**Research Article**

**Introduction to gb-Triple Systems**

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This paper introduces the category of gb-triple systems and studies some of their algebraic properties. Also provided is a functor from this category to the category of Leibniz algebras.

1. Introduction

A triple system is a vector space $g$ over a field $K$ together with a $K$-trilinear map $T: g^3 \to g$. Among the many examples known in the literature, one may mention 3-Lie algebras [1] and Lie triple systems [2] which are the generalizations of Lie algebras to ternary algebras, Jordan triple systems [2] which are the generalizations of Jordan algebras, and Leibniz 3-algebras [3] and Leibniz triple systems [4] which are generalizations of Leibniz algebras [5]. In this paper we enrich the family of triple systems by introducing the concept of gb-triple systems, presented as another generalization of Leibniz algebras with the particularity that, for all $a, b \in g$, the map $T: g \to g$, defined by $T(a, x, b) = T(a, x, b)$, is a derivation of $g$, a property of great importance in Nambu Mechanics. We investigate some of their algebraic properties and provide a functorial connection with Leibniz algebras and Lie algebras.

For the remaining of this paper, we assume that $K$ is a field of characteristic different to 2 and all tensor products are taken over $K$.

**Definition 1.** A gb-triple system is a $K$-vector space $g$ equipped with a trilinear operation

$$[-, -, -]_g: g^3 \to g$$

satisfying the identity

$$\left[ x, y, [a, b, c]_g \right]_g = \left[ a, [x, y, b]_g, c \right]_g - \left[ [a, x, c]_g, y, b \right]_g - \left[ x, [a, y, c]_g, b \right]_g$$

(2)

for fixed $p \neq q \in \{1, 2, \ldots, n\}$. It is easy to check that the identity (2) is satisfied. So $g$ is a gb-triple system when endowed with the operation $[-, -, -]_g$.

Because of the resemblance between the identity (2) and the generalized Leibniz identity [3], it is worth mentioning that, in general, Leibniz 3-algebras do not coincide with gb-triple systems. The following example provides a Leibniz 3-algebra that is not a gb-triple system.

**Example 3.** The two-dimensional complex Leibniz 3-algebra $\mathcal{L}$ (see [6, Theorem 2.14]) with basis $\{a_1, a_2\}$, $\dim([\mathcal{L}, \mathcal{L}, \mathcal{L}]) = 1$, and brackets

$$[a_i, a_j, a_k]_{\mathcal{L}} = \begin{cases} a_{p}, & \text{if } i = p, j = q, k \neq p, q \\ -a_{p}, & \text{if } i = q, j = p, k \neq p, q \\ 0, & \text{else} \end{cases}$$

(3)

for fixed $p \neq q \in \{1, 2\}$. It is easy to check that its bracket does not satisfy the identity (2).
Definition 4. Let \( g, g' \) be gb-triple systems. A function \( \alpha : g \to g' \) is said to be a homomorphism of gb-triple systems if
\[
\alpha \left( [x, y, z]_g \right) = [\alpha(x), \alpha(y), \alpha(z)]_{g'}, \quad \forall x, y, z \in g. \tag{5}
\]

We may thus form the category gb-TS of gb-triple systems and gb-triple system homomorphisms.

Recall that if \( g \) is a vector space endowed with a trilinear operation \( [\cdot, \cdot, \cdot]_g \), then a map \( D : g \to g \) is called a derivation with respect to \( \sigma \) if
\[
D(\sigma (a, b, c)) = \sigma (D(a), b, c) + \sigma (a, D(b), c) + \sigma (a, b, D(c)) \tag{6}
\]

Lemma 5. Let \( g \) be a gb-triple system and \( g, h \in g \). Then the map \( D_{gh} \) defined on \( g \) by
\[
D_{gh}(x) = [g, x, h]_g \tag{7}
\]
is a derivation with respect to the bracket \([-,-,-]_g\) of \( g \).

Proof. By setting \( \sigma = [-,-,-]_g \) and using the identity (2), we have
\[
\sigma(x, y, D_{gh}(a)) = \left[ x, y, [g, a, h]_g \right]_g
\]
\[
= [g, [x, y, a]_g, h]_g - [[g, x, h]_g, y, a]_g
\]
\[
= [g, y, x, a]_g - [g, x, y, h]_g - [x, y, a]_g
\]
\[
= D_{gh}(\sigma(x, y, a)) - \sigma(D_{gh}(x), y, a)
\]
\[
- \sigma(x, D_{gh}(y), a). \tag{8}
\]

Definition 6. A subspace \( I \) of a gb-triple system \( g \) is a subalgebra of \( g \) if \( I \) is a gb-triple system when endowed with the trilinear operation of \( g \).

Definition 7. A subalgebra \( I \) of a gb-triple system \( g \) is called ideal (resp., left ideal, resp., right ideal) of \( g \) if it satisfies the condition \([g, I, g]_g \subseteq I\) (resp., \([g, g, I]_g \subseteq I\), resp., \([I, g, g]_g \subseteq I\)). If \( I \) satisfies the three conditions, then \( I \) is called a 3-sided ideal.

Note that none of these three conditions implies the others as in the case of Lie triple systems.

Example 8. In Example 2, the subspace \( \mathfrak{F}_1 \) with basis \( \{a_p, a_q\} \) is an ideal of \( g \). However the subspace \( \mathfrak{F}_2 \) with basis \( \{a_p\} \) is not an ideal of \( g \), since, for \( k \neq p, q \), we have \([a_p, a_q, a_k] = a_p \notin \mathfrak{F}_2\).

Definition 9. Given a gb-triple system \( g \), one defines the center of \( g \) and the derived algebra of \( g \), respectively, by
\[
Z(g) = \{ x \in g : [g, x, g]_g = 0 \},
\]
\[
[g, g, g] = \{ [a_1, a_2, a_3]_g, a_1, a_2, a_3 \in g \}. \tag{9}
\]

Lemma 10. For a gb-triple system \( g \), \( Z(g) \) and \([g, g, g]\) are ideals of \( g \).

Proof. Clearly, \([g, Z(g), g]_g = 0 \). So \( Z(g) \) is an ideal of \( g \). That \([g, g, g]_g \) is an ideal follows from the fact that \( g \) is closed under the operation \([-,-,-]_g\). \qed

The following theorem classifies a subfamily of two-dimensional complex gb-triple systems. This result was obtained by Camacho et al. in [6] for Leibniz 3-algebras.

Theorem 11. Up to isomorphism, there are seven two-dimensional complex gb-triple systems with one-dimensional derived algebra.

Proof. The proof is similar to [6, Theorem 2.14]. Let \( g \) be a gb-triple system with basis \( \{a_i, a_k\} \), and assume that \( \dim([g, g, g]_g) = 1 \). Then write \([a_i, a_j, a_k]_g = \alpha_i a_j a_k, i, j, k = 1, 2 \). Then, using the identity (2), the only possible nonzero coefficients yield to the system of equations
\[
\alpha_{211} (\alpha_{122} + \alpha_{212} + \alpha_{221}) = 0,
\]
\[
\alpha_{122} (\alpha_{212} + \alpha_{221} + \alpha_{211}) = 0,
\]
\[
\alpha_{222} (\alpha_{212} + \alpha_{221}) = 0,
\]
for which the solution provides the following gb-triple systems with bracket operations:
\[
g_1 : [a_i, a_j, a_k]_g = \begin{cases} a a_i, & \text{if } i, k = 2, j = 1 \\ 0, & \text{else} \end{cases}
\]
\[
g_2 : [a_i, a_j, a_k]_g = \begin{cases} a a_i, & \text{if } i, j, k = 2 \\ 0, & \text{else} \end{cases}
\]
\[
g_3 : [a_i, a_j, a_k]_g = \begin{cases} a_i, & \text{if } i = 1, j, k = 2 \\ -a_i, & \text{if } i, j = 2, k = 1 \\ 0, & \text{else} \end{cases}
\]
\[
g_4 : [a_i, a_j, a_k]_g = \begin{cases} a_i, & \text{if } i, k = 2, j = 1 \\ -a_i, & \text{if } i, j = 2, k = 1 \\ 0, & \text{else} \end{cases}
\]
\[
g_5 : [a_i, a_j, a_k]_g = \begin{cases} a_i, & \text{if } i, k = 2, j = 1 \\ a a_i, & \text{if } i, j, k = 2 \\ 0, & \text{else} \end{cases}
\]
\[
g_6 : [a_i, a_j, a_k]_g = \begin{cases} a_i, & \text{if } i = 1, j, k = 2 \\ -a_i, & \text{if } i, j = 2, k = 1 \\ a a_i, & \text{if } i, j, k = 2 \end{cases}
\]
\[
g_7 : [a_i, a_j, a_k]_g = \begin{cases} - (1 + \alpha) a_i, & \text{if } i = 1, j, k = 2 \\ a_i, & \text{if } i, k = 2, j = 1 \\ a a_i, & \text{if } i, j = 2, k = 1 \end{cases}
\]
with \( \alpha \neq 0 \). \qed
Definition 12. Given a gb-triple system \( g \), one defines the left center and the right center of \( g \), respectively, by
\[
\begin{align*}
Z_L(g) &= \{ x \in g : [x, g, g]_g = 0 \}, \\
Z_R(g) &= \{ x \in g : [g, x, g]_g = 0 \}.
\end{align*}
\] (12)

Lemma 13. The left center \( Z_L(g) \) and the right center \( Z_R(g) \) are 3-sided ideals of \( g \).

Proof. To show that \( Z_L(g) \) is an ideal of \( g \), let \( g, g' \in g \) and let \( x \in Z_L(g) \). Then, for every \( u, v \in g \), we have, by the identity (2),
\[
\left[ [g, x, g']_g, u, v \right]_g = \left[ [g, [x, u, v]_g, g']_g - [x, [g, u, g']_g, v]_g \right]_g - [x, u, [g, v, g']_g]_g = 0.
\] (13)

So \( [g, Z_L(g), g]_g \subseteq Z_L(g) \). The proof that \( Z_R(g) \) is both left ideal and right ideal is similar, so is the case for \( Z_R(g) \).

Definition 14. Given a gb-triple system \( g \), we define left and right centralizers of a subalgebra \( S \) in \( g \) by
\[
\begin{align*}
C^l_g(S) &= \{ x \in g : [x, S, g]_g = 0 \}, \\
C^r_g(S) &= \{ x \in g : [g, S, x]_g = 0 \},
\end{align*}
\] (14) (15)

respectively.

Lemma 15. Let \( S \) be an ideal of a gb-triple system \( g \). Then \( C^r_g(S) \) and \( C^l_g(S) \) are also ideals of \( g \).

Proof. To show that \( C^r_g(S) \) is an ideal of \( g \), let \( x \in C^r_g(S), u \in S \), and \( g, a, b \in g \). Then, by the identity (2),
\[
\left[ [g, u, [a, x, b]]_g \right]_g = \left[ a, [g, u, x]_g, b \right]_g - \left[ [a, g, b]_g, u, x \right]_g = 0.
\] (16)

So \( [g, C^r_g(S), g]_g \subseteq C^r_g(S) \). The proof for \( C^l_g(S) \) is similar.

Definition 16. For a gb-triple system \( g \) and a subalgebra \( S \) of \( g \), we define the left normalizer of \( S \) in \( g \) by
\[
\mathcal{N}^l_g(S) := \{ x \in g : [x, S, g]_g \subseteq S \},
\] (17)

and the right normalizer of \( S \) in \( g \) by
\[
\mathcal{N}^r_g(S) := \{ x \in g : [g, S, x]_g \subseteq S \}.
\] (18)

Lemma 17. Let \( S \) be a subalgebra of a gb-triple system \( g \). Then \( \mathcal{N}^l_g(S) \) and \( \mathcal{N}^r_g(S) \) are also subalgebras of \( g \).

Proof. To show that \( \mathcal{N}^l_g(S) \) is a subalgebra of \( g \), let \( x, y, z \in \mathcal{N}^l_g(S), u \in S \), and \( g \in g \). Then, by the identity (2), we have
\[
\left[ [g, u, [x, y, z]]_g, y \right]_g - \left[ [x, g, z]_g, u, y \right]_g - \left[ [g, [u, x, z]]_g, y \right]_g \in S.
\] (19)

So \( [\mathcal{N}^l_g(S), \mathcal{N}^r_g(S), \mathcal{N}^r_g(S)]_g \subseteq \mathcal{N}^l_g(S) \). The proof for \( \mathcal{N}^r_g(S) \) is similar.

Remark 18. If \( S \) is an ideal, then \( \mathcal{N}^l_g(S) = g = \mathcal{N}^r_g(S) \).

2. From gb-Triple Systems to Leibniz Algebras

Recall that a Leibniz algebra (sometimes called Loday algebra, named after Jean-Louis Loday) is a \( K \) vector space with a bilinear product \([-,-]\) satisfying the Leibniz identity
\[
[x, [y, z]] = [[x, y] z] + [y, [x, z]].
\] (20)

Proposition 19. Let \( g \) be a gb-triple system. Define on \( g^2 \) the bracket operation \([-,-]\) by
\[
[a_1 \otimes a_2, b_1 \otimes b_2] = [a_1, b_1, a_2]_g \otimes b_2 + b_1 \otimes [a_1, b_2, a_2]_g.
\] (21)

Then \([-,-]\) satisfies the Leibniz identity.

Proof. On one hand, we have
\[
[a_1 \otimes a_2, [b_1 \otimes b_2, c_1 \otimes c_2]] = [a_1 \otimes a_2, [b_1, c_1, b_2]_g \otimes c_2] + [a_1 \otimes a_2, c_1 \otimes [b_1, c_2, b_2]_g] + [b_1, c_1, b_2]_g \otimes [a_1, c_2, a_2]_g + [a_1, c_1, a_2]_g \otimes [b_1, c_2, b_2]_g + c_1 \otimes [a_1, b_1, c_2]_g \otimes a_2.
\] (22)
Also,
\[
[\{a_1 \otimes a_2, b_1 \otimes b_2\}, c_1 \otimes c_2] = \left[\left\{a_1, b_1, a_2\right\}_g \otimes b_2, c_1 \otimes c_2\right]
\]
\[
+ \left[\left\{a_1, b_1, a_2\right\}_g \otimes c_2\right]
\]
\[
+ [c_1 \otimes \left\{a_1, b_1, a_2\right\}_g, c_2] \otimes b_2
\]
\[
+ \left[\left\{b_1, c_1, [a_1, b_2, a_2]\right\}_g \otimes c_2\right]
\]
\[
+ [c_1 \otimes \left\{b_1, c_2, [a_1, b_2, a_2]\right\}_g \otimes b_2].
\]
(23)

On the other hand,
\[
[\left\{b_1 \otimes b_2, [a_1 \otimes a_2, c_1 \otimes c_2]\right\} = \left[\left\{b_1 \otimes b_2, [a_1, c_1, a_2]\right\}_g \otimes c_2\right]
\]
\[
+ \left[\left\{b_1 \otimes b_2, c_1 \otimes [a_1, c_2, a_2]\right\}_g\right]
\]
\[
+ \left[\left\{a_1, c_1, a_2\right\}_g \otimes \left\{b_1, c_2, b_2\right\}_g \otimes c_2\right]
\]
\[
+ [c_1 \otimes \left\{b_1, c_2, [a_1, c_2, a_2]\right\}_g \otimes b_2].
\]
(24)

One checks using the identity (2) that the equality
\[
[a_1 \otimes a_2, [b_1 \otimes b_2, c_1 \otimes c_2]] = \left[\left\{a_1 \otimes a_2, b_1 \otimes b_2\right\}, c_1 \otimes c_2\right]
\]
\[
+ \left[\left\{b_1 \otimes b_2, \left\{a_1 \otimes a_2, c_1 \otimes c_2\right\}\right\}_g\right]
\]
(25)

holds.

Corollary 20. Let \(\mathfrak{g}\) be a gb-triple system; then \(\mathfrak{g}^{s_2}\) endowed with the bilinear map \([-,-]\) has a Leibniz algebra structure.

Proof. This is a consequence of Proposition 19.

Similarly, we have the following.

Corollary 21. Let \(\mathfrak{g}\) be a gb-triple system; then \(\mathfrak{g}^{s_2}\) has a Leibniz algebra structure, when endowed with the bilinear map defined by
\[
[a_1 \wedge a_2, b_1 \wedge b_2] = \left[\left\{a_1, b_1, a_2\right\}_g \wedge b_2, c_1 \wedge \left\{a_1, b_2, a_2\right\}_g\right].
\]
(26)

These determine two functors from the category gb-TS of gb-triple systems to the category \(\mathbf{LB}\) of Leibniz algebras.

Definition 22. Let \(\mathfrak{g}\) be a gb-triple system and \(L\) a Leibniz algebra. The action of \(L\) on \(\mathfrak{g}\) is a map \(A : \mathfrak{g} \otimes L \to \mathfrak{g}\) satisfying
\[
(1) \quad A \left(\left\{g_1, g_2, g_3\right\}_g \otimes x\right) = A \left(\left\{g_1, g_2 \otimes x\right\}_g\right)
\]
\[
+ \left[\left\{g_1, g_2 \otimes x\right\}_g, g_3\right]\]
(27)
\[
(2) \quad A \left(\left\{g_1, g_2, g_3\right\}_g \otimes y\right) = A \left(\left\{g_1, g_2 \otimes y\right\}_g\right) - A \left(\left\{g_1 \otimes y\right\}_g \otimes g_3\right)
\]
(28)

for all \(g_1, g_2, g_3 \in \mathfrak{g}\) and \(x, y \in L\).

Proposition 23. Let \(\mathfrak{g}\) be a gb-triple system; then the Leibniz algebra \(\mathfrak{g}^{s_2}\) acts on \(\mathfrak{g}\) via the map \(A : \mathfrak{g} \otimes \mathfrak{g}^{s_2} \to \mathfrak{g}\) defined by
\[
A(\left\{g_1 \otimes g_2\right\}_g) = [g_1, g_2]_g.
\]

Proof. The first condition of Definition 22 follows by (2). To show (28), we have
\[
A \left(\left\{z \otimes [a_1, a_2]_g \otimes b_1 \otimes b_2\right\}_g\right) = \left[\left\{g_1, g_2 \otimes z\right\}_g \otimes b_1 \otimes b_2\right]
\]
\[
+ [b_1 \otimes \left\{g_1, g_2 \otimes z\right\}_g \otimes b_2].
\]
(29)

Now let \(g_1, g_2 \in \mathfrak{g}\) and consider the map \(A : \mathfrak{g} \otimes \mathfrak{g}^{s_2} : \mathfrak{g} \to \mathfrak{g}\) defined by \(A_{g_1 \otimes g_2}(z) = [g_1, z]_g, z \in \mathfrak{g}\). Clearly, this map is a derivation of \(\mathfrak{g}\) as it is induced by the action (Proposition 23) defined above.

Proposition 24. For a gb-triple system \(\mathfrak{g}\), the subspace \(\mathfrak{U}(\mathfrak{g}) = \{A_{g_1 \otimes g_2} \mid g_1, g_2 \in \mathfrak{g}\}\) is a Lie algebra with respect to the product
\[
\left[A_{a_1 \otimes a_2}, A_{b_1 \otimes b_2}\right] = A_{a_1 \otimes a_2} \circ A_{b_1 \otimes b_2} - A_{b_1 \otimes b_2} \circ A_{a_1 \otimes a_2}.
\]

More precisely, it is an ideal of the Lie algebra \(\text{Der}(\mathfrak{g})\) of derivations of \(\mathfrak{g}\).

Proof. To show that \(\mathfrak{U}(\mathfrak{g})\) is a Lie subalgebra of \(\text{Der}(\mathfrak{g})\), let \(a_1, a_2, b_1, b_2 \in \mathfrak{g}\). Then, for all \(z \in \mathfrak{g}\),
\[
\left[A_{a_1 \otimes a_2}, A_{b_1 \otimes b_2}\right]_{\text{Der}(\mathfrak{g})}(z)
\]
\[
= A_{a_1 \otimes a_2} \circ A_{b_1 \otimes b_2}(z) - A_{b_1 \otimes b_2} \circ A_{a_1 \otimes a_2}(z)
\]
\[
= [a_1, [b_1, z, b_2]_g, a_2]_g - [b_1, [a_1, z, a_2]_g, b_2].
\]
(30)
\[ = A \left( z \otimes b_1 \otimes b_2 \otimes a_1 \otimes a_2 \right) \]
\[ - A \left( A \left( z \otimes a_1 \otimes a_2 \otimes b_1 \otimes b_2 \right) \right) \]
\[ = A \left( z \otimes \left[ a_1 \otimes a_2, b_1 \otimes b_2 \right] \right) \quad \text{by Proposition 23} \]
\[ = A_{[a_1 \otimes a_2, b_1 \otimes b_2]}(z). \]  

(31)

So \( \mathfrak{g}(g) \) is closed under the bracket of \( \text{Der}(g) \). Also, for any derivation \( \in \text{Der}(g) \), we have, for all \( z \in g \),
\[ \left[ d, A_{n, \Theta a_1} \right]_{\text{Der}(g)}(z) = \left( d \circ A_{n, \Theta a_1} \right)(z) - \left( A_{n, \Theta a_1} \circ d \right)(z) \]
\[ = d \left( A_{n, \Theta a_1}(z) \right) - A_{n, \Theta a_1} \left( d \left( z \right) \right) \]
\[ = d \left( \left[ a_1, z, a_2 \right]_g \right) - \left[ a_1, d \left( z \right), a_2 \right]_g \]
\[ = \left[ d \left( a_1 \right), z, a_2 \right]_g \]
\[ + \left[ a_1, z, d \left( a_2 \right) \right]_g \quad \text{by (6)} \]
\[ = A_{d(a_1) \otimes a_2}(z) + A_{a_1 \otimes d(a_2)}(z). \]  

(32)

Hence \( \left[ d, A_{n, \Theta a_1} \right]_{\text{Der}(g)} = A_{d(a_1) \otimes a_2} + A_{a_1 \otimes d(a_2)} \in \mathfrak{g}(g). \quad \square \)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


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