hindawi

Submit your manuscripts at
http://www.hindawi.com