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Research Article

Prime Decomposition of Three-Dimensional Manifolds into Boundary Connected Sum

Meili Zhang[1] and Bo Deng[2]

[1] *Department of Basis, Dalian Naval Academy, Jie Fang Road 667, Zhong Shan District , Dalian, Liaoning Province 116018, China*
[2] *Department of Basis, Dalian Institute of Science and Technology, Dalian, Liaoning Province, China*

Correspondence should be addressed to Meili Zhang; zhangmeili66@sina.com

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In 2003 Matveev suggested a new version of the Diamond Lemma suitable for topological applications. We apply this result to different situations and get a new conceptual proof of theorem on decomposition of three-dimensional manifolds into boundary connected sum of prime components.

1. Introduction

Since 2003 Matveev [1–3] had suggested a new version of the Diamond Lemma [4] of great importance for various fields of mathematics, which is suitable for and efficient solving topological problems. In this paper we apply this result to get a new conceptual proof of theorem on decomposition of three-dimensional manifolds into boundary connected sum of prime components.

2. Definition, Lemma, and ∂-Irreducible Manifolds

Diamond Lemma (see [5]). *If an oriented graph Γ has the properties (FP) and (MF), then each of its vertices has a unique root.*

Definition 1. Let D be a proper disk in a compact connected 3-manifold M. A disk reduction of M along D consists in cutting M along the disk D. Its result is a new manifold M'.

We apply nontrivial disk reductions to a given manifold M as many times as possible. If this process stops, then we obtain a set of new manifolds, which is called a root of M.

(1) If the disk is splitting, the manifold M is obtained by gluing the manifolds M_1 and M_2 together along disks on their boundaries. Then M is called a boundary connected sum of the manifolds M_1 and M_2 and denoted by $M = M_1 \#_\partial M_2$.

(2) If the disk is nonsplitting, then M' is also connected.

Definition 2. M is said to be ∂ irreducible if every properly embedded disk in M is trivial.

Lemma 3 (see [6]). *Let M be a ∂-irreducible manifold. Let N be the manifold obtained from M by attaching a 1-handle to make the boundary connected. Then N is a prime which is not ∂ irreducible.*

3. Proof of Theorem 4

Theorem 4. *Any connected irreducible compact 3-manifold M different from a ball and with nonempty boundary is homeomorphism to a boundary connected sum $M = M_1\#_\partial \cdots \#_\partial M_n$ of prime manifolds. All the summands are defined uniquely up to reordering and, if M is non-Orientale, replacing solid tori by solid Klein bottles $S^1\tilde{\times}D^2$.*

We apply the universal scheme [4] in two stages. First, by considering reductions along all disks we establish uniqueness of the ∂-irreducible manifolds M_i. Then we restrict ourselves to reductions only along splitting disks and by lemma [3] complete the proof of the theorem.

We construct the graph Γ [5] whose vertices are compact connected irreducible manifolds, considered up to addition or deletion of three-dimensional balls. The edges of the graph correspond to reductions along both splitting and nonsplitting disks.

Lemma 5 (see [7]). *Each essential disks reduction strictly decreases $\gamma(A)$, where $\gamma(A) = \sum_i g^2(F_i)$ $(A \in V(\Gamma))$, $g(F_i)$ is the genus of a component $F_i \subset \partial M$, and the sum is taken over all components of ∂M.*

Since every set of vertexes of nonnegative integers has a minimal vertex, by Lemma 5, the process stops and we get a root. Then Γ has the properties (FP).

Lemma 6. *The graph Γ has a mediator function; that is, Γ has the properties (MF).*

Proof. Let $(\overrightarrow{AB_1}, \overrightarrow{AB_2}) \in E^2(\Gamma)$ be a pair of edges with common beginning. Following the universal scheme, we define the value $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2})$ of the mediator function $\mu : E^2(\Gamma) \to N \cup \{0\}$ to be the minimal number $|D_1 \cap D_2|$ of connected components in the intersection of disks $D_1, D_2 \subset A$, defining the edges $\overrightarrow{AB_1}$ and $\overrightarrow{AB_2}$. This minimum is taken over all pairs of such disks. As usual, we assume that the disks are in general position, so that their intersection consists of a finite number of circles and arcs.

We first consider the property (MF1), that is, the case $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) = 0$. When the reduction is carried out along disjoint disks D_1 and D_2, then each of these disks survives under the reduction along the other disk. We can therefore assume that $D_i \subset B_j$ for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. By reducing each of the manifolds B_j along the disk D_i. We obtain the same vertex $C \in V(\Gamma)$. This proves the property (MF1).

Next, we verify the property (MF2), that is, the case $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) > 0$, when any two disks D_1 and D_2 defining the edges $\overrightarrow{AB_1}$ and $\overrightarrow{AB_2}$ intersect. Then these disks lie in the same connected manifold $Q \subset A$. According to the universal scheme, it is enough to show that there exists a mediator disk, that is, a nontrivial disk $D \subset A$ satisfying $|D_i \cap D| < |D_1 \cap D_2|$ for $i = 1, 2$.

Case 1. Among the circles in $D_1 \cap D_2$ we choose one, denoted by c, which is innermost with respect to the disk D_1. This means that the circle c bounds a disk D in D_1 such that $D_1 \cap D_2 = c$. We cut D_2 along c and glue up the boundaries of the cut by two parallel copies of the disk D. By applying a small perturbation we obtain a new disk D' and a sphere S'' whose intersection with D_2 is empty and whose intersection with D_1 consists of a smaller number of circles (since the circle c has disappeared). The disks D' must be nontrivial in Q, since otherwise the disk D_2 would be trivial. Therefore, D' can be taken as a mediator disk.

Case 2. Among the arcs in $D_1 \cap D_2$ we choose one, denoted by c, which is outermost with respect to the disk D_1. We cut D_2 along c and glue up the boundaries of the cut by two parallel copies of the disk D. By applying a small perturbation we obtain a new disk D' and a sphere D'' whose intersection with D_2 is empty and whose intersection with D_1 consists of a smaller number of arcs (since the circle c has disappeared). At least one of these two disks (say D') must be nontrivial in Q, since otherwise the disk D_2 would be trivial. Therefore, D' can be taken as a mediator disk. □

By applying the Diamond Lemma we prove that a root of any vertex of the graph exists and is unique.

We therefore obtain the following theorem, which is a particular case of Theorem 4 for manifolds without nonsplitting proper disks.

Theorem 7. *Any connected irreducible compact three-dimensional manifold M different from a ball and with nonempty boundary is decomposed by disk reductions into a union of ∂-irreducible parts. These parts are defined uniquely up to reordering.*

In order to finish the proof of Theorem 4 we change the graph Γ constructed by keeping the vertex set intact while disallowing reductions along nonsplitting disks. The edges of the new graph $\Gamma' \subset \Gamma$ therefore correspond to reductions only along splitting disks, so that the root of any vertex consists exactly of the prime summands of the manifolds corresponding to the vertex. The properties (FP) and (MF1) of Γ are automatically inherited by the graph Γ'. The only difficulty with the proof of the property (MF2) is that after applying to the disk D_2 a surgery along the innermost circle $c \subset D_1$ both new spheres S' and S'' and disk $D'' \subset Q$ may be nonsplitting and therefore can not be taken as mediator disks.

Case 1. Assume that $D_1 \cap D_2$ consists of $n > 3$ circles. Then we connect the spheres S' and S'' by a tube which does not intersect D_1.

Since each of these spheres is obtained by connecting S' and S'' by tubes contained in one of the parts into which $D'' \cup S' \cup S''$ divides Q, the disk D_3 is splitting and nontrivial (the latter follows from the simple observation that each simple arc connecting the sphere and disk on the boundary is trivial). It is important to note that, after a suitable small perturbation, D_3 intersects D_1 in $n - 1$ circles and intersects D_2 in two circles. Thus, D_3 is a mediator sphere.

The cases $n = 1$ and $n = 2$ need to be considered separately. We show that in each of these cases there exists a mediator disk. If $n = 1$, then the disks D_1 and D_2 split Q into 4 parts X_i, $1 \le i \le 4$. These parts are different, since they are separated by D_1 and D_2: any two parts lie on different sides with respect to at least one of the disks. The boundary of each part consists of a single splitting disk. At least one of these disks is nontrivial, since both D_1 and D_2 are nontrivial. Hence, it can be taken as a mediator disk.

Now let $n = 2$. Then the disks D_1 and D_2 split Q either into 5 parts X_i, $1 \le i \le 5$ (if the parts X_1 and X_3 are different), or into 4 parts (if the parts X_1 and X_3 coincide). None of the other parts can coincide, since only the parts X_1 and X_3 are not separated by D_1 and D_2. Since $X_1 \ne X_3$, then each of the parts X_2 and X_3 is bounded by a sphere X_1, X_4 is bounded by a disk, and at least one of these disks is nontrivial, since D_2 is nontrivial. This disk can be taken as a mediator. We also note that the part X_5 is bounded by a torus.

Assume that X_1 and X_3 coincide and constitute a single connected part X. Then we have three candidates for a mediator sphere: the spheres ∂X_2 and ∂X_4 and the new disk $D = \partial X_1 \#_\partial \partial X_3 \subset X$. The latter is obtained by connecting ∂X_1 and ∂X_3 by a tube inside X. We assert that at least one of the three disks is nontrivial and therefore is a mediator.

Arguing by contradiction, assume that all three disks are trivial. This means that X_2 is ball and X is homeomorphism to $D_2 \times I$. Then the reduction along the disk D_1 produces two manifolds $Y_1 = (X \cup X_4) \cup D_1$ and $Y_1' = (X_5 \cup X_2) \cup D_1'$, and the reduction along the disk D_2 also produces two manifolds $Y_2 = (X \cup X_2) \cup D_2$ and $Y_2' = (X_5 \cup X_4) \cup D_2'$. Here the balls X_2 are viewed as handle of index 1 (if they are attached to X) or as handle of index 2 (if they are attached to X_5). The balls D_1, D_1', D_2, D_2', and D_2' glue up the disks on the boundaries of the corresponding manifolds. Handles X_2 of index 1 connect different disks on the boundary of the manifold $X \approx D_2 \times I$, which implies that $Y_1 \approx Y_2 \approx S^1 \times D^2$. On the other hand, the bases of handles X_2 of index 2 in the torus ∂X_5 are isotopic; hence $Y_1' \approx Y_2'$. Therefore, the reductions along the disks D_1 and D_2 give the same result. This contradicts the fact that in our situation when $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) > 0$, the vertices B_1 and B_2 must be different.

Case 2. Assume that $D_1 \cap D_2$ consists of $n > 3$ arcs. The proof can be finished by an argument similar to that used in the proof of Case 1.

The arguments above are also applicable in the case of prime decompositions of non-Orientale manifolds, with the only difference that handles X_2 of index 1 can now be attached to the boundary of the manifold $X \approx D_2 \times I$ in two different ways. In one case the result is the direct product $S^1 \times D^2$, and in the other case we get the twisted product $S^1 \tilde{\times} D^2$. Nevertheless, since the manifold M is non-Orientale, all summands $S^1 \times D^2$ can be replaced by $S^1 \tilde{\times} D^2$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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