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Research Article

Introduction to *gb*-Triple Systems

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This paper introduces the category of *gb*-triple systems and studies some of their algebraic properties. Also provided is a functor from this category to the category of Leibniz algebras.

1. Introduction

A Triple system is a vector space $\mathfrak{g}$ over a field $\mathfrak{K}$ together with a $\mathfrak{K}$-trilinear map $T : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$. Among the many examples known in the literature, one may mention 3-Lie algebras [1] and Lie triple systems [2] which are the generalizations of Lie algebras to ternary algebras, Jordan triple systems [2] which are the generalizations of Jordan algebras, and Leibniz 3-algebras [3] and Leibniz triple systems [4] which are generalizations of Leibniz algebras [5]. In this paper we enrich the family of triple systems by introducing the concept of *gb*-triple systems, presented as another generalization of Leibniz algebras with the particularity that, for all $a, b \in \mathfrak{g}$, the map $T : \mathfrak{g} \to \mathfrak{g}$, defined by $T_{a,b}(x) = T(a, x, b)$, is a derivation of $\mathfrak{g}$, a property of great importance in Nambu Mechanics. We investigate some of their algebraic properties and provide a functorial connection with Leibniz algebras and Lie algebras.

For the remaining of this paper, we assume that $\mathfrak{K}$ is a field of characteristic different to 2 and all tensor products are taken over $\mathfrak{K}$.

Definition 1. A *gb*-triple system is a $\mathfrak{K}$-vector space $\mathfrak{g}$ equipped with a trilinear operation

$$[-,-,-]_{\mathfrak{g}} : \mathfrak{g}^{\otimes 3} \longrightarrow \mathfrak{g} \text{ satisfying the identity} \tag{1}$$

$$\begin{aligned}\left[x, y, [a,b,c]_{\mathfrak{g}}\right]_{\mathfrak{g}} = \left[a, [x,y,b]_{\mathfrak{g}}, c\right]_{\mathfrak{g}} - \left[[a,x,c]_{\mathfrak{g}}, y, b\right]_{\mathfrak{g}} \\ - \left[x, [a,y,c]_{\mathfrak{g}}, b\right]_{\mathfrak{g}}.\end{aligned} \tag{2}$$

Example 2. Let $\mathfrak{g}$ be an n-dimensional vector space with basis $\{a_1, a_2, \ldots, a_n\}$. Define on $\mathfrak{g}^{\otimes 3}$ the bracket $[-,-,-]$ by

$$\left[a_i, a_j, a_k\right]_{\mathfrak{g}} = \begin{cases} a_p, & \text{if } i = p,\ j = q,\ k \neq p, q \\ -a_p, & \text{if } i = q,\ j = p,\ k \neq p, q \\ 0, & \text{else} \end{cases} \tag{3}$$

for fixed $p \neq q \in \{1, 2, \ldots, n\}$. It is easy to check that the identity (2) is satisfied. So $\mathfrak{g}$ is a *gb*-triple system when endowed with the operation $[-,-,-]_{\mathfrak{g}}$.

Because of the resemblance between the identity (2) and the generalized Leibniz identity [3], it is worth mentioning that, in general, Leibniz 3-algebras do not coincide with *gb*-triple systems. The following example provides a Leibniz 3-algebra that is not a *gb*-triple system.

Example 3. The two-dimensional complex Leibniz 3-algebra Ł (see [6, Theorem 2.14]) with basis $\{a_1, a_2\}$, $\dim([Ł, Ł, Ł]_{Ł}) = 1$, and brackets

$$\left[a_i, a_j, a_k\right]_{\mathfrak{g}} = \begin{cases} a_1, & \text{if } i = 1,\ j, k = 2 \\ \alpha a_1, & \text{if } i, j, k = 2 \\ 0, & \text{else,} \end{cases} \tag{4}$$

with $\alpha \neq 0$, is not a *gb*-triple system. It is easy to check that its bracket does not satisfy the identity (2).

Definition 4. Let $\mathfrak{g}, \mathfrak{g}'$ be *gb*-triple systems. A function $\alpha : \mathfrak{g} \to \mathfrak{g}'$ is said to be a homomorphism of *gb*-triple systems if

$$\alpha\left([x,y,z]_{\mathfrak{g}}\right) = [\alpha(x), \alpha(y), \alpha(z)]_{\mathfrak{g}'} \quad \forall x,y,z \in \mathfrak{g}. \tag{5}$$

We may thus form the category *gb*-**TS** of *gb*-triple systems and *gb*-triple system homomorphisms.

Recall that if $\mathfrak{g}$ is a vector space endowed with a trilinear operation $\sigma : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, then a map $D : \mathfrak{g} \to \mathfrak{g}$ is called a derivation with respect to σ if

$$\begin{aligned} D(\sigma(a,b,c)) = {} & \sigma(D(a), b, c) + \sigma(a, D(b), c) \\ & + \sigma(a, b, D(c)). \end{aligned} \tag{6}$$

Lemma 5. *Let $\mathfrak{g}$ be a gb-triple system and $g, h \in \mathfrak{g}$. Then the map D_{gh} defined on $\mathfrak{g}$ by*

$$D_{gh}(x) = [g, x, h]_{\mathfrak{g}} \tag{7}$$

is a derivation with respect to the bracket $[-,-,-]_{\mathfrak{g}}$ of $\mathfrak{g}$.

Proof. By setting $\sigma = [-,-,-]_{\mathfrak{g}}$ and using the identity (2), we have

$$\begin{aligned} \sigma\left(x, y, D_{gh}(a)\right) &= \left[x, y, [g,a,h]_{\mathfrak{g}}\right]_{\mathfrak{g}} \\ &= \left[g, [x,y,a]_{\mathfrak{g}}, h\right]_{\mathfrak{g}} - \left[[g,x,h]_{\mathfrak{g}}, y, a\right]_{\mathfrak{g}} \\ &\quad - \left[x, [g,y,h]_{\mathfrak{g}}, a\right]_{\mathfrak{g}} \\ &= D_{gh}(\sigma(x,y,a)) - \sigma\left(D_{gh}(x), y, a\right) \\ &\quad - \sigma\left(x, D_{gh}(y), a\right). \end{aligned} \tag{8}$$

□

Definition 6. A subspace I of a *gb*-triple system $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ if I is a *gb*-triple system when endowed with the trilinear operation of $\mathfrak{g}$.

Definition 7. A subalgebra I of a *gb*-triple system $\mathfrak{g}$ is called ideal (resp., left ideal, resp., right ideal) of $\mathfrak{g}$ if it satisfies the condition $[\mathfrak{g}, I, \mathfrak{g}]_{\mathfrak{g}} \subseteq I$ (resp., $[\mathfrak{g}, \mathfrak{g}, I]_{\mathfrak{g}} \subseteq I$, resp., $[I, \mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}} \subseteq I$). If I satisfies the three conditions, then I is called a 3-sided ideal.

Note that none of these three conditions implies the others as in the case of Lie triple systems.

Example 8. In Example 2, the subspace $\mathfrak{I}_1$ with basis $\{a_p, a_q\}$ is an ideal of $\mathfrak{g}$. However the subspace $\mathfrak{I}_2$ with basis $\{a_q\}$ is not an ideal of $\mathfrak{g}$, since, for $k \neq p, q$, we have $[a_p, a_q, a_k] = a_p \notin \mathfrak{I}_2$.

Definition 9. Given a *gb*-triple system $\mathfrak{g}$, one defines the center of $\mathfrak{g}$ and the derived algebra of $\mathfrak{g}$, respectively, by

$$\begin{aligned} Z(\mathfrak{g}) &= \left\{x \in \mathfrak{g} : [\mathfrak{g}, x, \mathfrak{g}]_{\mathfrak{g}} = 0\right\}, \\ [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}] &= \left\{[a_1, a_2, a_3]_{\mathfrak{g}}, a_1, a_2, a_3 \in \mathfrak{g}\right\}. \end{aligned} \tag{9}$$

Lemma 10. *For a gb-triple system $\mathfrak{g}$, $Z(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]$ are ideals of $\mathfrak{g}$.*

Proof. Clearly, $[\mathfrak{g}, Z(\mathfrak{g}), \mathfrak{g}]_{\mathfrak{g}} = 0$. So $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. That $[\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$ is an ideal follows from the fact that $\mathfrak{g}$ is closed under the operation $[-,-,-]_{\mathfrak{g}}$. □

The following theorem classifies a subfamily of two-dimensional complex *gb*-triple systems. This result was obtained by Camacho et al. in [6] for Leibniz 3-algebras.

Theorem 11. *Up to isomorphism, there are seven two-dimensional complex gb-triple systems with one-dimensional derived algebra.*

Proof. The proof is similar to [6, Theorem 2.14]. Let $\mathfrak{g}$ be a *gb*-triple system with basis $\{a_1, a_2\}$, and assume that $\dim([\mathfrak{g}, \mathfrak{g}, \mathfrak{g}])_{\mathfrak{g}}) = 1$. Then write $[a_i, a_j, a_k]_{\mathfrak{g}} = \alpha_{ijk} a_1$, $i, j, k = 1, 2$. Then, using the identity (2), the only possible nonzero coefficients yield to the system of equations

$$\begin{aligned} \alpha_{221}(\alpha_{122} + \alpha_{212} + \alpha_{221}) &= 0, \\ \alpha_{122}(\alpha_{122} + \alpha_{212} + \alpha_{221}) &= 0, \\ \alpha_{222}(\alpha_{122} + \alpha_{221}) &= 0, \end{aligned} \tag{10}$$

for which the solution provides the following *gb*-triple systems with bracket operations:

$$\begin{aligned}
\mathfrak{g}_1 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} \alpha a_1, & \text{if } i, k = 2,\ j = 1 \\ 0, & \text{else,} \end{cases} \\
\mathfrak{g}_2 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} \alpha a_1, & \text{if } i, j, k = 2 \\ 0, & \text{else,} \end{cases} \\
\mathfrak{g}_3 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} a_1, & \text{if } i = 1,\ j, k = 2 \\ -a_1, & \text{if } i, j = 2,\ k = 1 \\ 0, & \text{else,} \end{cases} \\
\mathfrak{g}_4 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} a_1, & \text{if } i, k = 2,\ j = 1 \\ -a_1, & \text{if } i, j = 2,\ k = 1 \\ 0, & \text{else,} \end{cases} \\
\mathfrak{g}_5 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} a_1, & \text{if } i, k = 2,\ j = 1 \\ \alpha a_1, & \text{if } i, j, k = 2 \\ 0, & \text{else,} \end{cases} \\
\mathfrak{g}_6 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} a_1, & \text{if } i = 1,\ j, k = 2 \\ -a_1, & \text{if } i, j = 2,\ k = 1 \\ \alpha a_1, & \text{if } i, j, k = 2, \end{cases} \\
\mathfrak{g}_7 : \left[a_i, a_j, a_k\right]_{\mathfrak{g}} &= \begin{cases} -(1+\alpha) a_1, & \text{if } i = 1,\ j, k = 2 \\ a_1, & \text{if } i, k = 2,\ j = 1 \\ \alpha a_1, & \text{if } i, j = 2,\ k = 1, \end{cases}
\end{aligned} \tag{11}$$

with $\alpha \neq 0$. □

Definition 12. Given a *gb*-triple system $\mathfrak{g}$, one defines the left center and the right center of $\mathfrak{g}$, respectively, by

$$Z_L(\mathfrak{g}) = \left\{x \in \mathfrak{g} : [x, \mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}} = 0\right\},$$
$$Z_R(\mathfrak{g}) = \left\{x \in \mathfrak{g} : [\mathfrak{g}, \mathfrak{g}, x]_{\mathfrak{g}} = 0\right\}. \tag{12}$$

Lemma 13. *The left center $Z_L(\mathfrak{g})$ and the right center $Z_R(\mathfrak{g})$ are 3-sided ideals of $\mathfrak{g}$.*

Proof. To show that $Z_L(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, let $g, g' \in \mathfrak{g}$ and let $x \in Z_L(\mathfrak{g})$. Then, for every $u, v \in \mathfrak{g}$, we have, by the identity (2),

$$\left[\left[g, x, g'\right]_{\mathfrak{g}}, u, v\right]_{\mathfrak{g}} = \left[g, \underbrace{[x, u, v]_{\mathfrak{g}}}_{0}, g'\right]_{\mathfrak{g}} - \underbrace{\left[x, \left[g, u, g'\right], v\right]_{\mathfrak{g}}}_{0} - \underbrace{\left[x, u, \left[g, v, g'\right]_{\mathfrak{g}}\right]_{\mathfrak{g}}}_{0} = 0. \tag{13}$$

So $[\mathfrak{g}, Z_L(\mathfrak{g}), \mathfrak{g}]_{\mathfrak{g}} \subseteq Z_L(\mathfrak{g})$. The proof that $Z_L(\mathfrak{g})$ is both left ideal and right ideal is similar, so is the case for $Z_R(\mathfrak{g})$. □

Definition 14. Given a *gb*-triple system $\mathfrak{g}$, we define left and right centralizers of a subalgebra S in $\mathfrak{g}$ by

$$C^l_{\mathfrak{g}}(S) = \left\{x \in \mathfrak{g} : [x, S, \mathfrak{g}]_{\mathfrak{g}} = 0\right\}, \tag{14}$$

$$C^r_{\mathfrak{g}}(S) = \left\{x \in \mathfrak{g} : [\mathfrak{g}, S, x]_{\mathfrak{g}} = 0\right\}, \tag{15}$$

respectively.

Lemma 15. *Let S be an ideal of a gb-triple system $\mathfrak{g}$. Then $C^r_{\mathfrak{g}}(S)$ and $C^l_{\mathfrak{g}}(S)$ are also ideals of $\mathfrak{g}$.*

Proof. To show that $C^r_{\mathfrak{g}}(S)$ is an ideal of $\mathfrak{g}$, let $x \in C^r_{\mathfrak{g}}(S)$, $u \in S$, and $g, a, b \in \mathfrak{g}$. Then, by the identity (2),

$$\left[g, u, [a, x, b]_{\mathfrak{g}}\right]_{\mathfrak{g}} = \left[a, \underbrace{[g, u, x]}_{0}, b\right] - \underbrace{\left[[a, g, b]_{\mathfrak{g}}, u, x\right]_{\mathfrak{g}}}_{0} - \underbrace{\left[g, \underbrace{[a, u, b]_{\mathfrak{g}}}_{\in S}, x\right]_{\mathfrak{g}}}_{0} = 0. \tag{16}$$

So $[\mathfrak{g}, C^r_{\mathfrak{g}}(S), \mathfrak{g}]_{\mathfrak{g}} \subseteq C^r_{\mathfrak{g}}(S)$. The proof for $C^l_{\mathfrak{g}}(S)$ is similar. □

Definition 16. For a *gb*-triple system $\mathfrak{g}$ and a subalgebra S of $\mathfrak{g}$, we define the left normalizer of S in $\mathfrak{g}$ by

$$\mathfrak{N}^l_{\mathfrak{g}}(S) := \{x \in \mathfrak{g} : [x, S, \mathfrak{g}] \subseteq S\}, \tag{17}$$

and the right normalizer of S in $\mathfrak{g}$ by

$$\mathfrak{N}^r_{\mathfrak{g}}(S) := \{x \in \mathfrak{g} : [\mathfrak{g}, S, x] \subseteq S\}. \tag{18}$$

Lemma 17. *Let S be a subalgebra of a gb-triple system $\mathfrak{g}$. Then $\mathfrak{N}^l_{\mathfrak{g}}(S)$ and $\mathfrak{N}^r_{\mathfrak{g}}(S)$ are also subalgebras of $\mathfrak{g}$.*

Proof. To show that $\mathfrak{N}^r_{\mathfrak{g}}(S)$ is a subalgebra of $\mathfrak{g}$, let $x, y, z \in \mathfrak{N}^r_{\mathfrak{g}}(S)$, $u \in S$, and $g \in \mathfrak{g}$. Then, by the identity (2), we have

$$\left[g, u, [x, y, z]_{\mathfrak{g}}\right]_{\mathfrak{g}} = \left[x, \underbrace{[g, u, y]_{\mathfrak{g}}}_{\in S}, z\right]_{\mathfrak{g}} - \left[[x, g, z]_{\mathfrak{g}}, u, y\right]_{\mathfrak{g}} - \left[g, \underbrace{[x, u, z]_{\mathfrak{g}}}_{\in S}, y\right]_{\mathfrak{g}} \in S. \tag{19}$$

So $[\mathfrak{N}^r_{\mathfrak{g}}(S), \mathfrak{N}^r_{\mathfrak{g}}(S), \mathfrak{N}^r_{\mathfrak{g}}(S)]_{\mathfrak{g}} \subseteq \mathfrak{N}^r_{\mathfrak{g}}(S)$. The proof for $\mathfrak{N}^l_{\mathfrak{g}}(S)$ is similar. □

Remark 18. If S is an ideal, then $\mathfrak{N}^l_{\mathfrak{g}}(S) = \mathfrak{g} = \mathfrak{N}^r_{\mathfrak{g}}(S)$.

2. From *gb*-Triple Systems to Leibniz Algebras

Recall that a Leibniz algebra (sometimes called Loday algebra, named after Jean-Louis Loday) is a $\mathfrak{K}$ vector space with a bilinear product $[-,-]$ satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y] z] + [y, [x, z]]. \tag{20}$$

Proposition 19. *Let $\mathfrak{g}$ be a gb-triple system. Define on $\mathfrak{g}^{\otimes 2}$ the bracket operation $[-,-]$ by*

$$[a_1 \otimes a_2, b_1 \otimes b_2] = [a_1, b_1, a_2]_{\mathfrak{g}} \otimes b_2 + b_1 \otimes [a_1, b_2, a_2]_{\mathfrak{g}}. \tag{21}$$

Then $[-,-]$ satisfies the Leibniz identity.

Proof. On one hand, we have

$$\begin{aligned}
[a_1 \otimes a_2, [b_1 \otimes b_2, c_1 \otimes c_2]] &= \left[a_1 \otimes a_2, [b_1, c_1, b_2]_{\mathfrak{g}} \otimes c_2\right] \\
&\quad + \left[a_1 \otimes a_2, c_1 \otimes [b_1, c_2, b_2]_{\mathfrak{g}}\right] \\
&= \left[a_1, [b_1, c_1, b_2]_{\mathfrak{g}}, a_2\right]_{\mathfrak{g}} \otimes c_2 \\
&\quad + [b_1, c_1, b_2]_{\mathfrak{g}} \otimes [a_1, c_2, a_2]_{\mathfrak{g}} \\
&\quad + [a_1, c_1, a_2]_{\mathfrak{g}} \otimes [b_1, c_2, b_2]_{\mathfrak{g}} \\
&\quad + c_1 \otimes \left[a_1, [b_1, c_2, b_2]_{\mathfrak{g}}, a_2\right]_{\mathfrak{g}}.
\end{aligned} \tag{22}$$

Also,

$$\begin{aligned}[[a_1\otimes a_2, b_1\otimes b_2], c_1\otimes c_2] &= [[a_1,b_1,a_2]_{\mathfrak{g}}\otimes b_2, c_1\otimes c_2] \\ &\quad + [b_1\otimes [a_1,b_2,a_2]_{\mathfrak{g}}, c_1\otimes c_2] \\ &= [[a_1,b_1,a_2]_{\mathfrak{g}}, c_1, b_2]_{\mathfrak{g}}\otimes c_2 \\ &\quad + c_1\otimes [[a_1,b_1,a_2]_{\mathfrak{g}}, c_2, b_2]_{\mathfrak{g}} \\ &\quad + [b_1, c_1, [a_1,b_2,a_2]_{\mathfrak{g}}]_{\mathfrak{g}}\otimes c_2 \\ &\quad + c_1\otimes [b_1, c_2, [a_1,b_2,a_2]_{\mathfrak{g}}]_{\mathfrak{g}}.\end{aligned} \tag{23}$$

On the other hand,

$$\begin{aligned}[b_1\otimes b_2, [a_1\otimes a_2, c_1\otimes c_2]] &= [b_1\otimes b_2, [a_1,c_1,a_2]_{\mathfrak{g}}\otimes c_2] \\ &\quad + [b_1\otimes b_2, c_1\otimes [a_1,c_2,a_2]_{\mathfrak{g}}] \\ &= [b_1, [a_1,c_1,a_2]_{\mathfrak{g}}, b_2]_{\mathfrak{g}}\otimes c_2 \\ &\quad + [a_1,c_1,a_2]_{\mathfrak{g}}\otimes [b_1,c_2,b_2]_{\mathfrak{g}} \\ &\quad + [b_1,c_1,b_2]_{\mathfrak{g}}\otimes [a_1,c_2,a_2]_{\mathfrak{g}} \\ &\quad + c_1\otimes [b_1, [a_1,c_2,a_2]_{\mathfrak{g}}, b_2]_{\mathfrak{g}}.\end{aligned} \tag{24}$$

One checks using the identity (2) that the equality

$$\begin{aligned}[a_1\otimes a_2, [b_1\otimes b_2, c_1\otimes c_2]] &= [[a_1\otimes a_2, b_1\otimes b_2], c_1\otimes c_2] \\ &\quad + [b_1\otimes b_2, [a_1\otimes a_2, c_1\otimes c_2]]\end{aligned} \tag{25}$$

holds. □

Corollary 20. *Let $\mathfrak{g}$ be a gb-triple system; then $\mathfrak{g}^{\otimes 2}$ endowed with the bilinear map $[-,-]$ has a Leibniz algebra structure.*

Proof. This is a consequence of Proposition 19. □

Similarly, we have the following.

Corollary 21. *Let $\mathfrak{g}$ be a gb-triple system; then $\mathfrak{g}^{\wedge 2}$ has a Leibniz algebra structure, when endowed with the bilinear map defined by*

$$[a_1\wedge a_2, b_1\wedge b_2] = [a_1,b_1,a_2]_{\mathfrak{g}}\wedge b_2 + b_1\wedge [a_1,b_2,a_2]_{\mathfrak{g}}. \tag{26}$$

These determine two functors from the category *gb*-**TS** of *gb*-triple systems to the category **Lb** of Leibniz algebras.

Definition 22. Let $\mathfrak{g}$ be a *gb*-triple system and Ł a Leibniz algebra. The action of Ł on $\mathfrak{g}$ is a map $A:\mathfrak{g}\otimes Ł\to\mathfrak{g}$ satisfying

$$\begin{aligned}(1)\quad A([g_1,g_2,g_3]_{\mathfrak{g}}\otimes x) &= [A(g_1\otimes x), g_2, g_3]_{\mathfrak{g}} \\ &\quad + [g_1, A(g_2\otimes x), g_3]_{\mathfrak{g}} \\ &\quad + [g_1, g_2, A(g_3\otimes x)]_{\mathfrak{g}},\end{aligned} \tag{27}$$

$$(2)\quad A(g\otimes [x,y]) = A(A(g\otimes y)\otimes x) - A(A(g\otimes x)\otimes y) \tag{28}$$

for all $g_1,g_2,g_3\in\mathfrak{g}$ and $x,y\in Ł$.

Proposition 23. *Let $\mathfrak{g}$ be a gb-triple system; then the Leibniz algebra $\mathfrak{g}^{\otimes 2}$ acts on $\mathfrak{g}$ via the map $A:\mathfrak{g}\otimes\mathfrak{g}^{\otimes 2}\to\mathfrak{g}$ defined by $A(z\otimes g_1\otimes g_2) = [g_1,z,g_2]_{\mathfrak{g}}$.*

Proof. The first condition of Definition 22 follows by (2). To show (28), we have

$$\begin{aligned}&A(A(z\otimes b_1\otimes b_2)\otimes a_1\otimes a_2) - A(A(z\otimes a_1\otimes a_2)\otimes b_1\otimes b_2) \\ &\quad = [a_1, [b_1,z,b_2]_{\mathfrak{g}}, a_2]_{\mathfrak{g}} - [b_1, [a_1,z,a_2]_{\mathfrak{g}}, b_2]_{\mathfrak{g}} \\ &\quad = [[a_1,b_1,a_2]_{\mathfrak{g}}, z, b_2]_{\mathfrak{g}} + [b_1, z, [a_1,b_2,a_2]]_{\mathfrak{g}} \quad \text{by (2)} \\ &\quad = A(z\otimes [a_1,b_1,a_2]_{\mathfrak{g}}\otimes b_2) + A(z\otimes b_1\otimes [a_1,b_2,a_2]_{\mathfrak{g}}) \\ &\quad = A(z\otimes ([a_1,b_1,a_2]_{\mathfrak{g}}\otimes b_2 + b_1\otimes [a_1,b_2,a_2]_{\mathfrak{g}})) \\ &\quad = A(z\otimes [a_1\otimes a_2, b_1\otimes b_2]) \quad \text{by (21)}.\end{aligned} \tag{29}$$

□

Now let $g_1,g_2\in\mathfrak{g}$ and consider the map $A_{g_1\otimes g_2}:\mathfrak{g}\to\mathfrak{g}$ defined by $A_{g_1\otimes g_2}(z) = [g_1,z,g_2]_{\mathfrak{g}}$, $z\in\mathfrak{g}$. Clearly, this map is a derivation of $\mathfrak{g}$ as it is induced by the action (Proposition 23) defined above.

Proposition 24. *For a gb-triple system $\mathfrak{g}$, the subspace $\mathfrak{A}(\mathfrak{g}) = \{A_{g_1\otimes g_2} \mid g_1,g_2\in\mathfrak{g}\}$ is a Lie algebra with respect to the product*

$$[A_{a_1\otimes a_2}, A_{b_1\otimes b_2}] = A_{a_1\otimes a_2}\circ A_{b_1\otimes b_2} - A_{b_1\otimes b_2}\circ A_{a_1\otimes a_2}. \tag{30}$$

More precisely, it is an ideal of the Lie algebra Der($\mathfrak{g}$) of derivations of $\mathfrak{g}$.

Proof. To show that $\mathfrak{A}(\mathfrak{g})$ is a Lie subalgebra of Der($\mathfrak{g}$), let $a_1,a_2,b_1,b_2\in\mathfrak{g}$. Then, for all $z\in\mathfrak{g}$,

$$\begin{aligned}&[A_{a_1\otimes a_2}, A_{b_1\otimes b_2}]_{\mathrm{Der}(\mathfrak{g})}(z) \\ &\quad = A_{a_1\otimes a_2}\circ A_{b_1\otimes b_2}(z) - A_{b_1\otimes b_2}\circ A_{a_1\otimes a_2}(z) \\ &\quad = [a_1, [b_1,z,b_2]_{\mathfrak{g}}, a_2]_{\mathfrak{g}} - [b_1, [a_1,z,a_2]_{\mathfrak{g}}, b_2]_{\mathfrak{g}}\end{aligned}$$

$$\begin{aligned}
&= A\left(A\left(z\otimes b_1\otimes b_2\right)\otimes a_1\otimes a_2\right)\\
&\quad - A\left(A\left(z\otimes a_1\otimes a_2\right)\otimes b_1\otimes b_2\right)\\
&= A\left(z\otimes\left[a_1\otimes a_2, b_1\otimes b_2\right]\right) \quad \text{by Proposition 23}\\
&= A_{[a_1\otimes a_2, b_1\otimes b_2]}\left(z\right).
\end{aligned} \tag{31}$$

So $\mathfrak{A}(\mathfrak{g})$ is closed under the bracket of $\mathrm{Der}(\mathfrak{g})$. Also, for any derivation $\in \mathrm{Der}(\mathfrak{g})$, we have, for all $z \in \mathfrak{g}$,

$$\begin{aligned}
\left[d, A_{a_1\otimes a_2}\right]_{\mathrm{Der}(\mathfrak{g})}(z) &= \left(d\circ A_{a_1\otimes a_2}\right)(z) - \left(A_{a_1\otimes a_2}\circ d\right)(z)\\
&= d\left(A_{a_1\otimes a_2}(z)\right) - A_{a_1\otimes a_2}\left(d(z)\right)\\
&= d\left(\left[a_1, z, a_2\right]_{\mathfrak{g}}\right) - \left[a_1, d(z), a_2\right]_{\mathfrak{g}}\\
&= \left[d\left(a_1\right), z, a_2\right]_{\mathfrak{g}}\\
&\quad + \left[a_1, z, d\left(a_2\right)\right]_{\mathfrak{g}} \quad \text{by (6)}\\
&= A_{d(a_1)\otimes a_2}(z) + A_{a_1\otimes d(a_2)}(z).
\end{aligned} \tag{32}$$

Hence $[d, A_{a_1\otimes a_2}]_{\mathrm{Der}(\mathfrak{g})} = A_{d(a_1)\otimes a_2} + A_{a_1\otimes d(a_2)} \in \mathfrak{A}(\mathfrak{g})$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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