Research Article

Wiener-Itô Chaos Expansion of Hilbert Space Valued Random Variables

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The notion of n -fold iterated Itô integral with respect to a cylindrical Hilbert space valued Wiener process is introduced and the Wiener-Itô chaos expansion is obtained for a square Bochner integrable Hilbert space valued random variable. The expansion can serve a basis for developing the Hilbert space valued analog of Malliavin calculus of variations which can then be applied to the study of stochastic differential equations in Hilbert spaces and their solutions.

1. Introduction

The Wiener-Itô chaos expansion of a square integrable random variable which was first proved in [1] plays fundamental role in Malliavin calculus of variations [2, 3] which appeared to be a powerful instrument in the analysis of functionals of Brownian motion. The Malliavin calculus has found extensive applications to stochastic differential equations arising as models of various random phenomena. One of the important sources of such equations is markets modeling in financial mathematics [4, 5].

In the last decades, many researchers' interest has been drawn to stochastic differential equations in infinite dimensional Hilbert spaces driven by a cylindrical Wiener process or, equivalently, by countable set of Brownian motions [6, 7]. For example, in financial mathematics, such equations are used in modeling interest rates term structure or zero-coupon bond market [8, 9]. The present work was motivated by the need to make the Malliavin calculus applicable to Hilbert space valued stochastic processes. The first step in this direction is to obtain the generalization of the Wiener-Itô chaos expansion for Hilbert space valued random variables.

In order to achieve this, we first prove the Hilbert space valued version of the Itô representation theorem in Section 2. This generalization is established in Theorem 9 and Corollary 11.

In Section 3, we introduce iterated Itô integrals and multiple Itô integrals with respect to a cylindrical Wiener

process. In the Hilbert space valued case, the integrand of an n -fold iterated Itô integral is a function defined on a certain subset of $\mathbb{R}^n$ and taking values in a certain space of Hilbert space valued continuous n -linear forms defined on the n th Cartesian power of the Hilbert space where the Wiener process takes values.

Section 4 contains main results of the paper which are stated in Theorem 9 and Corollary 11. The proof of the theorem follows the scheme of the proof of the Wiener-Itô chaos expansion in the $\mathbb{R}$ -valued case in [5].

2. Itô Representation Theorem for Hilbert Space Valued Random Variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For any separable Hilbert space $\mathbb{H}$, we denote by $L^2(\Omega, \mathcal{F}, P; \mathbb{H})$, or $L^2(\Omega; \mathbb{H})$ for short, the space of all Bochner square integrable $\mathbb{H}$ -valued random variables on $(\Omega, \mathcal{F}, P)$. Let $\{e_j\}$ be an orthonormal basis in $\mathbb{H}$ and let $\{\beta_j(t) \mid t \in \mathbb{R}\}_{j=1}^\infty$ be a sequence of independent identically distributed Brownian motions on the probability space. Consider the corresponding cylinder Wiener process, defined by

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j, \quad t \in \mathbb{R}. \quad (1)$$

It is easy to see that the series is not convergent in $L^2(\Omega; \mathbb{H})$; however, for any $x \in \mathbb{H}$, we have that $(W(t), x) := \sum_{j=1}^{\infty} \beta_j(e_j, x)_{\mathbb{H}}$ is a random variable belonging to $L^2(\Omega; \mathbb{R})$. Denote by $\mathcal{B}_t^a$, $t \geq a$ the σ -algebra, generated by the Wiener process at $[a; t]$, that is, the σ -algebra, generated by the set of random variables $(W(s), x)$, where $a \leq s \leq t$, $x \in \mathbb{H}$. (Note that the series (1) is not convergent in $L^2(\Omega; \mathbb{H})$ at $t \neq 0$ although $(W(s), x) \in L^2(\Omega; \mathbb{R})$ for any $s \in \mathbb{R}$, $x \in \mathbb{H}$.) The family $\{\mathcal{B}_t^a\}$ is called the filtration generated by the Wiener process $W(t)$, $t \geq a$. Note that the Brownian motions $\beta_j(t)$, $j \in \mathbb{N}$, $t \geq a$ are martingales with respect to the filtration $\mathcal{B}_t^a$.

Let H be another separable Hilbert space and let $\{g_i\}_{i=1}^{\infty}$ be an orthonormal basis in H . Then, the family of operators $\{g_i \otimes e_j\}_{i,j=1}^{\infty}$, defined by the equality

$$(g_i \otimes e_j)x := g_i(e_j, x)_{\mathbb{H}}, \quad x \in \mathbb{H}, \quad (2)$$

forms an orthonormal basis in the space $\mathcal{L}_2(\mathbb{H}, H)$ of all Hilbert-Schmidt operators acting from $\mathbb{H}$ to H . Any $A \in \mathcal{L}_2(\mathbb{H}, H)$ has the decomposition

$$A = \sum_{i,j=1}^{\infty} a_{ij} (g_i \otimes e_j), \quad (3)$$

where $a_{ij} = (g_i, Ae_j)_H$ and

$$\|A\|_{\mathcal{L}_2(\mathbb{H}, H)}^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty. \quad (4)$$

For any $\mathcal{L}_2(\mathbb{H}; H)$ -valued random process $\varphi(t)$ adapted to the filtration $\mathcal{B}_t^a$, where $a \leq t \leq b$, satisfying the property

$$E \left[\int_a^b \|\varphi(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2 dt \right] < \infty, \quad (5)$$

the stochastic Itô integral with respect to the cylindrical Wiener process

$$\int_a^b \varphi(t, \omega) dW(t) \quad (6)$$

is well-defined and is an element of the space $L^2(\Omega; H)$ (see the definition and the properties in [6]). Note that if the function $\varphi(t) = \varphi(t, \omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$ -measurable, the equality (5) implies

$$E \left[\int_a^b \|\varphi(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2 dt \right] = \int_a^b E \left[\|\varphi(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2 \right] dt. \quad (7)$$

Theorem 1. For any $\mathcal{B}_b^a$ -measurable random variable $F \in L^2(\Omega; H)$, there exists a unique $\mathcal{B}_t^a$ -adapted $\mathcal{L}_2(\mathbb{H}, H)$ -valued random process $\varphi(t)$ satisfying (5) such that

$$F = E[F] + \int_a^b \varphi(t, \omega) dW(t), \quad (8)$$

$$\|F\|_{L^2(\Omega; H)}^2 = \|E[F]\|_H^2 + E \left[\int_a^b \|\varphi(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2 dt \right].$$

Proof. For any $i \in \mathbb{N}$, we have $F_i := (F, g_i)_H \in L^2(\Omega; \mathbb{R})$. Therefore, we have the following decomposition for F :

$$F = \sum_{i=1}^{\infty} F_i g_i. \quad (9)$$

Denote $F_{in} = E[F_i | \mathcal{B}_b^{a,(n)}]$, where $\mathcal{B}_b^{a,(n)}$ is the filtration generated by the n -dimensional Wiener process

$$W_n(t) = \sum_{j=1}^n \beta_j(t) e_j, \quad a \leq t \leq b. \quad (10)$$

By the finite dimensional Itô representation theorem ([10], Theorem 4.3.3.), there exists a unique $\mathbb{R}^n$ -valued random process

$$\varphi_{in}(t) = [\nu_{i1}^{(n)}(t), \dots, \nu_{in}^{(n)}(t)], \quad a \leq t \leq b, \quad (11)$$

such that the following conditions hold:

- (i) all the mappings $\nu_{ik}^{(n)} : (t, \omega) \mapsto \nu_{ik}^{(n)}(t, \omega)$ are $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$ -measurable,
- (ii) $E[\int_a^b \nu_{ik}^{(n)}(t)^2 dt] < \infty$, $k = 1, \dots, n$,
- (iii) the processes $\nu_{ik}^{(n)}(t)$, $k = 1, \dots, n$ are adapted to the filtration $\mathcal{B}_t^a$.

Moreover, we have

$$\begin{aligned} F_{in} &= E[F_{in}] + \int_a^b \varphi_{in}(t) dB_n(t) \\ &= E[F_i] + \sum_{k=1}^n \int_a^b \nu_{ik}^{(n)}(t) d\beta_k(t). \end{aligned} \quad (12)$$

Here, $B_n(t)$ denotes the n -dimensional Brownian motion

$$B_n(t) = \begin{bmatrix} \beta_1(t) \\ \vdots \\ \beta_n(t) \end{bmatrix}. \quad (13)$$

Since $\mathcal{B}_b^{a,(n)} \subseteq \mathcal{B}_b^{a,(m)}$ for $m > n$, we have

$$\begin{aligned} F_{jn} &= E[F_j | \mathcal{B}_b^{a,(n)}] = E[E[F_j | \mathcal{B}_b^{a,(m)}] | \mathcal{B}_b^{a,(n)}] \\ &= E[F_{jm} | \mathcal{B}_b^{a,(n)}] \\ &= E \left[E[F_{in}] + \sum_{k=1}^m \int_a^b \nu_{ik}^{(m)}(t) d\beta_k(t) \mid \mathcal{B}_b^{a,(n)} \right] \\ &= E[F_i] + \sum_{k=1}^n \int_a^b \nu_{ik}^{(m)}(t) d\beta_k(t). \end{aligned} \quad (14)$$

It follows from the uniqueness of the representation (12) that $\nu_{ik}^{(m)} \equiv \nu_{ik}^{(n)}$ for $k \leq n < m$. Thus, the equality (12) can be rewritten as

$$\begin{aligned} F_{in} &= E[F_{in}] + \int_a^b \varphi_{in}(t) dB_n(t) \\ &= E[F_i] + \sum_{k=1}^n \int_a^b \nu_{ik}(t) d\beta_k(t). \end{aligned} \quad (15)$$

By the Jensen inequality, we have

$$F_{in}^2 = E[F_i | \mathcal{B}_b^{a,(n)}]^2 \leq E[F_i^2 | \mathcal{B}_b^{a,(n)}]. \quad (16)$$

Therefore,

$$E[F_{in}^2] \leq E[E[F_i^2 | \mathcal{B}_b^{a,(n)}]] = E[F_i^2] < \infty. \quad (17)$$

Thus,

$$\begin{aligned} E[F_{in}^2] &= E\left[\left(E[F_i] + \sum_{k=1}^n \int_a^b \nu_{ik}(t) d\beta_k(t)\right)^2\right] \\ &= E[F_i^2] + \sum_{k=1}^n E\left[\int_a^b \nu_{ik}^2(t) dt\right] \leq E[F_i^2] < \infty. \end{aligned} \quad (18)$$

Consequently,

$$\sum_{k=1}^{\infty} E\left[\int_a^b \nu_{ik}^2(t) dt\right] < \infty. \quad (19)$$

Since $\mathcal{B}_b^{a,(n)} \subseteq \mathcal{B}_b^{a,(n+1)}$ and $\mathcal{B}_b^a = \bigcup_{n \in \mathbb{N}} \mathcal{B}_b^{a,(n)}$, we have

$$\begin{aligned} F_i &= \lim_{n \rightarrow \infty} E[F_i | \mathcal{B}_b^{a,(n)}] = E[F_i] + \sum_{k=1}^{\infty} \int_a^b \nu_{ik}(t) d\beta_k(t), \\ E[F_i^2] &= \lim_{n \rightarrow \infty} E[F_{in}^2] = E[F_i^2] + \sum_{k=1}^{\infty} E\left[\int_a^b \nu_{ik}^2(t) dt\right]. \end{aligned} \quad (20)$$

For any $N \in \mathbb{N}$, it holds that

$$E\left[\sum_{i=1}^N F_i^2\right] = \sum_{i=1}^N E[F_i^2] + \sum_{i=1}^N \sum_{k=1}^{\infty} E\left[\int_a^b \nu_{ik}^2(t) dt\right] \leq E[\|F\|_H^2]. \quad (21)$$

It follows from (20) that

$$\sum_{i=1}^{\infty} F_i g_i = \sum_{i=1}^{\infty} E[F_i] g_i + \sum_{i=1}^{\infty} g_i \sum_{k=1}^{\infty} \int_a^b \nu_{ik}(t) d\beta_k(t), \quad (22)$$

since the series in the right-hand side of the equality are convergent in $L^2(\Omega; H)$ by the estimate (21). Thus, the equality (8) holds true. It also follows from this estimate by the Levy theorem that we can pass to the limits at $N \rightarrow \infty$ in the equality. As a result, we obtain

$$\begin{aligned} E[\|F\|_H^2] &= \sum_{i=1}^{\infty} E[F_i^2] \\ &\quad + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} E\left[\int_a^b \nu_{ik}^2(t) dt\right] \\ &= \|E[F]\|_H^2 + E\left[\int_a^b \sum_{i,k=1}^{\infty} \nu_{ik}^2(t) dt\right] \\ &= \|E[F]\|_H^2 + E\left[\int_a^b \|\varphi(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2 dt\right], \end{aligned} \quad (23)$$

where

$$\varphi(t) := \sum_{i,k=1}^{\infty} \nu_{ik}(t) (g_i \otimes e_k). \quad (24)$$

By the polarization identity, we obtain the following assertion. $\square$

Corollary 2. Let $\xi(t)$ and $\eta(t)$ be $\mathcal{L}_2(\mathbb{H}, H)$ -valued random processes, adapted to the filtration $\mathcal{B}_t^a$, $a \leq t \leq b$, such that

$$\int_a^b E[\|\xi(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2] < \infty, \quad \int_a^b E[\|\eta(t)\|_{\mathcal{L}_2(\mathbb{H}, H)}^2] < \infty. \quad (25)$$

Then,

$$\begin{aligned} E\left[\left(\int_a^b \xi(t) dW(t), \int_a^b \eta(t) dW(t)\right)_H\right] \\ = \int_a^b E[(\xi(t), \eta(t))_{\mathcal{L}_2(\mathbb{H}, H)}] dt. \end{aligned} \quad (26)$$

3. Multiple Itô Integrals with respect to the Cylindrical Wiener Process

Let $\xi(t_1, t_2, \omega)$ be a function defined for $a \leq t_1 \leq t_2 \leq b$, $\omega \in \Omega$ taking values in the space $\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))$ and $\mathcal{B}(\mathbb{R}^2) \times \mathcal{F}$ -measurable. If for any $t_2 \in [a; b]$ the random process $\xi(t, t_2)$, $a \leq t \leq t_2$ is adapted to the filtration $\mathcal{B}_t^a$, $0 \leq t \leq t_2$ and the condition

$$E\left[\int_a^{t_2} \|\xi(t, t_2)\|_{\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))}^2 dt\right] < \infty \quad (27)$$

holds true, then the Itô integral

$$\int_a^{t_2} \xi(t_1, t_2) dW(t_1) \quad (28)$$

is well defined and is an element of the space $L^2(\Omega; \mathcal{L}_2(\mathbb{H}; H))$ for all $t_2 \in [a; b]$. As a function of t_2 , it is an $\mathcal{L}_2(\mathbb{H}; H)$ -valued random process adapted to the filtration $\mathcal{B}_t^a$. We also have

$$\begin{aligned} E\left[\left\|\int_a^{t_2} \xi(t_1, t_2) dW(t_1)\right\|_{\mathcal{L}_2(\mathbb{H}; H)}^2\right] \\ = E\left[\int_a^{t_2} \|\xi(t_1, t_2)\|_{\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))}^2 dt_1\right] < \infty. \end{aligned} \quad (29)$$

If moreover we have

$$E\left[\int_a^b \int_a^{t_2} \|\xi(t_1, t_2)\|_{\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))}^2 dt_1 dt_2\right] < \infty, \quad (30)$$

then the following iterated Itô integral is well defined:

$$\int_a^b \int_a^{t_2} \xi(t_1, t_2) dW(t_1) dW(t_2) \quad (31)$$

and it satisfies the equality

$$\begin{aligned} E \left[\left\| \int_a^b \int_a^{t_2} \xi(t_1, t_2) dW(t_1) dW(t_2) \right\|_H^2 \right] \\ = E \left[\int_a^b \int_a^{t_2} \|\xi(t_1, t_2)\|_{\mathcal{L}_2(\mathbb{H}; H)}^2 dt_1 dt_2 \right]. \end{aligned} \quad (32)$$

Note that the operators $g_i \otimes e_j \otimes e_k$ defined by the equality

$$(g_i \otimes e_j \otimes e_k)x := (g_i \otimes e_j)(e_k, x)_{\mathbb{H}}, \quad i, j, k \in \mathbb{N} \quad (33)$$

form an orthonormal basis in the space $\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))$ and any operator $A \in \mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))$ has the following decomposition:

$$\begin{aligned} A &= \sum_{i,j,k} a_{ijk} (g_i \otimes e_j \otimes e_k), \\ \|A\|_{\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))}^2 &= \sum_{i,j,k} a_{ijk}^2 < \infty. \end{aligned} \quad (34)$$

We can identify the decomposition (34) with the H -valued bilinear form on $\mathbb{H} \times \mathbb{H}$ defined by the equality $(g_i \otimes e_j \otimes e_k)(x, y) := g_i(e_j, x)_{\mathbb{H}}(e_k, y)_{\mathbb{H}}$. Thus, the space $\mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H))$ can be identified with the Hilbert space $\mathcal{L}_2^2(\mathbb{H} \times \mathbb{H}; H)$ of all continuous bilinear forms A of the form (34) with the norm generated by the scalar product

$$(A, B)_{\mathcal{L}_2^2(\mathbb{H} \times \mathbb{H}; H)} = \sum_{i,j,k} a_{ijk} b_{ijk}, \quad (35)$$

where $A = \sum_{i,j,k} a_{ijk} (g_i \otimes e_j \otimes e_k)$, $B = \sum_{i,j,k} b_{ijk} (g_i \otimes e_j \otimes e_k)$. So, the iterated integral (31) with respect to the cylindrical Wiener process $W(t)$ is well defined for any $\mathcal{L}_2^2(\mathbb{H} \times \mathbb{H}; H)$ -valued $\mathcal{B}(\mathbb{R}^2) \times \mathcal{F}$ -measurable function $\xi(t_1, t_2, \omega)$, defined on $S_2^{a,b} \times \Omega$, where

$$S_2^{a,b} = \{(t_1, t_2) \mid a \leq t_1 \leq t_2 \leq b\}, \quad (36)$$

if it satisfies the condition (30).

One can easily extend the above definition to the case of arbitrary $n > 2$, defining the n times iterated integral

$$\begin{aligned} J_n(\xi) \\ = \int_a^b \int_a^{t_2} \dots \int_a^{t_{n-1}} \xi(t_1, t_2, \dots, t_n, \omega) dW(t_1), \dots, dW(t_n) \end{aligned} \quad (37)$$

inductively for any function $\xi : S_n \times \Omega \rightarrow \mathcal{L}_2^n(\mathbb{H}^{\times n}; H)$, where

$$S_n^{a,b} = \{(t_1, \dots, t_n) \mid a \leq t_1 \leq \dots, t_n \leq b\}, \quad (38)$$

$\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)$ is the space of all continuous H -valued n -linear forms on $\mathbb{H} \times \dots \times \mathbb{H}$ having form

$$A = \sum_{i,k_1, \dots, k_n} a_{i,k_1, \dots, k_n} (g_i \otimes e_{k_1} \otimes \dots \otimes e_{k_n}) \quad (39)$$

such that

$$\|A\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 = \sum_{i,k_1, \dots, k_n} a_{i,k_1, \dots, k_n}^2 < \infty, \quad (40)$$

if it satisfies the following conditions:

- (n-i) ξ is $\mathcal{B}(\mathbb{R}^n) \times \mathcal{F}$ -measurable;
- (n-ii) $\xi(t_1, t_2, \dots, t_n)$ is $\mathcal{B}_{t_1}^a$ -measurable for any $(t_1, t_2, \dots, t_n) \in S_n^{a,b}$;
- (n-iii) $E[\int_a^b \int_a^{t_2} \dots \int_a^{t_{n-1}} \|\xi(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_1, \dots, dt_n] < \infty$.

For the defined iterated integral, we have

$$\begin{aligned} E[\|J_n(\xi)\|_H^2] \\ = \int_a^b \int_a^{t_2} \dots \int_a^{t_{n-1}} E[\|\xi(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2] dt_1, \dots, dt_n. \end{aligned} \quad (41)$$

We state without proof the next two lemmas which are straightforward generalizations of the corresponding properties of the $\mathbb{R}$ -valued iterated Itô integrals.

Lemma 3. Let $\xi : S_{n_1}^{a,b} \times \Omega \rightarrow \mathcal{L}_2^{n_1}(\mathbb{H}^{\times n_1}; H)$ and $\eta : S_{n_2}^{a,b} \times \Omega \rightarrow \mathcal{L}_2^{n_2}(\mathbb{H}^{\times n_2}; H)$ satisfy the conditions (n-i), (n-ii) and (n-iii) with $n = n_1$ and $n = n_2$ correspondingly. Then,

$$E[(J_{n_1}(\xi), J_{n_2}(\eta))_H] = \begin{cases} 0, & n_1 \neq n_2, \\ (\xi, \eta)_{L^2(S_n^{a,b}; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}, & n_1 = n_2 = n. \end{cases} \quad (42)$$

Lemma 4. Let $\xi : S_n^{a,b} \times \Omega \rightarrow \mathcal{L}_2^n(\mathbb{H}^{\times n}; H)$ satisfy the conditions (n-i), (n-ii) and (n-iii). Then,

$$\begin{aligned} E[J_n(\xi) | \mathcal{B}_t^{a,b}] \\ = \int_a^t \int_a^{t_2} \dots \int_a^{t_{n-1}} \xi(t_1, t_2, \dots, t_n, \omega) dW(t_1), \dots, dW(t_n). \end{aligned} \quad (43)$$

Denote by $\widehat{L}^2([a; b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$ the space of all symmetric functions $f \in [a; b]^n \rightarrow \mathcal{L}_2^n(\mathbb{H}^{\times n}; H)$ satisfying the condition

$$\begin{aligned} \|f\|_{L^2([a; b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 \\ = \int \dots \int_{[a; b]^n} \|f_n(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_1, \dots, dt_n < \infty. \end{aligned} \quad (44)$$

Definition 5. For any $f \in \widehat{L}^2([a; b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$, define the multiple n -fold Itô integral by the equality

$$I_n(f) := n! J_n(f), \quad (45)$$

if the right-hand side iterated integral exists.

4. The Decomposition Theorem

To establish the main result, we need a few lemmas.

Lemma 6. *The set of random variables*

$$\left\{ \varphi \left(\beta_{j_1}(t_{i_1}), \dots, \beta_{j_n}(t_{i_n}) \right) \mid j_k \in \mathbb{N}, t_k \in [a; b], \right. \\ \left. \varphi \in C_0^\infty(\mathbb{R}^n), n \in \mathbb{N} \right\} \quad (46)$$

is dense in $L^2(\mathcal{B}_b^a, P)$.

Proof. Let $\{t_i\}_{i=1}^\infty$ be a dense subset in $[a; b]$. Let $\{\zeta_k\}_{k=1}^\infty$ be a fixed ordering of the countable set of random variables $\{\beta_j(t_i)\}_{j,i=1}^\infty$. Denote by $\mathcal{H}_p$ be the σ -algebra generated by $\{\zeta_k\}_{k=1}^p$. We have $\mathcal{H}_p \subset \mathcal{H}_{p+1}$ for all $p \in \mathbb{N}$ and $\mathcal{B}_b^a$ is the smallest σ -algebra containing all $\mathcal{H}_p$.

For any $g \in L^2(\mathcal{B}_b^a, P)$, we have

$$g = E(g \mid \mathcal{B}_b^a) = \lim_{p \rightarrow \infty} E(g \mid \mathcal{H}_p), \quad (47)$$

where the limit is pointwise a.e. with respect to P and in $L^2(\mathcal{B}_b^a, P)$. By the Doob-Dynkin lemma for any $p \in \mathbb{N}$, there exists a Borel function $g_p: \mathbb{R}^p \rightarrow \mathbb{R}$ such that

$$E[g \mid \mathcal{H}_p] = g_p(\zeta_1, \dots, \zeta_p). \quad (48)$$

Let $P_{\zeta_1, \dots, \zeta_p}$ be the probability measure on $\mathbb{R}^p$ generated by $\zeta_1, \dots, \zeta_p$. Since g_p can be approximated in $L^2(\mathbb{R}^p, P_{\zeta_1, \dots, \zeta_p})$ by functions $\varphi \in C_0^\infty(\mathbb{R}^p)$, the assertion follows. $\square$

Consider the following exponential functionals of Brownian motions β_j :

$$\mathcal{E}_\theta^{j,a,b} := \exp \left[\int_a^b \theta(t) d\beta_j(t) - \frac{|\theta|_0^2}{2} \right], \quad (49)$$

$$j \in \mathbb{N}, \theta \in L^2[a; b].$$

Lemma 7. *The linear span of the set*

$$\left\{ \prod_{k=1}^p \mathcal{E}_{\theta_k}^{j_k, a, b} \mid j_k \in \mathbb{N}, \theta_k \in L^2[a; b], k = 1, \dots, p, p \in \mathbb{N} \right\} \quad (50)$$

is dense in $L^2(\mathcal{B}_b^a, P)$.

Proof. Let $g \in L^2(\mathcal{B}_b^a, P)$ be orthogonal to all functions of the family (50). Take $\theta_k(t) = \lambda_k 1_{[a; t_k]}$, where $\{t_k\}_{k=1}^p \subset [a; b]$, $\bar{\lambda} = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$. Then, we have

$$\prod_{k=1}^p \mathcal{E}_{\theta_k}^{j_k, a, b} \\ = \prod_{k=1}^p \exp \left[\lambda_k \int_a^{t_k} d\beta_{j_k}(t) - \frac{t_k - a}{2} \right] \quad (51) \\ = \exp \left[-\frac{1}{2} \sum_{k=1}^p (t_k - a) \right] \cdot \exp \left[\sum_{k=1}^p \lambda_k \beta_{j_k}(t_k) \right].$$

It follows that

$$G(\bar{\lambda}) := \int_{\Omega} \exp \left[\sum_{k=1}^p \lambda_k \beta_{j_k}(t_k) \right] g(\omega) dP(\omega) = 0 \quad (52)$$

for all $p \in \mathbb{N}$, $\bar{\lambda} \in \mathbb{R}^p$, $j_1, \dots, j_p \in \mathbb{N}$, $t_1, \dots, t_p \in [a; b]$.

Fix $p, j_1, \dots, j_p, t_1, \dots, t_p$. The corresponding function $G(\bar{\lambda})$ is real analytic. Consider its analytic extension onto $\mathbb{C}^p$. It is also equal to zero. Taking $\bar{\lambda} = (iy_1, \dots, iy_p)$, we obtain

$$\int_{\Omega} \exp \left[i \sum_{k=1}^p y_k \beta_{j_k}(t_k) \right] g(\omega) dP(\omega) = 0. \quad (53)$$

For any $\varphi \in C_0^\infty(\mathbb{R}^p)$ denoting by $\hat{\varphi}$ its Fourier transform, we obtain

$$\int_{\Omega} \varphi(\beta_{j_1}(t_1), \dots, \beta_{j_p}(t_p)) g(\omega) dP(\omega) \\ = \int_{\Omega} \frac{1}{(2\pi)^{p/2}} \\ \times \int_{\mathbb{R}^p} \hat{\varphi}(\bar{y}) \exp \left[i \sum_{k=1}^p y_k \beta_{j_k}(t_k) \right] d\bar{y} g(\omega) dP(\omega) \\ = \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \hat{\varphi}(\bar{y}) \\ \times \int_{\Omega} \exp \left[i \sum_{k=1}^p y_k \beta_{j_k}(t_k) \right] g(\omega) dP(\omega) d\bar{y} = 0. \quad (54)$$

Thus, g is orthogonal to a dense subset in $L^2(\mathcal{B}_b^a, P)$. $\square$

Lemma 8. *For any $\alpha_1, \dots, \alpha_n \in \mathbb{N} \cup \{0\}$, $\theta_1, \dots, \theta_n \in \mathcal{S}$, the product*

$$\prod_{j=1}^n h_{\alpha_j} \left(\frac{\int_a^b \theta_j(t) d\beta_j(t)}{|\theta_j \cdot 1_{[a; b]}|_0} \right) \quad (55)$$

is a linear combination of the iterated Itô integrals of the form

$$\int_a^b \int_a^{t_p} \dots \int_a^{t_2} \theta_{j_1}(t_1), \dots, \theta_{j_p}(t_p) d\beta_{j_1}(t_1), \dots, d\beta_{j_p}(t_p), \quad (56)$$

where $p = \alpha_1 + \dots + \alpha_n$ and $j_1, \dots, j_p \in \{1, \dots, n\}$.

Proof. We use induction with respect to p . The assertion is evident for $p = 1$. Suppose it is true for some $p \in \mathbb{N}$. Let $\alpha_1 + \dots + \alpha_n = p + 1$. We introduce the following notation:

$$\begin{aligned} X_j(t) &:= \frac{|\theta_j \cdot 1_{[a;t]}|_0^{\alpha_j}}{\alpha_j!} h_{\alpha_j} \left(\frac{\int_a^t \theta_j(\tau) d\beta_j(\tau)}{|\theta_j \cdot 1_{[a;t]}|_0} \right) \\ &= \int_a^t \int_a^{t_{\alpha_j}} \dots \\ &\quad \int_a^{t_2} \theta_j(t_1), \dots, \theta_j(t_{\alpha_j}) d\beta_j(t_1), \dots, d\beta_j(t_{\alpha_j}) \\ &=: \int_a^t Y_j(t_{\alpha_j}, \omega) d\beta_j(t_{\alpha_j}), \quad j = 1, \dots, n. \end{aligned} \quad (57)$$

This means that X_j satisfy the following stochastic differential equations:

$$dX_j(t) = Y_j(t, \omega) d\beta_j(t_{\alpha_j}), \quad j = 1, \dots, n, \quad (58)$$

and we have

$$Y_j(t) = \frac{|\theta_j \cdot 1_{[a;t]}|_0^{\alpha_j-1}}{(\alpha_j - 1)!} h_{\alpha_j-1} \left(\frac{\int_a^t \theta_j(\tau) d\beta_j(\tau)}{|\theta_j \cdot 1_{[a;t]}|_0} \right). \quad (59)$$

By the Itô formula, we obtain

$$\begin{aligned} d \left(\prod_{j=1}^n X_j(t) \right) &= \sum_{j=1}^n \left[\prod_{i \neq j} X_i(t) \right] dX_j(t) \\ &\quad + \sum_{i \neq j} dX_i(t) \cdot dX_j(t). \end{aligned} \quad (60)$$

Since $dX_i(t) \cdot dX_j(t) = Y_i(t, \omega) Y_j(t, \omega) d\beta_i(t) d\beta_j(t) = 0$, we come to the equality

$$\begin{aligned} \prod_{j=1}^n X_j(b) &= \sum_{j=1}^n \int_a^b \left[\prod_{i \neq j} X_i(t) \right] Y_j(t, \omega) d\beta_j(t) \\ &= \sum_{j=1}^n \int_a^b \left[\prod_{i \neq j} \frac{|\theta_i \cdot 1_{[a;t]}|_0^{\alpha_i}}{\alpha_i!} h_{\alpha_i} \left(\frac{\int_a^t \theta_i(\tau) d\beta_i(\tau)}{|\theta_i \cdot 1_{[a;t]}|_0} \right) \right. \\ &\quad \cdot \frac{|\theta_j \cdot 1_{[a;t]}|_0^{\alpha_j-1}}{(\alpha_j - 1)!} h_{\alpha_j-1} \left(\frac{\int_a^t \theta_j(\tau) d\beta_j(\tau)}{|\theta_j \cdot 1_{[a;t]}|_0} \right) \Big] \\ &\quad \times \theta_j(t) d\beta_j(t). \end{aligned} \quad (61)$$

Since $\sum_{i \neq j} \alpha_i + \alpha_j - 1 = p$, we can apply the assumption to the products in the brackets, thus completing the proof. $\square$

Theorem 9. Any $\mathcal{B}_t^a$ -measurable random variable $F \in L^2(\Omega; H)$ has the unique decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad (62)$$

where $f_n \in \hat{L}^2([a; b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$, $n \in \mathbb{N}$, $I_0(f_0) := f_0 \in H$. It holds that

$$\|F\|_{L^2(\Omega; H)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2([a; b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2, \quad (63)$$

where $\|f_0\|_{L^2([a; b]^0; \mathcal{L}_2^0(\mathbb{H}^{\times 0}; H))} := \|f_0\|_H^2$.

Proof. By Theorem 1, we have

$$F = E[F] + \int_a^b \varphi_1(t_1, \omega) dW(t_1), \quad (64)$$

where $\varphi_1(t)$ is a $\mathcal{L}_2^1(\mathbb{H}; H)$ -valued random variable, $\mathcal{B}_t^a$ -measurable for any t . For its norm, we have

$$\|F\|_{L^2(\Omega; H)}^2 = \|E[F]\|_H^2 + E \left[\int_a^b \|\varphi_1(t)\|_{\mathcal{L}_2^1(\mathbb{H}; H)}^2 dt \right]. \quad (65)$$

Let $\psi_0 := E[F]$. For any $t_1 \in [a, b]$ by Theorem 1, taking t_1 for b and $\mathcal{L}_2(\mathbb{H}; H)$ for H , we obtain the following representation:

$$\varphi_1(t_1) = E[\varphi_1(t_1)] + \int_a^{t_1} \varphi_2(t_1, t_2, \omega) dW(t_2), \quad (66)$$

where $\varphi_2 : S_2^{a,b} \times \Omega \rightarrow \mathcal{L}_2(\mathbb{H}; \mathcal{L}_2(\mathbb{H}; H)) \cong \mathcal{L}_2^2(\mathbb{H}^{\times 2}; H)$, with the following equality for the norm:

$$\begin{aligned} \|\varphi_1(t_1)\|_{\mathcal{L}_2(\Omega; \mathcal{L}_2(\mathbb{H}; H))}^2 &= \|E[\varphi_1(t_1)]\|_{\mathcal{L}_2(\mathbb{H}; H)}^2 \\ &\quad + E \left[\int_a^{t_1} \|\varphi_2(t_1, t_2)\|_{\mathcal{L}_2^2(\mathbb{H}^{\times 2}; H)}^2 dt_2 \right]. \end{aligned} \quad (67)$$

Set $\psi_1(t_1) := E[\varphi_1(t_1)]$. Substituting the representation (66) into (64), we come to the equality

$$\begin{aligned} F &= J_0(\psi_0) + \int_a^b \psi_1(t_1) dW(t_1) \\ &\quad + \int_a^b \int_a^{t_1} \varphi_2(t_1, t_2, \omega) dW(t_2) dW(t_1). \end{aligned} \quad (68)$$

From (65) and (67), it follows that

$$\begin{aligned} \|F\|_{L^2(\Omega; H)}^2 &= \|\psi_0\|_H^2 + \int_a^b \|\psi_1(t_1)\|_{\mathcal{L}_2^1(\mathbb{H}; H)}^2 dt_1 \\ &\quad + E \left[\int_a^b \int_a^{t_1} \|\varphi_2(t_1, t_2)\|_{\mathcal{L}_2^2(\mathbb{H}^{\times 2}; H)}^2 dt_2 dt_1 \right] \\ &= \|\psi_0\|_H^2 + \int_a^b \|\psi_1(t_1)\|_{\mathcal{L}_2^1(\mathbb{H}; H)}^2 dt_1 \\ &\quad + \int_a^b \int_a^{t_1} E \left[\|\varphi_2(t_1, t_2)\|_{\mathcal{L}_2^2(\mathbb{H}^{\times 2}; H)}^2 \right] dt_2 dt_1. \end{aligned} \quad (69)$$

Applying further in similar manner Theorem 1 and setting

$$\psi_n(t_1, \dots, t_n) := E[\varphi_n(t_1, \dots, t_n)] \in \mathcal{L}_2^n(\mathbb{H}^{\times n}; H), \quad (70)$$

$$(t_1, \dots, t_n) \in S_n^{a,b}$$

at the n th step of this process, we obtain

$$F = J_0(\psi_0) + J_1(\psi_1) + \dots + J_n(\psi_n) + J_{n+1}(\varphi_{n+1}) \quad (71)$$

with

$$\begin{aligned} & \|F\|_{L^2(\Omega; H)}^2 \\ &= \|\psi_0\|_H^2 + \int_a^b \|\psi_1(t_1)\|_{\mathcal{L}_2^1(\mathbb{H}; H)}^2 dt_1 \\ &+ \int_a^b \int_a^{t_1} \|\psi_2(t_1, t_2)\|_{\mathcal{L}_2^2(\mathbb{H}^{\times 2}; H)}^2 dt_2 dt_1 \\ &+ \dots + \int_a^b \int_a^{t_1} \dots \\ &\quad \int_a^{t_{n-1}} \|\psi_n(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_n, \dots, dt_1 \\ &+ \int_a^b \int_a^{t_1} \dots \\ &\quad \int_a^{t_n} E\left[\|\varphi_{n+1}(t_1, \dots, t_{n+1})\|_{\mathcal{L}_2^{n+1}(\mathbb{H}^{\times(n+1)}; H)}^2\right] dt_{n+1}, \dots, dt_1. \end{aligned} \quad (72)$$

It follows from here that

$$\begin{aligned} & \|\psi_0\|_H^2 \\ &+ \sum_{k=1}^n \int_a^b \int_a^{t_1} \dots \\ &\quad \int_a^{t_{k-1}} \|\psi_k(t_1, \dots, t_k)\|_{\mathcal{L}_2^k(\mathbb{H}^{\times k}; H)}^2 dt_k, \dots, dt_1 \\ &\leq \|F\|_{L^2(\Omega; H)}^2 \end{aligned} \quad (73)$$

and consequently

$$\begin{aligned} & \|\psi_0\|_H^2 \\ &+ \sum_{k=1}^{\infty} \int_a^b \int_a^{t_1} \dots \\ &\quad \int_a^{t_{k-1}} \|\psi_k(t_1, \dots, t_k)\|_{\mathcal{L}_2^k(\mathbb{H}^{\times k}; H)}^2 dt_k, \dots, dt_1 \\ &< \infty. \end{aligned} \quad (74)$$

This means that the series

$$\sum_{n=0}^{\infty} J_n(\psi_n) \quad (75)$$

is convergent in the space $L^2(\Omega; H)$ and

$$\Phi := \lim_{n \rightarrow \infty} J_{n+1}(\varphi_{n+1}) \quad (76)$$

is an element of this space. By Lemma 3, the integral $J_{n+1}(\varphi_{n+1})$ is orthogonal in $L^2(\Omega; H)$ to the integrals $J_0(\psi_0), \dots, J_n(\psi_n)$ for all $\psi_0 \in H, \psi_1 \in L^2(S_1^{a,b}; \mathcal{L}_2^1(\mathbb{H}; H)), \dots, \psi_n \in L^2(S_n^{a,b}; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$. It follows that Φ is orthogonal to $J_n(\psi_n)$ for all $n \in \mathbb{N} \cup \{0\}, \psi_n \in L^2(S_n^{a,b}; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$.

Note that by the well-known connection between iterated Itô integrals with respect to a Brownian motion and the Hermite polynomials, we have

$$\begin{aligned} & J_n(g_i \otimes e_j^{\otimes n} \theta^{\otimes n}) \\ &= g_i \int_a^b \int_a^{t_1} \dots \int_a^{t_{n-1}} \theta^{\otimes n}(t_1, \dots, t_n) d\beta_j(t_n) \dots d\beta_j(t_1) \\ &= g_i |\theta \cdot 1_{[a,b]}|_0^n h_n \left(\frac{\int_a^b \theta(t) d\beta_j(t)}{|\theta \cdot 1_{[a,b]}|_0} \right) \end{aligned} \quad (77)$$

for any $n, i, j \in \mathbb{N}, \theta \in \mathcal{S}$.

Let $\Phi = \sum_{i=1}^{\infty} \Phi_i g_i$, where $\Phi_i \in L^2(\Omega; \mathbb{R})$. For any $n, i, j_1, \dots, j_n \in \mathbb{N}$ and any $\theta_{j_1}, \dots, \theta_{j_n} \in \mathcal{S}$, we have

$$\begin{aligned} & E \left[\Phi_i \cdot \int_a^b \int_a^{t_n} \dots \right. \\ &\quad \left. \int_a^{t_2} \theta_{j_1}(t_1), \dots, \theta_{j_n}(t_n) d\beta_{j_1}(t_1), \dots, d\beta_{j_n}(t_n) \right] \\ &= E \left[\left(\Phi, g_i \int_a^b \int_a^{t_n} \dots \right. \right. \\ &\quad \left. \left. \int_a^{t_2} \theta_{j_1}(t_1), \dots, \theta_{j_n}(t_n) d\beta_{j_1}(t_1), \dots, d\beta_{j_n}(t_n) \right) \right]_H \\ &= (\Phi, J_n(g_i \otimes e_{j_1} \otimes \dots \otimes e_{j_n} \otimes \theta_{j_1} \otimes \dots \otimes \theta_{j_n})) = 0, \end{aligned} \quad (78)$$

from where, by Lemma 8 and the equality (77), it follows that, for any $i \in \mathbb{N}, \Phi_i$ is orthogonal in $L^2(\Omega; \mathbb{R})$ to the products of the form

$$\prod_{j=1}^n h_{\alpha_j} \left(\frac{\int_a^b \theta_j(t) d\beta_j(t)}{|\theta_j \cdot 1_{[a,b]}|_0} \right). \quad (79)$$

It follows from here and the connection between the Hermite polynomials and general powers that

$$E \left[\Phi_i \prod_{j=1}^n \left(\int_a^b \theta_j(t) d\beta_j(t) \right)^{\alpha_j} \right] = 0. \quad (80)$$

It follows from the expansion

$$\begin{aligned} \mathcal{E}_\theta^{j,a,b} &= \exp \left[\int_a^b \theta(t) d\beta_j(t) - \frac{|\theta|_0^2}{2} \right] \\ &= e^{-(|\theta|_0^2/2)} \sum_{k=0}^{\infty} \frac{\left(\int_a^b \theta(t) d\beta_j(t) \right)^k}{k!} \end{aligned} \quad (81)$$

that a product of the form

$$\prod_{k=1}^n \mathcal{E}_{\theta_k}^{j_k, a, b} \quad (82)$$

has a decomposition into the series of the products of the form

$$\prod_{j=1}^n \left(\int_a^b \theta_j(t) d\beta_j(t) \right)^{\alpha_j}, \quad \alpha_j \in \mathbb{N} \cup \{0\}, \quad j = 1, \dots, n, \quad (83)$$

convergent in the space $L^2(\Omega; \mathbb{R})$. Consequently, Φ_i is orthogonal to any product of the form (82). By Lemma 7, this means that $\Phi_i = 0$ for any $i \in \mathbb{N}$; that is, $\Phi = 0$. Passing to the limit in the equality (71), we obtain

$$F = \sum_{n=0}^{\infty} J_n(\psi_n). \quad (84)$$

Setting the functions ψ_n to be equal to zero on $[a; b]^n \setminus S_n^{a,b}$ and defining $f_n := \hat{\psi}_n$, where $\hat{\psi}_n$ is the symmetrization of ψ_n , we obtain (62). By the equality

$$\begin{aligned} \|f_n\|_{L^2([a;b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 &= \int \dots \int_{[a;b]^n} \|f_n(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_1, \dots, dt_n \\ &= n! \int \dots \int_{S_n^{a,b}} \|f_n(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_1, \dots, dt_n \\ &= n! \int \dots \int_{S_n^{a,b}} \left\| \frac{\psi_n(t_1, \dots, t_n)}{n!} \right\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_1, \dots, dt_n \\ &= \frac{1}{n!} \int \dots \int_{S_n^{a,b}} \|\psi_n(t_1, \dots, t_n)\|_{\mathcal{L}_2^n(\mathbb{H}^{\times n}; H)}^2 dt_1, \dots, dt_n, \end{aligned} \quad (85)$$

passing to the limit in (72), we obtain (63). $\square$

Definition 10. For any $f \in \hat{L}^2(\mathbb{R}^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$, define the n -fold Itô integral $I_n(f)$ by the equality (45), where the iterated integral

$$\begin{aligned} J_n(f) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{t_2} \dots \\ &\quad \int_{-\infty}^{t_{n-1}} \xi(t_1, t_2, \dots, t_n, \omega) dW(t_1), \dots, dW(t_n) \end{aligned} \quad (86)$$

is understood as the limit of the integrals $J_n(f \cdot 1_{[-A; A]^n})$ at $A \rightarrow +\infty$ in the space $L^2(\Omega; H)$ if the limit exists.

Corollary 11. Any $F \in L^2(\Omega; H)$ can be uniquely decomposed into the series

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad (87)$$

where $f_n \in \hat{L}^2(\mathbb{R}^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$, $n \in \mathbb{N}$, $I_0(f_0) := f_0 \in H$. The equality

$$\|F\|_{L^2(\Omega; H)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\mathbb{R}^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 \quad (88)$$

holds true, where $\|f_0\|_{L^2(\mathbb{R}^0; \mathcal{L}_2^0(\mathbb{H}^{\times 0}; H))} := \|f_0\|_H^2$.

Proof. Let $F_N = E[F | \mathcal{B}_N^{-N}]$, $N \in \mathbb{N}$. For any $N \in \mathbb{N}$, it can be uniquely represented in the form

$$F_N = \sum_{n=0}^{\infty} I_n(f_{N,n}) = \sum_{n=0}^{\infty} J_n(\psi_{N,n}), \quad (89)$$

where $\psi_{N,n} \in L^2(S_n^{-N,N}; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$ and $f_{N,n}$ is the symmetrization of the function $\psi_{N,n}$ set to be equal to zero at the set $[-N; N]^n \setminus S_n^{-N,N}$. For its norm, we have

$$\begin{aligned} \|F_N\|_{L^2(\Omega; H)}^2 &= \sum_{n=0}^{\infty} n! \|f_{N,n}\|_{L^2([a;b]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 \\ &= \sum_{n=0}^{\infty} \|\psi_{N,n}\|_{L^2(S_n^{-N,N}; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2. \end{aligned} \quad (90)$$

Suppose that the functions $f_{N,n}$ and $\psi_{N,n}$ are defined everywhere in $\mathbb{R}^n$ and are equal to zero on $\mathbb{R}^n \setminus [-N, N]^n$ and $\mathbb{R}^n \setminus S_n^{-N,N'}$, correspondingly.

It follows from the equality $E[F_{N''} | \mathcal{B}_{N'}^{-N'}] = F_{N'}$ where $N'' > N'$, from the uniqueness of the representation (89), and from the assertion of Lemma 4 that the equality $\psi_{N',n} = \psi_{N'',n} \cdot 1_{S_n^{-N',N'}}$ holds true and therefore $f_{N',n} = f_{N'',n} \cdot 1_{[-N', N']^n}$ for $N'' > N'$ and any $n \in \mathbb{N}$. It follows from here that for any $n \in \mathbb{N}$ there exists a unique function ψ_n such that for any $N \in \mathbb{N}$ the equality $\psi_n = \psi_{N,n} \cdot 1_{S_n^{-N,N'}}$ holds true and there exists a unique function f_n such that for any $N \in \mathbb{N}$ it holds $f_n = f_{N,n} \cdot 1_{[-N', N']^n}$. By the estimates

$$\begin{aligned} \|\psi_{N,n}\|_{L^2(\mathbb{R}^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 &= \|\psi_{N,n}\|_{L^2(S_n^{-N,N}; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 \leq \|F_N\|_{L^2(\Omega; H)}^2 \\ \|f_{N,n}\|_{L^2(\mathbb{R}^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 &= \|f_{N,n}\|_{L^2([-N, N]^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))}^2 \leq \|F_N\|_{L^2(\Omega; H)}^2, \end{aligned} \quad (91)$$

it follows that

$$\psi_n := \lim_{N \rightarrow \infty} \psi_{N,n}, \quad f_n := \lim_{N \rightarrow \infty} f_{N,n}, \quad (92)$$

in the space $L^2(\mathbb{R}^n; \mathcal{L}_2^n(\mathbb{H}^{\times n}; H))$. Consequently,

$$J_n(\psi_n) := \lim_{N \rightarrow \infty} J_n(\psi_{N,n}), \quad I_n(f_n) = \lim_{N \rightarrow \infty} I_n(f_{N,n}) \quad (93)$$

exist in the space $L^2(\Omega; H)$. Passing to the limits in the equalities (89) and (90) we obtain (87) and (88). $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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