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Research Article

On the Spectrum and Spectral Norms of r-Circulant Matrices with Generalized k-Horadam Numbers Entries

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This work is concerned with the spectrum and spectral norms of r-circulant matrices with generalized k-Horadam numbers entries. By using Abel transformation and some identities we obtain an explicit formula for the eigenvalues of them. In addition, a sufficient condition for an r-circulant matrix to be normal is presented. Based on the results we obtain the precise value for spectral norms of normal r-circulant matrix with generalized k-Horadam numbers, which generalize and improve the known results.

1. Introduction

There is no doubt that the r-circulant matrices have been one of the most interesting research areas in computation mathematics. It is well known that these matrices have a wide range of applications in signal processing, digital image disposal, coding theory, linear forecast, and design of self-regress.

There are many works concerning estimates for spectral norms of r-circulant matrices with special entries. For example, Solak [1] established lower and upper bounds for the spectral norms of circulant matrices with Fibonacci and Lucas numbers entries. subsequently, Ipek [2] investigated some improved estimations for spectral norms of these matrices. Bani-Domi and Kittaneh [3] established two general norm equalities for circulant and skew circulant operator matrices. Shen and Cen [4] gave the bounds of the spectral norms of r-circulant matrices whose entries are Fibonacci and Lucas numbers. In [5] they defined r-circulant matrices involving k-Lucas and k-Fibonacci numbers and also investigated the upper and lower bounds for the spectral norms of these matrices.

Recently, Yazlik and Taskara [6] define a generalization $\{H_{k,n}\}$ of the special second-order sequences such as Fibonacci, Lucas, k-Fibonacci, k-Lucas, generalized k-Fibonacci and k-Lucas, Horadam, Pell, Jacobsthal, and Jacobsthal-Lucas sequences. For any integer number $k \geqslant 1$, the generalized k-Horadam sequence $\{H_{k,n}\}$ is defined by the following recursive relation:

$$\begin{gathered} H_{k,n+2} = f(k) H_{k,n+1} + g(k) H_{k,n}, \\ H_{k,0} = a, \quad H_{k,1} = b, \end{gathered} \tag{1}$$

where $f(k)$ and $g(k)$ are scaler-value polynomials, $f^2(k) + 4g(k) > 0$. The following are some particular cases.

(i) If $f(k) = k$, $g(k) = 1$ and $a = 0, b = 1$, the k-Fibonacci sequence is obtained:

$$F_{k,n+2} = kF_{k,n+1} + F_{k,n}, \qquad F_{k,0} = 0, \qquad F_{k,1} = 1. \tag{2}$$

(ii) If $f(k) = k$, $g(k) = 1$ and $a = 2$, $b = k$, the k-Lucas sequence is obtained:

$$L_{k,n+2} = kL_{k,n+1} + L_{k,n}, \qquad F_{k,0} = 0, \qquad F_{k,1} = k. \tag{3}$$

(iii) If $f(k) = 1$, $g(k) = 1$ and $a = 0, b = 1$, the Fibonacci sequence is obtained:

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1. \tag{4}$$

(iv) If $f(k) = 1$, $g(k) = 1$ and $a = 2$, $b = 1$, the Lucas sequence is obtained:

$$L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 2, \qquad L_1 = 1. \tag{5}$$

(v) If $f(k) = 1$, $g(k) = 2$ and $a = 0$, $b = 1$, the Jacobsthal sequence is obtained:

$$J_{n+2} = J_{n+1} + 2J_n, \qquad J_0 = 0, \qquad J_1 = 1. \tag{6}$$

In [7], the authors present new upper and lower bounds for the spectral norm of an r-circulant matrix $C_r(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1})$, and they study the spectral norm of circulant matrix with generalized k-Horadam numbers in [8]. In this paper, we first give an explicit formula for the eigenvalues of r-circulant matrix with generalized k-Horadam numbers entries using different methods in [7]. Afterwards, we present a sufficient condition for an r-circulant matrix to be normal. Based on the results, the precise value for spectral norms of normal r-circulant matrix whose entries are generalized k-Horadam numbers is obtained, which generalize and improve the main results in [1, 2, 4, 5].

2. Preliminaries

In this section, we present some known lemmas and results that will be used in the following study.

Definition 1. For any given $c_0, c_1, \ldots, c_{n-1} \in \mathbb{C}$, the r-circulant matrix C, denoted by $C = C_r(c_0, c_1, \ldots, c_{n-1})$, is of the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{pmatrix}. \tag{7}$$

It is obvious that the matrix C_r turns into a classical circulant matrix for $r = 1$.

Lemma 2 (see [9]). *Let $C = C_r(c_0, c_1, \ldots, c_{n-1})$ be an r-circulant matrix; then the eigenvalues of C are given by*

$$\lambda_i = \sum_{j=0}^{n-1} c_j \mu_i^j, \quad \mu_i = r^{1/n}\omega^i, \quad i = 0, 1, \ldots, n-1, \tag{8}$$

where $\omega = e^{-2\pi i/n}$ is the nth root of unity.

Let us take any matrix $A = [a_{ij}]$ of order n; it is well known that the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{0 \leqslant i \leqslant n-1} \lambda_i \left(A^H A\right)}, \tag{9}$$

where A^H is the conjugate transpose of A and $\lambda_i(A^H A)$ is the eigenvalue of $A^H A$.

For a normal matrix A (i.e., $AA^H = A^H A$), we have the following lemma.

Lemma 3 (see [10]). *Let A be a normal matrix with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. Then the spectral norm of A is*

$$\|A\|_2 = \max_{0 \leqslant i \leqslant n-1} \left|\lambda_i\right|. \tag{10}$$

The following lemma can be found in [11].

Lemma 4 (see [11], Abel transformation). *Suppose that $\{a_i\}$ and $\{b_i\}$ are two sequences, and $S_i = a_1 + a_2 + \cdots + a_i$ ($i = 1, 2, \ldots$); then*

$$\sum_{i=1}^{n} a_i b_i = S_n b_n - \sum_{i=1}^{n-1} \left(b_{i+1} - b_i\right) S_i. \tag{11}$$

3. Spectrum of r-Circulant Matrix with Generalized k-Horadam Numbers

We start this section by giving the following lemma.

Lemma 5. *Suppose that $\{H_{k,i}\}_{i \in \mathbb{N}}$ is a generalized k-Horadam sequence defined in* (1). *The following conclusions hold.*

(1) *If $f(k) + g(k) \neq 1$, then*

$$\sum_{i=0}^{n} H_{k,i} = \frac{H_{k,n+1} + g(k) H_{k,n} + f(k) a - a - b}{f(k) + g(k) - 1}. \tag{12}$$

(2) *If $f(k) + g(k) = 1$, then*

$$\sum_{i=0}^{n} H_{k,i} = \frac{g(k) H_{k,n} + n\left[g(k) a + b\right] + a}{g(k) + 1}. \tag{13}$$

Proof. (1) According to (1), we have

$$\sum_{i=0}^{n} H_{k,i} = f(k) \sum_{i=0}^{n} H_{k,i-1} + g(k) \sum_{i=0}^{n} H_{k,i-2}. \tag{14}$$

Changing the summation index in (14), we have

$$\begin{aligned} \sum_{i=0}^{n} H_{k,i} &= f(k) \left(\sum_{i=0}^{n} H_{k,i} - H_{k,n} + H_{k,-1} \right) \\ &\quad + g(k) \left(\sum_{i=0}^{n} H_{k,i} - H_{k,n-1} - H_{k,n} + H_{k,-1} + H_{k,-2} \right). \end{aligned} \tag{15}$$

By direct calculation, together with recursive relation (1), one can obtain that

$$\begin{aligned} &\left[f(k) + g(k) - 1\right] \sum_{i=0}^{n} H_{k,i} \\ &\quad = H_{k,n+1} + g(k) H_{k,n} + f(k) a - a - b. \end{aligned} \tag{16}$$

Therefore we immediately obtain (12) from $f(k) + g(k) \neq 1$.

(2) Suppose that $f(k) + g(k) = 1$; we first illustrate that $H_{k,i+1} + g(k) H_{k,i} \equiv g(k)a + b$. Let $V_i = H_{k,i+1} + g(k) H_{k,i}$; then $V_0 = g(k)a + b$. Combining (1) and $f(k) + g(k) = 1$, one can obtain that

$$\begin{aligned} V_{i+1} &= H_{k,i+2} + g(k) H_{k,i+1} \\ &= \left(f(k) H_{k,i+1} + g(k) H_{k,i}\right) + g(k) H_{k,i+1} \\ &= H_{k,i+1} + g(k) H_{k,i} = V_i, \end{aligned} \tag{17}$$

which shows that $\{V_i\}$ is a constant sequence, and therefore

$$H_{k,i+1} + g(k) H_{k,i} = V_i = V_0 = g(k) a + b. \tag{18}$$

Evaluating summation from 0 to n, we have

$$\sum_{i=0}^{n} H_{k,i+1} + g(k) \sum_{i=0}^{n} H_{k,i} = (n+1)\left[g(k) a + b\right]. \tag{19}$$

Changing the summation index in (19) gives

$$\begin{aligned}\left(\sum_{i=0}^{n} H_{k,i} + H_{k,n+1} - a\right) + g(k) \sum_{i=0}^{n} H_{k,i} \\ = (n+1)\left[g(k) a + b\right].\end{aligned} \tag{20}$$

Therefore

$$\left[g(k)+1\right] \sum_{i=0}^{n} H_{k,i} = g(k) H_{k,n} + n\left[g(k) a + b\right] + a. \tag{21}$$

In view of assumptions $f^2(k) + g(k) > 1$ and $f(k) + g(k) = 1$, we know that $g(k) + 1 \neq 0$. Thus we obtain (13) from (21). □

From Lemma 5 we have the following theorem.

Theorem 6. *Let $A = C_r(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1})$ be an r-circulant matrix with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$; then for $i = 0, 1, 2, \ldots, n-1$ the following hold.*

(1) *If $f(k) + g(k) \neq 1$, then*

$$\begin{aligned}\lambda_i = {} & \Big(rH_{k,n} + g(k) r^{1+(1/n)} \omega^i H_{k,n-1} \\ & + r^{1/n} \left[f(k) a - b\right] \omega^i - a\Big) \\ & \times \left(r^{1/n} \omega^i f(k) + r^{2/n} \omega^{2i} g(k) - 1\right)^{-1}.\end{aligned} \tag{22}$$

(2) *If $f(k) + g(k) = 1$, then*

$$\begin{aligned}\lambda_i = {} & \Big(\left(g(k) rH_{k,n-1} + a\right)\left(1 - r^{1/n}\omega^i\right) \\ & + \left[g(k) a + b\right]\left(r^{1/n} w^i - r\right)\Big) \\ & \times \left(\left(1 - r^{1/n}\omega^i\right)\left[g(k) r^{1/n} \omega^i + 1\right]\right)^{-1}.\end{aligned} \tag{23}$$

Proof. According to Lemma 2, we have

$$\lambda_i = \sum_{i=0}^{n-1} H_{k,i} \mu_i^j, \qquad \mu_i = r^{1/n} w^i. \tag{24}$$

Using Abel transformation (Lemma 4), we have

$$\begin{aligned}\lambda_i &= \mu_i^{n-1} \sum_{j=0}^{n-1} H_{k,j} - \sum_{j=0}^{n-2} \left(\left(\mu_i^{j+1} - \mu_i^j\right) \sum_{s=0}^{j} H_{k,s}\right) \\ &= \mu_i^{n-1} \sum_{j=0}^{n-1} H_{k,j} - \left(\mu_i - 1\right) \sum_{j=0}^{n-2} \left(\mu_i^j \sum_{s=0}^{j} H_{k,s}\right).\end{aligned} \tag{25}$$

(1) In the light of (12) and (25), one can obtain that

$$\begin{aligned}\lambda_i &= \mu_i^{n-1} \sum_{j=0}^{n-1} H_{k,j} - \frac{\mu_i - 1}{f(k) + g(k) - 1} \\ &\quad \times \sum_{j=0}^{n-2} \mu_i^j \left[H_{k,j+1} + g(k) H_{k,j} + f(k) a - a - b\right] \\ &= \mu_i^{n-1} \frac{H_{k,n} + g(k) H_{k,n-1} + f(k) a - a - b}{f(k) + g(k) - 1} \\ &\quad - \frac{\mu_i - 1}{f(k) + g(k) - 1} \\ &\quad \times \left(\sum_{j=0}^{n-2} H_{k,j+1} \mu_i^j + g(k) \sum_{j=0}^{n-2} H_{k,j} \mu_i^j \right. \\ &\quad \left. + \left[f(k) a - a - b\right] \sum_{j=0}^{n-2} \mu_i^j\right).\end{aligned} \tag{26}$$

It is clear that

$$\begin{aligned}\sum_{j=0}^{n-2} H_{k,j+1} \mu_i^j &= \frac{\lambda_i - a}{\mu_i}, \\ \sum_{j=0}^{n-2} H_{k,j} \mu_i^j &= \lambda_i - \mu_i^{n-1} H_{k,n-1}.\end{aligned} \tag{27}$$

Substituting (27) into (26), we obtain that

$$\begin{aligned}\lambda_i &= \mu_i^{n-1}\frac{H_{k,n}+g(k)H_{k,n-1}+f(k)a-a-b}{f(k)+g(k)-1}\\&\quad-\frac{\mu_i-1}{f(k)+g(k)-1}\\&\quad\times\left(\frac{\lambda_i-a}{\mu_i}+g(k)\left(\lambda_i-\mu_i^{n-1}H_{k,n-1}\right)\right.\\&\qquad\left.+\left[f(k)a-a-b\right]\sum_{j=0}^{n-2}\mu_i^j\right)\\&=\frac{(1-\mu_i)\left[1+g(k)\mu_i\right]}{\mu_i\left[f(k)+g(k)-1\right]}\lambda_i\\&\quad+\frac{\mu_i^{n-1}H_{k,n}+\mu_i^n g(k)H_{k,n-1}}{f(k)+g(k)-1}\\&\quad+\frac{\mu_i^n\left(f(k)a-a-b\right)-a\left(1-\mu_i\right)}{\mu_i\left[f(k)+g(k)-1\right]}\\&\quad+\frac{\left(1-\mu_i^{n-1}\right)\left(f(k)a-a-b\right)}{f(k)+g(k)-1}.\end{aligned}\tag{28}$$

Therefore we have

$$\begin{aligned}&\left[g(k)\mu_i^2+f(k)\mu_i-1\right]\lambda_i\\&\quad=\mu_i^nH_{k,n}+g(k)\mu_i^{n+1}H_{k,n-1}+\mu_i^n\left(f(k)a-a-b\right)\\&\qquad-a\left(1-\mu_i\right)+\left(\mu_i-r\right)\left(f(k)a-a-b\right)\\&\quad=rH_{k,n}+g(k)r^{1+(1/n)}\omega^iH_{k,n-1}\\&\qquad+r^{1/n}\left[f(k)a-b\right]\omega^i-a.\end{aligned}\tag{29}$$

We immediately obtain formula (22) from (29).

(2) Taking into account (13) and (25), we have

$$\begin{aligned}\lambda_i &= \mu_i^{n-1}\sum_{j=0}^{n-1}H_{k,j}-\frac{\mu_i-1}{g(k)+1}\\&\quad\times\sum_{j=0}^{n-2}\mu_i^j\left[g(k)H_{k,j}+j\cdot\left(g(k)a+b\right)+a\right]\\&=\mu_i^{n-1}\sum_{j=0}^{n-1}H_{k,j}+\frac{g(k)\left(1-\mu_i\right)}{g(k)+1}\\&\quad\times\sum_{j=0}^{n-2}H_{k,j}\mu_i^j+\frac{1-\mu_i}{g(k)+1}\sum_{j=0}^{n-2}\left[j\cdot\left(g(k)a+b\right)+a\right]\mu_i^j\\&=\frac{\mu_i^{n-1}\left[g(k)H_{k,n-1}+(n-1)\left(g(k)a+b\right)+a\right]}{g(k)+1}\\&\quad+\frac{g(k)\left(1-\mu_i\right)}{g(k)+1}\left(\lambda_i-H_{k,n-1}\mu_i^{n-1}\right)\\&\quad+\frac{1-\mu_i}{g(k)+1}\left[\left((n-2)\left(g(k)a+b\right)+a\right)\right.\\&\qquad\left.\times\sum_{j=0}^{n-2}\mu_i^j-\left(g(k)a+b\right)\sum_{j=0}^{n-3}\sum_{s=0}^{j}\mu_i^s\right].\end{aligned}\tag{30}$$

It follows that

$$\begin{aligned}\lambda_i &= \frac{g(k)\mu_i^n}{g(k)+1}H_{k,n-1}+\frac{g(k)\left(1-\mu_i\right)}{g(k)+1}\lambda_i\\&\quad+\frac{\mu_i^{n-1}\left[(n-1)\left(g(k)a+b\right)+a\right]}{g(k)+1}\\&\quad+\frac{1-\mu_i}{g(k)+1}\left[\left((n-2)\left(g(k)a+b\right)+a\right)\right.\\&\qquad\left.\times\sum_{j=0}^{n-2}\mu_i^j-\frac{\left(g(k)a+b\right)}{1-\mu_i}\sum_{j=0}^{n-3}\left(1-\mu_i^{j+1}\right)\right]\\&=\frac{g(k)\left(1-\mu_i\right)}{g(k)+1}\lambda_i\\&\quad+\frac{g(k)\mu_i^nH_{k,n-1}+\mu_i^{n-1}\left(g(k)a+b\right)+a}{g(k)+1}\\&\quad+\frac{\left(g(k)a+b\right)\left(\mu_i-\mu_i^n\right)}{\left(g(k)+1\right)\left(1-\mu_i\right)}.\end{aligned}\tag{31}$$

Therefore we obtain (23). This concludes the proof. □

4. Spectral Norms of Normal r-Circulant Matrices

In this section, we consider the spectral norms of normal r-circulant matrix whose entries are generalized k-Horadam numbers. Our results generalize and improve the results in [1, 2, 4, 5]. The following lemma can be found in [9], and we give a concise proof.

Lemma 7. *Let $A = C_r(a_0, a_1, \ldots, a_{n-1})$ be an r-circulant matrix. If $|r| = 1$, then A is normal matrix.*

Proof. It is well known that

$$A=\sum_{i=0}^{n-1}a_iP^i,\qquad P=\begin{pmatrix}0 & I_{n-1}\\ r & 0\end{pmatrix}.\tag{32}$$

If $|r| = 1$, then

$$PP^H = \begin{pmatrix} 0 & I_{n-1} \\ r & 0 \end{pmatrix} \begin{pmatrix} 0 & \overline{r} \\ I_{n-1} & 0 \end{pmatrix} = I_n. \tag{33}$$

That is, $P^H = P^{-1}$. According to (32), we obtain that

$$\begin{aligned} AA^H &= \left(\sum_{i=0}^{n-1} a_i P^i\right)\left(\sum_{j=0}^{n-1} a_j \left(P^H\right)^j\right) \\ &= \left(\sum_{i=0}^{n-1} a_i P^i\right)\left(\sum_{j=0}^{n-1} a_j \left(P^{-1}\right)^j\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_i a_j P^{i-j}, \end{aligned}$$

$$\begin{aligned} A^H A &= \left(\sum_{j=0}^{n-1} a_j \left(P^H\right)^j\right)\left(\sum_{i=0}^{n-1} a_i P^i\right) \\ &= \left(\sum_{j=0}^{n-1} a_j P^{-j}\right)\left(\sum_{i=0}^{n-1} a_j P^i\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_i a_j P^{i-j}. \end{aligned} \tag{34}$$

Therefore $AA^H = A^H A$, which shows that A is normal. □

According to Theorem 6 and Lemma 7, we have the following theorem.

Theorem 8. *Suppose that $A = C_r(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1})$ is an r-circulant matrix. If $|r| = 1$ and $H_{k,i} \geqslant 0$, $i = 0, 1, 2, \ldots, n-1$, then the spectral norm of A is*

$$\|A\|_2 = \begin{cases} \max\limits_{0\leqslant i\leqslant n-1} \left| \dfrac{rH_{k,n} + g(k) r^{1+(1/n)} \omega^i H_{k,n-1} + r^{1/n} \left[f(k) a - b\right] \omega^i - a}{r^{1/n}\omega^i f(k) + r^{2/n}\omega^{2i} g(k) - 1} \right|, & f(k) + g(k) \neq 1, \\ \max\limits_{0\leqslant i\leqslant n-1} \left| \dfrac{\left(g(k) rH_{k,n-1} + a\right)\left(1 - r^{1/n}\omega^i\right) + \left[g(k) a + b\right]\left(r^{1/n}w^i - r\right)}{\left(1 - r^{1/n}\omega^i\right)\left[g(k) r^{1/n}\omega^i + 1\right]} \right|, & f(k) + g(k) = 1. \end{cases} \tag{35}$$

The following theorem simplifies and generalizes the results of Theorem 2.2 in [12].

Theorem 9. *Let $A = Circ(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1})$ be a circulant matrix; then*

$$\|A\|_2 = \begin{cases} \dfrac{H_{k,n} + g(k) H_{k,n-1} + f(k) a - a - b}{f(k) + g(k) - 1}, \\ \qquad\qquad f(k) + g(k) \neq 1, \\ \dfrac{g(k) H_{k,n-1} + (n-1)\left[g(k) a + b\right] + a}{g(k) + 1}, \\ \qquad\qquad f(k) + g(k) = 1. \end{cases} \tag{36}$$

Proof. Suppose that $r = 1$; it follows from Lemma 7 that A is normal. Notice that

$$\left|\lambda_i\right| = \left|\sum_{i=0}^{n-1} H_{k,i}\mu_i^j\right| \leqslant \sum_{i=0}^{n-1} H_{k,i}\left|\mu_i\right|^j = \lambda_0. \tag{37}$$

It follows from Lemma 3 that $\|A\|_2 = \lambda_0$. According to Theorem 6, if $f(k) + g(k) \neq 1$ and $r = 1$, we obtain that

$$\|A\|_2 = \lambda_0 = \frac{H_{k,n} + g(k) H_{k,n-1} + f(k) a - a - b}{f(k) + g(k) - 1}. \tag{38}$$

Similarly, if $f(k) + g(k) = 1$, it follows that

$$\|A\|_2 = \lambda_0 = \frac{g(k) H_{k,n-1} + (n-1)\left[g(k) a + b\right] + a}{g(k) + 1}. \tag{39}$$

This completes the proof. □

Taking into account formulae (4)–(6), we have the following corollary.

Corollary 10. *Let $A_1 = Circ(F_0, F_1, \ldots, F_{n-1})$ be a circulant matrix; then*

$$\left\|A_1\right\|_2 = F_{n+1} - 1. \tag{40}$$

Corollary 11. *Let $A_2 = Circ(L_0, L_1, \ldots, L_{n-1})$ be a circulant matrix; then*

$$\left\|A_2\right\|_2 = F_{n+2} + F_n - 1. \tag{41}$$

Corollary 12. *Let $A_3 = Circ(J_0, J_1, \ldots, J_{n-1})$ be a circulant matrix; then*

$$\left\|A_3\right\|_2 = \frac{J_{n+1} - 1}{2}. \tag{42}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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