Research Article

Variational Statement and Domain Decomposition Algorithms for Bitsadze-Samarskii Nonlocal Boundary Value Problem for Poisson's Two-Dimensional Equation

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The Bitsadze-Samarskii nonlocal boundary value problem is considered. Variational formulation is done. The domain decomposition and Schwarz-type iterative methods are used. The parallel algorithm as well as sequential ones is investigated.

1. Introduction

In applied sciences different problems with nonlocal boundary conditions arise very often. In some nonlocal problems, unlike classical boundary value problems, instead of boundary conditions, the dependence between the value of an unknown function on the boundary and some of its values inside of the domain is given.

Modern investigation of nonlocal elliptic boundary value problems originates from Bitsadze and Samarskii work [1], in which by means of the method of integral equations the theorems are proved on the existence and uniqueness of a solution for the second order multidimensional elliptic equations in rectangular domains. Some classes of problems for which the proposed method works are given.

Many works are devoted to the investigation of nonlocal problems for elliptic equations (see, e.g., [2–18] and references therein).

It is known how a great role takes place in the variational formulation of classical and nonlocal boundary value problems in modern mathematics (see, e.g., [13–15, 19–27]).

It is also well known that in order to find the approximate solutions, it is important to construct useful economical algorithms. For constructing such algorithms, the method

of domain decomposition has a great importance (see, e.g., [23, 28, 29]).

In the work [6] the iterative method of proving the existence of a solution of Bitsadze-Samarskii problem for Laplace equation was proposed. This iterative method is based on the idea of Schwarz alternating method [30, pages 249–254]. It should be noted that the usage of Schwarz alternating method not only gives us the existence of a solution, but also allows finding effective algorithms for numerical resolution of such problems. By this approach the nonlocal problem reduces to classical Dirichlet problems on whole domain that yields the possibility to apply the already developed effective methods for numerical resolution of these problems. In [7, 11–13, 15] using Schwarz alternating method and domain decomposition algorithms Bitsadze-Samarskii nonlocal problem is studied for Laplace equation. The domain decomposition algorithms are more economical than the method which was proposed in [6].

In the work [6] the reduction of nonlocal problem to the sequence of Dirichlet problems is studied. For investigating author used Schwarz lemma but not domain decomposition. At first domain decomposition for Bitsadze-Samarskii nonlocal boundary value problem was introduced in [7]. In the abstract [11] the convergence of the domain decomposition

method for the second order nonlinear elliptic equation is given. In [12] the domain decomposition sequential and parallel algorithms are fixed. In [13] the sequential and parallel iterative algorithms are given. In [13] attention is devoted to the operator decomposition method and to the possibility of the variational formulation of the problem as well. In [15] the convergence of the domain decomposition parallel algorithm is fixed. Note that in the works [7, 11–13, 15] results are mainly given without proof.

In the works [9, 10, 16, 21, 22, 31–33] different methods are displayed for study of nonlocal problems in the theory of ordinary differential equations and in the theory of equations with partial derivatives.

The present work is devoted to the variational formulation and domain decomposition and Schwarz-type iterative methods for Bitsadze-Samarskii nonlocal boundary value problem for Poisson's two-dimensional equation. Here we investigate the parallel algorithm as well as sequential ones. The rate of convergence is presented too.

The outline of this paper is as follows. In Section 2 for the Poisson equation in a rectangle we state Bitsadze-Samarskii nonlocal problem. In Section 3 the variational formulation of this problem is discussed. The convergence of the Schwarz-type iterative sequential algorithm is studied in Section 4. The same question for parallel algorithm is considered in Section 5. In Section 6 some conclusions are given.

2. Formulation of Problem

In the plane Oxy , let us consider the rectangle $G = \{(x, y) \mid -a < x < 0, 0 < y < b\}$, where a and b are the given positive constants. By ∂G we denote the boundary of the rectangle G and by Γ_t the intersection of the line $x = t$ with the set $\bar{G} = G \cup \partial G$.

Consider the nonlocal Bitsadze-Samarskii boundary value problem [1]

$$-\Delta u(x, y) = f(x, y), \quad (x, y) \in G, \quad (1)$$

$$u(x, y)|_{\Gamma} = 0, \quad (2)$$

$$u(x, y)|_{\Gamma_{-\xi}} = u(x, y)|_{\Gamma_0}, \quad (3)$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, $\Gamma = \partial G \setminus \Gamma_0$, $\xi \in (0, a)$; $f(x, y)$ is a given function, $f(x, y) \in C(\bar{G})$; and $u(x, y) \in C(\bar{G}) \cap C^{(2)}(G)$ is an unknown function.

Uniqueness of the solution of problem (1)–(3) follows from the extremum principle. It is known [1] that if f is a continuous function on $\bar{G}$, then there exists unique regular solution of problem (1)–(3) $u(x, y) \in C^2(G) \cap C(\bar{G})$.

3. Variational Statement of Problem

We use usual L_2 and Sobolev spaces H^k and $\dot{H}^k$. Let us denote by $D(\bar{G})$ the vector space of all real functions satisfying the following conditions: $v(x, y)$ is defined almost everywhere on $\bar{G}$, the boundary value $v(0, y)$ is defined almost everywhere on Γ_0 , and $v(x, y) \in L_2(G)$, $v(0, y) \in L_2(0, b)$.

Functions $v_1(x, y)$ and $v_2(x, y)$ are assumed as the same element of $D(\bar{G})$ if $v_1(x, y) = v_2(x, y)$ almost everywhere on $\bar{G}$ and $v_1(0, y) = v_2(0, y)$ almost everywhere on Γ_0 .

Let $\bar{Q} = \{(x, y) \mid 0 \leq x \leq \xi, 0 \leq y \leq b\}$ and $\mathcal{T}$ the operator which extends elements of $D(\bar{G})$ as follows:

$$\mathcal{T}v(x, y) = \begin{cases} v(x, y), & \text{if } (x, y) \in \bar{G}, \\ -v(-x, y) + 2v(0, y), & \text{if } (x, y) \in \bar{Q}. \end{cases} \quad (4)$$

Operator $\mathcal{T}$ associates to every function $v(x, y)$ of the vector space $D(\bar{G})$ the following function $\tilde{v}(x, y) = \mathcal{T}v(x, y)$ in such a way that the function $\tilde{v}(x, y) - v(0, y)$ is the odd function with respect to the variable x almost everywhere for almost all $y \in [0, b]$.

Let us define on vector space $D(\bar{G})$ the scalar product

$$[v, w] = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^x \tilde{v}(s, y) \tilde{w}(s, y) ds dx dy. \quad (5)$$

Introducing the scalar product (5), let us denote vector space $D(\bar{G})$ by $H(\bar{G})$ where the norm is defined as follows:

$$\|v\|_H^2 = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^x \tilde{v}^2(s, y) ds dx dy. \quad (6)$$

The following statements take place [24].

Theorem 1. *The norm defined in $H(\bar{G})$ by the formula*

$$\|v\|^2 = \|v(x, y)\|_{L_2(G)}^2 + \|v(0, y)\|_{L_2(0, b)}^2 \quad (7)$$

is equivalent to the norm $\|\cdot\|_H$.

Theorem 2. *Space $H(\bar{G})$ is complete with the metric $\rho(v, w) = \|v - w\|_H$.*

Let the domain of definition of the operator $A = -\Delta$ be the vector space $D_A(\bar{G}) \subset H(\bar{G})$ of the elements $v(x, y)$ defined on $\bar{G}$ for which the following conditions are fulfilled:

$$(1) \ v(x, y) \in C^{(\infty)}(\bar{G}), \ \partial^k v(0, y)/\partial x^k = 0, \ y \in [0, b], \ k = 1, 2, \dots;$$

$$(2) \ v(x, y)|_{\Gamma} = 0, \ v(x, y)|_{\Gamma_{-\xi}} = v(x, y)|_{\Gamma_0}.$$

Theorem 3. *The vector space $D_A(\bar{G})$ is dense in the space $H(\bar{G})$.*

Thus, the operator A acts from the dense vector space $D_A(\bar{G})$ of the Hilbert space $H(\bar{G})$ to the space $H(\bar{G})$.

Theorem 4. *Operator A is positively defined on the vector space $D_A(\bar{G})$.*

To show the symmetry of the operator A we use the following two lemmas, whose proofs are not difficult.

Lemma 5. *For an arbitrary function $v(x, y)$ of the vector space $D_A(\bar{G})$ the following identity is valid:*

$$\mathcal{T}Av = A\mathcal{T}v. \quad (8)$$

Lemma 6. For two arbitrary functions $v(x, y)$ and $w(x, y)$ of the vector space $D_A(\bar{G})$ we have

$$\int_{-\xi}^{\xi} \frac{\partial \bar{v}(x, y)}{\partial x} \bar{w}(x, y) dx = 0, \quad y \in [0, b]. \quad (9)$$

The scalar product given by (5) can be represented in the form

$$\begin{aligned} [v, w] &= \int_0^b \int_{-\xi}^0 \int_{-a}^x v(s, y) w(s, y) ds dx dy \\ &+ \xi \int_0^b \int_{-a}^0 v(s, y) w(s, y) dx dy \\ &+ \int_0^b \int_0^{\xi} \int_{-x}^0 (2v(0, y) - v(s, y)) \\ &\quad \times (2w(0, y) - w(s, y)) ds dx dy. \end{aligned} \quad (10)$$

In the case of the scalar product (5) we have the positively defined operator A , but it is not symmetric.

As A is positive definite operator defined on the vector space $D_A(G)$ which is dense in the space $H(\bar{G})$, for the problem (1)–(3) we can use the standard way of the variational formulation [27].

Let us introduce the new scalar product on $D_A(\bar{G})$:

$$\begin{aligned} [v, w]_A &= [Av, w] \\ &= \int_0^b \int_{-\xi}^{\xi} \int_{-a}^x \left(\frac{\partial \bar{v}(s, y)}{\partial s} \frac{\partial \bar{w}(s, y)}{\partial s} \right. \\ &\quad \left. + \frac{\partial \bar{v}(s, y)}{\partial y} \frac{\partial \bar{w}(s, y)}{\partial y} \right) ds dx dy. \end{aligned} \quad (11)$$

Denote by $\|\cdot\|_A$ the corresponding norm and by $\rho_A(v, w)$ the corresponding metric. By $H_A(\bar{G})$ we denote the space obtained after completion of $D_A(\bar{G})$ by the metric $\rho_A(v, w)$.

The following statement is true [24].

Theorem 7. The function $v(x, y) \in H(\bar{G})$ belongs to the space $H_A(\bar{G})$ if and only if the following relations are fulfilled:

$$\begin{aligned} v(x, y) &\in H^1(G), \quad v(0, y) \in \dot{H}^1(0, b), \\ v(x, y)|_{\Gamma} &= 0, \quad v(x, y)|_{\Gamma_{-\xi}} = v(x, y)|_{\Gamma_0} = v(0, y). \end{aligned} \quad (12)$$

Thus, functions of the space $H_A(\bar{G})$ satisfy boundary conditions. For every function $f(x, y) \in H(\bar{G})$ there exists a unique function $u(x, y)$ in the space $H_A(\bar{G})$, which minimizes the quadratic functional

$$F(v) = \|v\|_A^2 - 2[f, v]. \quad (13)$$

For any function $v(x, y) \in H_A(\bar{G})$ the following relation is fulfilled

$$[u, v]_A = [f, v]. \quad (14)$$

The function $u(x, y)$ from the space $H_A(\bar{G})$ which minimizes the functional (14) is called the generalized solution of the equation $Au = f$.

If the function $u(x, y)$ is sufficiently smooth then $u(x, y)$ is a solution in a classical sense of problem (1)–(3).

4. Domain Decomposition and Sequential Algorithm

In this section and next sections, for simplicity, let us consider Laplace equation with nonlocal (3) and again for simplicity homogeneous Dirichlet (2) conditions. So, we study the following problem:

$$\Delta u(x, y) = 0, \quad (x, y) \in G, \quad (15)$$

$$u(x, y)|_{\Gamma} = 0, \quad (16)$$

$$u(x, y)|_{\Gamma_{-\xi}} = u(x, y)|_{\Gamma_0}. \quad (17)$$

For problem (15)–(17) let us consider the following sequential iterative procedure:

$$\begin{aligned} \Delta u_1^n(x, y) &= 0, \quad (x, y) \in G_1, \\ u_1^n(x, y)|_{\Gamma^1} &= 0, \\ u_1^n(x, y)|_{\Gamma_{-\xi_1}} &= u_2^{n-1}(x, y)|_{\Gamma_{-\xi_1}}, \\ n &= 1, 2, \dots; \\ \Delta u_2^n(x, y) &= 0, \quad (x, y) \in G_2, \\ u_2^n(x, y)|_{\Gamma^2} &= 0, \\ u_2^n(x, y)|_{\Gamma_{-\xi}} &= u_2^n(x, y)|_{\Gamma_0} = u_1^n(x, y)|_{\Gamma_{-\xi}}, \\ n &= 1, 2, \dots \end{aligned} \quad (18)$$

Here we utilize the following notations:

$$\begin{aligned} G_1 &= \{-a < x < -\xi_1, 0 < y < b\}, \\ G_2 &= \{-\xi < x < 0, 0 < y < b\}, \end{aligned} \quad (19)$$

where $-\xi_1$ is a fixed point of the interval $(-\xi, 0)$, $\Gamma^1 = \partial G_1 \setminus \Gamma_{-\xi_1}$, $\Gamma^2 = \partial G_2 \setminus (\Gamma_{-\xi} \cup \Gamma_0)$, and $u_2^0(-\xi_1, y) = \varphi(y)$ is any continuous function on the segment $[0, b]$, which satisfies the following conditions: $\varphi(0) = \varphi(b) = 0$.

The iterative procedure (18) reduces our nonlocal non-classical problem (15)–(17) to the investigation of the sequence of classical Dirichlet boundary value problems on every step of the iteration.

The following statement takes place.

Theorem 8. The sequential iterative process (18) converges to a solution of problem (15)–(17) uniformly in the domain $\bar{G}$, and the following estimations are true:

$$\begin{aligned} |u(x, y) - u_1^n(x, y)| &\leq Cq^{n-1}, \quad (x, y) \in \bar{G}_1, \\ |u(x, y) - u_2^n(x, y)| &\leq Cq^{n-1}, \quad (x, y) \in \bar{G}_2, \end{aligned} \quad (20)$$

where $q \in (0, 1)$ is a constant independent of functions $u(x, y)$, $u_1^n(x, y)$, $u_2^n(x, y)$, and constant C depends on $\varphi(y)$.

Proof. Note that solving problem (18), we get two sequences $\{u_1^n\}$, $\{u_2^n\}$ of harmonic functions, which are defined on the domains $\overline{G}_1$ and $\overline{G}_2$, respectively.

We have the following relations:

if $x = -\xi$ and $y \in [0, b]$, then $u_2^n = u_1^n$, that is $u_1^{n+1} - u_2^n = u_2^{n+1} - u_2^n$;

if $x = -\xi_1$ and $y \in [0, b]$, then $u_2^n = u_1^{n+1}$, that is $u_1^{n+1} - u_1^n = u_2^n - u_2^{n-1}$.

Let us introduce the notations:

$$\begin{aligned} M_n &= \max_{y \in [0, b]} |u_1^{n+1}(-\xi, y) - u_1^n(-\xi, y)| \\ &= \max_{y \in [0, b]} |u_2^{n+1}(-\xi, y) - u_2^n(-\xi, y)|, \\ M_n^1 &= \max_{y \in [0, b]} |u_1^{n+1}(-\xi_1, y) - u_1^n(-\xi_1, y)| \\ &= \max_{y \in [0, b]} |u_2^n(-\xi_1, y) - u_2^{n-1}(-\xi_1, y)|. \end{aligned} \quad (21)$$

If $M_n^1 \neq 0$, then, for the harmonic function $v_1^n = (u_1^{n+1} - u_1^n)/M_n^1$, applying the lemma from [30, pages 250–254] to the domain $\overline{G}_1$, we have

$$M_n \leq qM_n^1, \quad (22)$$

where $q \in (0, 1)$ and depends only on domain G_1 .

If $M_n \neq 0$, then, for the harmonic function $v_2^n = (u_2^{n+1} - u_2^n)/M_n$, applying the extremum principle [30, pages 218] to the domain G_2 , we have

$$M_{n+1}^1 \leq M_n. \quad (23)$$

If $M_n^1 = 0$ for some index n , then from the extremum principle we get $M_n = 0$. So, the inequality (22) is also true.

If $M_n = 0$ for some index n , then from the extremum principle we get again $M_{n+1}^1 = 0$. So, the inequality (23) is also true.

From the estimations (22) and (23) obtained above it follows that

$$M_{n+1} \leq qM_n, \quad M_{n+1}^1 \leq qM_n^1. \quad (24)$$

This means that the sequences $\{M_n\}$ and $\{M_n^1\}$ tend to zero, and we obtain the uniform convergence of the series

$$u_1^1 + \sum_{k=1}^{\infty} (u_1^{k+1} - u_1^k) = \lim_{n \rightarrow \infty} u_1^n = u_1, \quad (25)$$

$$u_2^1 + \sum_{k=1}^{\infty} (u_2^{k+1} - u_2^k) = \lim_{n \rightarrow \infty} u_2^n = u_2,$$

in the domains $\overline{G}_1$ and $\overline{G}_2$, respectively.

According to Weierstrass theorem [30, pages 232, 233], the functions u_1 and u_2 are harmonic ones, defined, respectively, in G_1 and G_2 , and satisfy the condition (16). As for domain $G_{12} = G_1 \cap G_2$, we have the following relations:

if $x = -\xi$ and $y \in [0, b]$, then $u_2^n - u_1^n = 0$;

if $x = -\xi_1$ and $y \in [0, b]$, then $u_2^n - u_1^n = u_2^n - u_2^{n-1}$.

The latter difference tends to zero uniformly.

Again, according to the extremum principle, we obtain that the functions u_1 and u_2 coincide with each other in the domain G_{12} and define a regular harmonic function on G , which represents the solution of the problem (15)–(17).

Now, let us estimate the rate of convergence of the iterative process (18). Using the triangle inequality, the extremum principle and the second inequality of (24), we have

$$\begin{aligned} |u_1^{n+p} - u_1^n| &= |u_1^{n+p} - u_1^{n+p-1} + u_1^{n+p-1} - u_1^{n+p-2} \\ &\quad + \dots + u_1^{n+2} - u_1^{n+1} + u_1^{n+1} - u_1^n| \\ &\leq M_{n+p-1}^1 + M_{n+p-2}^1 + \dots + M_{n+1}^1 + M_n^1 \\ &\leq q^{n+p-2} M_1^1 + q^{n+p-3} M_1^1 + \dots + q^n M_1^1 + q^{n-1} M_1^1 \\ &= q^{n-1} M_1^1 (q^{p-1} + q^{p-2} + \dots + q + 1) \\ &= q^{n-1} M_1^1 \frac{1 - q^p}{1 - q}, \\ M_1^1 &= \max_{y \in [0, b]} |u_1^2(-\xi_1, y) - u_1^1(-\xi_1, y)| \\ &\leq \max_{y \in [0, b]} (|u_1^2(-\xi_1, y)| + |u_1^1(-\xi_1, y)|) \\ &= \max_{y \in [0, b]} |u_2^1(-\xi_1, y)| + \max_{y \in [0, b]} |u_2^0(-\xi_1, y)| \\ &\leq 2 \max_{y \in [0, b]} |\varphi(y)| = M. \end{aligned} \quad (26)$$

Thus,

$$|u_1^{n+p} - u_1^n| \leq q^{n-1} M \frac{1 - q^p}{1 - q}. \quad (27)$$

If in this inequality we tend p to ∞ , then we obtain in the domain $\overline{G}_1$

$$|u_1 - u_1^n| \leq q^{n-1} \frac{M}{1 - q}. \quad (28)$$

Analogous estimation is true for u_2 in the domain $\overline{G}_2$. Consequently, we get

$$\begin{aligned} |u_1(x, y) - u_1^n(x, y)| &\leq Cq^{n-1}, \quad (x, y) \in \overline{G}_1, \\ |u_2(x, y) - u_2^n(x, y)| &\leq Cq^{n-1}, \quad (x, y) \in \overline{G}_2. \end{aligned} \quad (29)$$

This completes the proof of Theorem 8. $\square$

5. Domain Decomposition and Parallel Algorithm

Algorithm (18) for the solution of the problem (15) has a sequential form. Now, let us consider one more approach to the solution of the problem (15)–(17). In this case the search of approximate solutions on domains $\overline{G}_1$ and $\overline{G}_2$ will be carried out not by means of a sequential algorithm but in a parallel way.

Consider the following overlapping parallel iterative process:

$$\begin{aligned} \Delta u_1^n(x, y) &= 0, \quad (x, y) \in G_1, \\ u_1^n(x, y)|_{\Gamma^1} &= 0, \\ u_1^n(x, y)|_{\Gamma_{-\xi_1}} &= u_2^{n-1}(x, y)|_{\Gamma_{-\xi_1}}, \\ n &= 1, 2, \dots; \\ \Delta u_2^n(x, y) &= 0, \quad (x, y) \in G_2, \\ u_2^n(x, y)|_{\Gamma^2} &= 0, \\ u_2^n(x, y)|_{\Gamma_{-\xi}} &= u_2^n(x, y)|_{\Gamma^0} = u_1^{n-1}(x, y)|_{\Gamma_{-\xi}}, \\ n &= 1, 2, \dots, \end{aligned} \quad (30)$$

where $u_2^0(-\xi_1, y) = \varphi_1(y)$ and $u_1^0(\xi, y) = \varphi_2(y)$ are any continuous functions on the segment $[0, b]$, which satisfy the following conditions: $\varphi_1(0) = \varphi_1(b) = 0$, $\varphi_2(0) = \varphi_2(b) = 0$.

The following statement takes place.

Theorem 9. *The parallel iterative process (30) converges to a solution of the problem (15)–(17) uniformly in the domain $\overline{G}$, and the following estimations are true:*

$$\begin{aligned} |u(x, y) - u_1^n(x, y)| &\leq Cq^{(n/2)-1}, \quad (x, y) \in \overline{G}_1, \\ |u(x, y) - u_2^n(x, y)| &\leq Cq^{(n/2)-1}, \quad (x, y) \in \overline{G}_2, \end{aligned} \quad (31)$$

where $q \in (0, 1)$ is a constant independent of functions $u(x, y)$, $u_1^n(x, y)$, and $u_2^n(x, y)$, and constant C depends on $\varphi_1(y)$, $\varphi_2(y)$.

Proof. Let us prove this theorem in a similar way as Theorem 8 was proven. We should note that the boundary value problems (30) are simultaneously solved in domains G_1 and G_2 , respectively. Sequences of harmonic functions $\{u_1^n\}$, $\{u_2^n\}$ are defined on $\overline{G}_1$ and $\overline{G}_2$, respectively.

The following relations are satisfied:

$$\text{if } x = \xi \text{ and } y \in [0; b], \text{ then } u_2^n = u_1^{n-1}; \text{ that is, } u_1^n - u_1^{n-1} = u_2^{n+1} - u_2^n;$$

$$\text{if } x = -\xi_1 \text{ and } y \in [0; b], \text{ then } u_2^{n-1} = u_1^n; \text{ that is, } u_1^{n+1} - u_1^n = u_2^n - u_2^{n-1}.$$

If we introduce the notation

$$\begin{aligned} M_{n-1} &= \max_{y \in [0; b]} |u_1^n(-\xi, y) - u_1^{n-1}(-\xi, y)| \\ &= \max_{y \in [0; b]} |u_2^{n+1}(-\xi, y) - u_2^n(-\xi, y)|, \\ M_n^1 &= \max_{y \in [0; b]} |u_1^{n+1}(-\xi_1, y) - u_1^n(-\xi_1, y)| \\ &= \max_{y \in [0; b]} |u_2^n(-\xi_1, y) - u_2^{n-1}(-\xi_1, y)|, \end{aligned} \quad (32)$$

and, for the harmonic function $v_1^n = (u_1^{n+1} - u_1^n)/M_n^1$, apply the lemma from [30, pages 250–254] to the domain G_1 , we will have the estimation

$$M_n \leq qM_n^1, \quad (33)$$

where $q \in (0, 1)$ and depends on domain G_1 only.

If $M_n^1 = 0$, then from the extremum principle we get that $M_n = 0$ and the inequality (33) is clear.

If $M_n \neq 0$, then, for the harmonic function $v_2^{n-1} = (u_2^{n+1} - u_2^n)/M_{n-1}$, from the extremum principle in the domain G_2 we have

$$M_{n+1}^1 \leq M_{n-1}. \quad (34)$$

If $M_{n-1} = 0$, then, using again the extremum principle, we have $M_n^1 = 0$ and the inequality (34) holds.

From the estimations (33) and (34), we obtain

$$M_{n+1} \leq qM_{n-1}, \quad M_{n+2}^1 \leq qM_n^1. \quad (35)$$

That means that the sequences $\{M_n\}$ and $\{M_n^1\}$ tend to zero.

Thus, in this case, the series analogous to the series from (25) are also uniformly convergent. The corresponding harmonic functions u_1 and u_2 are defined in $\overline{G}_1$ and $\overline{G}_2$, respectively, and satisfy condition (16). As for the common part of these domains, we have

$$\text{if } x = -\xi \text{ and } y \in [0, b], \text{ then } u_2^n - u_1^n = u_2^n - u_2^{n+1};$$

$$\text{if } x = -\xi_1 \text{ and } y \in [0, b], \text{ then } u_2^n - u_1^n = u_2^n - u_2^{n-1}$$

and this difference tends to zero, when $n \rightarrow \infty$. Thus, the functions u_1 and u_2 again coincide in the domain G_{12} and define a regular harmonic function on G , which represents the solution of the problem (15)–(17).

Let us estimate the rate of convergence of the constructed sequences.

We should remark that from the second inequality of (35) we have the following: if $n = 2k$, then

$$M_n^1 \leq q^{(n-2)/2} M_2^1; \quad (36)$$

if $n = 2k - 1$, then

$$M_n^1 \leq q^{(n-1)/2} M_1^1. \quad (37)$$

Using the triangle inequality, we get

$$\begin{aligned} |u_1^{n+p} - u_1^n| &= |u_1^{n+p} - u_1^{n+p-1} + u_1^{n+p-1} - u_1^{n+p-2} \\ &\quad + \dots + u_1^{n+2} - u_1^{n+1} + u_1^{n+1} - u_1^n| \\ &\leq M_{n+p-1}^1 + M_{n+p-2}^1 + \dots + M_{n+1}^1 + M_n^1. \end{aligned} \quad (38)$$

Taking into account inequalities (36) and (37) for $n = 2k$ and $p = 2m$, we obtain from (36)

$$\begin{aligned}
 & |u_1^{n+p} - u_1^n| \\
 & \leq q^{(n+p-1)/2} M_1^1 + q^{(n+p-2)/2} M_2^1 \\
 & \quad + q^{(n+p-3)/2} M_1^1 + q^{(n+p-4)/2} M_2^1 \\
 & \quad + \dots + q^{(n+1)/2} M_1^1 + q^{(n-2)/2} M_2^1 \\
 & = (M_1^1 + M_2^1) q^{n/2} (q^{(p-4)/2} + q^{(p-6)/2} + \dots + q + 1) \\
 & \quad + q^{(n+p-2)/2} M_1^1 + q^{(n-2)/2} M_2^1 \\
 & = q^{(n-2)/2} \left[q (M_1^1 + M_2^1) \frac{1 - q^{(p-2)/2}}{1 - q} + q^{p/2} M_1^1 + M_2^1 \right]. \quad (39)
 \end{aligned}$$

If $n = 2k - 1$ and $p = 2m - 1$, then from (38) we have

$$\begin{aligned}
 & |u_1^{n+p} - u_1^n| \leq q^{(n+p-1)/2} M_1^1 + q^{(n+p-2)/2} M_2^1 \\
 & \quad + q^{(n+p-3)/2} M_1^1 + q^{(n+p-4)/2} M_2^1 \\
 & \quad + \dots + q^{(n+1)/2} M_2^1 + q^{(n-1)/2} M_1^1 \\
 & = q^{(n+p-1)/2} M_1^1 + (M_1^1 + M_2^1) q^{(n-2)/2} \\
 & \quad \times (q^{(p-2)/2} + q^{(p-4)/2} + \dots + q^{3/2} + q^{1/2}) \\
 & = q^{(n-2)/2} \left[q^{p/2} M_1^1 + (M_1^1 + M_2^1) \frac{q^{1/2} - q^{p/2}}{1 - q} \right]. \quad (40)
 \end{aligned}$$

If $n = 2k$ and $p = 2m - 1$, then

$$\begin{aligned}
 & |u_1^{n+p} - u_1^n| \leq q^{(n+p-1)/2} M_2^1 + q^{(n+p-2)/2} M_1^1 \\
 & \quad + \dots + q^{(n+3)/2} M_1^1 + q^{(n+2)/2} M_2^1 \\
 & \quad + q^{(n+1)/2} M_1^1 + q^{(n-2)/2} M_2^1 \\
 & = q^{(n-2)/2} \left[(M_1^1 + M_2^1) (q^{(p-1)/2} + q^{(p-3)/2} \right. \\
 & \quad \left. + \dots + q) + M_2^1 \right] \\
 & = q^{(n-2)/2} \left[(M_1^1 + M_2^1) \frac{q - q^{(p+1)/2}}{1 - q} + M_2^1 \right], \quad (41)
 \end{aligned}$$

and at last, supposing $n = 2k - 1$ and $p = 2m$, we have

$$\begin{aligned}
 & |u_1^{n+p} - u_1^n| \leq q^{(n+p-1)/2} M_2^1 + q^{(n+p-2)/2} M_1^1 \\
 & \quad + \dots + q^{(n+3)/2} M_2^1 + q^{(n+2)/2} M_1^1 \\
 & \quad + q^{(n+1)/2} M_2^1 + q^{(n-1)/2} M_1^1 \\
 & = q^{(n-2)/2} (M_1^1 + M_2^1) q^{(n-2)/2} \\
 & \quad \times (q^{(p-1)/2} + q^{(p-3)/2} + \dots + q^{3/2} + q^{1/2}) \\
 & = q^{(n-2)/2} (M_1^1 + M_2^1) \frac{q^{1/2} - q^{(p+1)/2}}{1 - q}. \quad (42)
 \end{aligned}$$

If in the obtained inequalities we tend to the limit, when $p \rightarrow \infty$, we get the following estimations.

If $n = 2k$ and $p = 2m$, then

$$|u - u_1^n| \leq q^{(n-2)/2} \left[\frac{(M_1^1 + M_2^1) q}{1 - q} + M_2^1 \right]; \quad (43)$$

if $n = 2k - 1$ and $p = 2m - 1$, then

$$|u - u_1^n| \leq q^{(n-1)/2} \frac{(M_1^1 + M_2^1)}{1 - q}; \quad (44)$$

if $n = 2k$ and $p = 2m - 1$, then

$$|u - u_1^n| \leq q^{(n-2)/2} \left[\frac{(M_1^1 + M_2^1) q}{1 - q} + M_2^1 \right]; \quad (45)$$

and if $n = 2k - 1$ and $p = 2m$, then

$$|u - u_1^n| \leq q^{(n-1)/2} \frac{(M_1^1 + M_2^1)}{1 - q}. \quad (46)$$

Let us estimate M_1^1 and M_2^1 . We have

$$\begin{aligned}
 M_1^1 &= \max_{y \in [0; b]} |u_1^2(-\xi_1, y) - u_1^1(-\xi_1, y)| \\
 &\leq \max_{y \in [0; b]} (|u_1^2(-\xi_1, y)| + |u_1^1(-\xi_1, y)|) \\
 &= \max_{y \in [0; b]} (|u_2^1(-\xi_1, y)| + |u_2^0(-\xi_1, y)|) \\
 &\leq 2 \max_{y \in [0; b]} \{|\varphi_1(y)|, |\varphi_2(y)|\}, \\
 M_2^1 &= \max_{y \in [0; b]} |u_1^3(-\xi_1, y) - u_1^2(-\xi_1, y)| \\
 &\leq \max_{y \in [0; b]} (|u_1^3(-\xi_1, y)| + |u_1^2(-\xi_1, y)|) \\
 &= \max_{y \in [0; b]} (|u_1^2(-\xi_1, y)| + |u_1^1(-\xi_1, y)|) \\
 &\leq 2 \max_{y \in [0; b]} \{|\varphi_1(y)|, |\varphi_2(y)|\}. \quad (47)
 \end{aligned}$$

So, for any n , we obtain the following relation in the domain $\bar{G}_1$:

$$|u_1(x, y) - u_1^n(x, y)| \leq Cq^{(n-2)/2}. \quad (48)$$

Analogous estimation is true for u_2 in the domain $\bar{G}_2$

$$|u_2(x, y) - u_2^n(x, y)| \leq Cq^{(n-2)/2}. \quad (49)$$

Theorems similar to Theorems 8 and 9 are valid for the sequential as well as parallel algorithms for multigrid domain decomposition case too. $\square$

6. Conclusion

Because of importance of nonlocal problems many scientific articles are devoted for their investigation. We gave one problem to illustrate the variational formulation and domain decomposition of the problem. Nonlocal problems for the second and the fourth order ordinary and partial differential equations are also studied by authors. Nonlocal boundary value problems with some kind of nonlocal integral conditions are studied as well (see, e.g., [15, 24–26] and references therein).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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