Research Article

On Some Integral Inequalities Related to Hermite-Hadamard-Fejér Inequalities for Coordinated Convex Functions

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Several new mappings associated with coordinated convexity are proposed, by which we obtain some new Hermite-Hadamard-Fejér type inequalities for coordinated convex functions. We conclude that the results obtained in this work are the generalizations of the earlier results.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$; then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2} \quad (1)$$

is known as the Hermite-Hadamard inequality.

In [1], Fejér established the following weighted generalization of inequality (1).

Theorem 1. If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then the inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx &\leq \int_a^b f(x) p(x) dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b p(x) dx \end{aligned} \quad (2)$$

holds, where $p : [a, b] \rightarrow \mathbb{R}$ is positive, integrable, and symmetric about $x = (a+b)/2$.

Inequalities (1) and (2) have been extended, generalized, and improved by a number of authors (e.g., [2–9]).

In [4], Dragomir proposed the following Hermite-Hadamard type inequalities which refine the first inequality of (1).

Theorem 2 (see [4]). Let f be convex on $[a, b]$. Then H is convex, increasing on $[0, 1]$, and, for all $t \in [0, 1]$,

$$f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx, \quad (3)$$

where

$$H(t) = \frac{1}{b-a} \int_a^b f\left(ts + (1-t)\frac{a+b}{2}\right) ds. \quad (4)$$

An analogous result for convex functions which refines the second inequality of (1) is obtained by Yang and Hong in [10] as follows.

Theorem 3 (see [10]). Let f be convex on $[a, b]$. Then P is convex, increasing on $[0, 1]$, and, for all $t \in [0, 1]$,

$$\frac{1}{b-a} \int_a^b f(x) dx = P(0) \leq P(t) \leq P(1) = \frac{f(a) + f(b)}{2}, \quad (5)$$

where

$$P(t) = \frac{1}{2(b-a)} \times \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)s\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)s\right) \right] ds. \quad (6)$$

A modification for convex functions which is also known as coordinated convex functions was introduced as follows by Dragomir in [5].

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$; a mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w) \quad (7)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be coordinated convex on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$, and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $y \in [c, d]$ and $x \in [a, b]$.

A formal definition for coordinated convex functions may be stated as follows.

Definition 4. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on coordinates on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leq \lambda tf(x, y) + \lambda(1-t)f(x, w) + (1-\lambda)tf(z, y) + (1-t)(1-\lambda)f(z, w) \quad (8)$$

holds for all $(x, y), (z, y), (x, w),$ and $(z, w) \in \Delta$ and $t, \lambda \in [0, 1]$.

Dragomir in [5] established the following Hermite-Hadamard type inequalities for coordinated convex functions in a rectangle from the plane $\mathbb{R}^2$.

Theorem 5. Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is convex on the coordinates on Δ . Then one has the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (9)$$

The mapping connected with the first inequality of (9) is considered in [5].

If $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a coordinated convex function, then the following mapping on $[0, 1]$ can be defined by

$$\begin{aligned} H(t, s) &= \frac{1}{(b-a)(d-c)} \\ &\times \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dy dx. \end{aligned} \quad (10)$$

The mapping H has the following properties:

- (i) H is coordinated convex and monotonic nondecreasing on $[0, 1]^2$;
- (ii) we have the following bounds for H :

$$\begin{aligned} \sup_{(t,s) \in [0,1]^2} H(t, s) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &= H(1, 1), \end{aligned} \quad (11)$$

$$\inf_{(t,s) \in [0,1]^2} H(t, s) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(0, 0).$$

Recently, Hwang et al. [11] established a monotonic non-decreasing mapping connected with the Hadamard's inequality for coordinated convex functions in a rectangle from the plane as follows.

Theorem 6 (see [11]). Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is coordinated convex on Δ and the mapping $F : [0, 1]^2 \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} F(t, s) &= \frac{1}{4(b-a)(d-c)} \\ &\times \int_a^b \int_c^d \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)c + \left(\frac{1-s}{2}\right)y\right) \right. \\ &\quad + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)d + \left(\frac{1-s}{2}\right)y\right) \\ &\quad + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)c + \left(\frac{1-s}{2}\right)y\right) \\ &\quad \left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)d + \left(\frac{1-s}{2}\right)y\right) \right] dy dx. \end{aligned} \quad (12)$$

Then

- (i) the mapping F is coordinated convex on $[0, 1]^2$;
- (ii) the mapping F is coordinated monotonic nondecreasing on $[0, 1]^2$;
- (iii) we have the bounds

$$\begin{aligned} \inf_{(t,s) \in [0,1]^2} F(t,s) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx \\ &= F(0,0), \\ \sup_{(t,s) \in [0,1]^2} F(t,s) &= \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\ &= F(1,1). \end{aligned} \quad (13)$$

Fejér-type inequality for coordinated convex mappings is established in [12] as follows.

Theorem 7 (see [12]). Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a coordinated convex function on Δ . Then one has the following inequalities:

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y) dy dx \\ &\leq \int_a^b \int_c^d f(x,y) p(x,y) dy dx \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\ &\quad \times \int_a^b \int_c^d p(x,y) dy dx, \end{aligned} \quad (14)$$

where $p : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is positive, integrable, and symmetric about $x = (a+b)/2$ and $y = (c+d)/2$.

Theorem 8 (see [12]). Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a coordinated convex function, and consider the function $P : [a, b] \times [c, d] \rightarrow \mathbb{R}$ which is positive, integrable, and symmetric about $x = (a+b)/2$ and $y = (c+d)/2$. Let S be a function defined on $[0, 1]^2$ by

$$\begin{aligned} S(t,s) &= \frac{1}{(b-a)(d-c)} \\ &\quad \times \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \\ &\quad \times p(x,y) dy dx. \end{aligned} \quad (15)$$

Then S is a coordinated convex function on $[0, 1]^2$, nondecreasing on $[0, 1]^2$. Moreover,

$$\begin{aligned} \inf_{(t,s) \in [0,1]^2} S(t,s) &= S(0,0) \\ &= f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y) dy dx, \\ \sup_{(t,s) \in [0,1]^2} S(t,s) &= S(1,1) = \int_a^b \int_c^d f(x,y) p(x,y) dy dx. \end{aligned} \quad (16)$$

Theorem 9 (see [12]). Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a coordinated convex function, and consider the function $P : [a, b] \times [c, d] \rightarrow \mathbb{R}$ which is positive, integrable, and symmetric about $x = (a+b)/2$ and $y = (c+d)/2$. Let A be a function defined on $[0, 1]^2$ by

$$\begin{aligned} A(t,s) &= \frac{1}{4} \int_a^b \int_c^d \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) \right. \\ &\quad \times p\left(\frac{x+a}{2}, \frac{y+c}{2}\right) \\ &\quad + f\left(\frac{1+t}{2}a + \frac{1-t}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) \\ &\quad \times p\left(\frac{x+a}{2}, \frac{y+d}{2}\right) \\ &\quad + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x, \frac{1+s}{2}c + \frac{1-s}{2}y\right) \\ &\quad \times p\left(\frac{x+b}{2}, \frac{y+c}{2}\right) \\ &\quad + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x, \frac{1+s}{2}d + \frac{1-s}{2}y\right) \\ &\quad \times p\left(\frac{x+b}{2}, \frac{y+d}{2}\right) \left. \right] dy dx. \end{aligned} \quad (17)$$

Then, A is coordinated convex on $[0, 1]^2$, symmetric about $(1/2, 1/2)$, and non-decreasing on $[0, 1]^2$, and

$$\begin{aligned} \inf_{(t,s) \in [0,1]^2} A(t,s) &= A(0,0) \\ &= \frac{1}{(b-a)(d-c)} \\ &\quad \times \int_a^b \int_c^d f(x,y) p(x,y) dy dx, \\ \sup_{(t,s) \in [0,1]^2} A(t,s) &= A(1,1) \\ &= \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \\ &\quad \times \int_a^b \int_c^d p(x,y) dy dx. \end{aligned} \quad (18)$$

In this paper, we establish some new results about the Hermite-Hadamard-Fejér type inequality for coordinated convex mappings which generalize the results (10), (12), (15), and (17).

2. Main Results

We will use the following lemma to prove our results.

Lemma 10 (see [13]). Let $f : [a, b] \rightarrow R$ be a convex function, $0 < \alpha < 1$, $0 < \beta < 1$, $A = \alpha a + (1 - \alpha)b$, $u_0 = (b - a) \min\{\alpha/(1 - \beta), (1 - \alpha)/\beta\}$, and let h be defined by $h(t) = (1 - \beta)f(A - \beta t) + \beta f(A + (1 - \beta)t)$, $t \in [0, u_0]$. Then h is convex, increasing on $[0, u_0]$, and, for all $t \in [0, u_0]$,

$$f(\alpha a + (1 - \alpha)b) \leq h(t) \leq \alpha f(a) + (1 - \alpha)f(b). \quad (19)$$

Lemma 11. Let $f : [a, b] \times [c, d] \rightarrow R$ be a coordinated convex function, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \lambda < 1$, $0 < \mu < 1$, $A = \alpha a + (1 - \alpha)b$, $B = \lambda c + (1 - \lambda)d$, $u_0 = (b - a) \min\{\alpha/(1 - \beta), (1 - \alpha)/\beta\}$, $v_0 = (d - c) \min\{\lambda/(1 - \mu), (1 - \lambda)/\mu\}$ and let h be defined by

$$\begin{aligned} h(t, s) = & (1 - \beta)(1 - \mu)f(A - \beta t, B - \mu s) \\ & + (1 - \beta)\mu f(A - \beta t, B + (1 - \mu)s) \\ & + \beta(1 - \mu)f(A + (1 - \beta)t, B - \mu s) \\ & + \beta\mu f(A + (1 - \beta)t, B + (1 - \mu)s), \end{aligned} \quad (20)$$

$t \in [0, u_0]$, $s \in [0, v_0]$. Then h is coordinated convex, coordinated monotonic nondecreasing on $[0, u_0] \times [0, v_0]$, and, for all $(t, s) \in [0, u_0] \times [0, v_0]$,

$$\begin{aligned} & f(\alpha a + (1 - \alpha)b, \lambda c + (1 - \lambda)d) \\ & \leq h(t, s) \\ & \leq \alpha \lambda f(a, c) + (1 - \alpha)\lambda f(b, c) + \alpha(1 - \lambda)f(a, d) \\ & \quad + (1 - \alpha)(1 - \lambda)f(b, d). \end{aligned} \quad (21)$$

Proof. We note that if f is convex and g is linear, then the composition $f \circ g$ is convex. Also we note that a positive constant multiple of a convex function and a sum of two convex functions are convex. Hence, it is easily observed that h is coordinated convex, coordinated monotonic nondecreasing on $[0, u_0] \times [0, v_0]$, from the coordinated convexity of f and Lemma 10. Next,

$$\begin{aligned} h(t, s) = & (1 - \beta)(1 - \mu)f(A - \beta t, B - \mu s) \\ & + (1 - \beta)\mu f(A - \beta t, B + (1 - \mu)s) \\ & + \beta(1 - \mu)f(A + (1 - \beta)t, B - \mu s) \\ & + \beta\mu f(A + (1 - \beta)t, B + (1 - \mu)s) \\ \geq & f[(1 - \beta)(A - \beta t) + \beta(A + (1 - \beta)t), \\ & (1 - \mu)(B - \mu s) + \mu(B + (1 - \mu)s)] \\ = & f(A, B) = f(\alpha a + (1 - \alpha)b, \lambda c + (1 - \lambda)d). \end{aligned} \quad (22)$$

Also,

$$\begin{aligned} h(t, s) = & (1 - \beta)(1 - \mu)f \\ & \times \left[\frac{\alpha(b - a) + \beta t}{b - a}a + \frac{(1 - \alpha)(b - a) - \beta t}{b - a}b, \right. \\ & \left. \frac{\lambda(d - c) + \mu s}{d - c}c + \frac{(1 - \lambda)(d - c) - \mu s}{d - c}d \right] \\ & + (1 - \beta)\mu f \\ & \times \left[\frac{\alpha(b - a) + \beta t}{b - a}a + \frac{(1 - \alpha)(b - a) - \beta t}{b - a}b, \right. \\ & \left. \frac{\lambda(d - c) - (1 - \mu)s}{d - c}c \right. \\ & \left. + \frac{(1 - \lambda)(d - c) + (1 - \mu)s}{d - c}d \right] \\ & + \beta(1 - \mu)f \\ & \times \left[\frac{\alpha(b - a) - (1 - \beta)t}{b - a}a + \frac{(1 - \alpha)(b - a) + (1 - \beta)t}{b - a}b, \right. \\ & \left. \frac{\lambda(d - c) + \mu s}{d - c}c + \frac{(1 - \lambda)(d - c) - \mu s}{d - c}d \right] \\ & + \beta\mu f \left[\frac{\alpha(b - a) - (1 - \beta)t}{b - a}a \right. \\ & \left. + \frac{(1 - \alpha)(b - a) + (1 - \beta)t}{b - a}b, \right. \\ & \left. \frac{\lambda(d - c) - (1 - \mu)s}{d - c}c \right. \\ & \left. + \frac{(1 - \lambda)(d - c) + (1 - \mu)s}{d - c}d \right] \\ = & (1 - \beta)(1 - \mu)I_1 + (1 - \beta)\mu I_2 + \beta(1 - \mu)I_3 + \beta\mu I_4. \end{aligned} \quad (23)$$

From the coordinated convexity of f , we obtain

$$\begin{aligned} I_1 = & f \left[\frac{\alpha(b - a) + \beta t}{b - a}a + \frac{(1 - \alpha)(b - a) - \beta t}{b - a}b, \right. \\ & \left. \frac{\lambda(d - c) + \mu s}{d - c}c + \frac{(1 - \lambda)(d - c) - \mu s}{d - c}d \right] \\ \leq & \left(\frac{\alpha(b - a) + \beta t}{b - a} \right) \left(\frac{\lambda(d - c) + \mu s}{d - c} \right) f(a, c) \\ & + \left(\frac{\alpha(b - a) + \beta t}{b - a} \right) \left(\frac{(1 - \lambda)(d - c) - \mu s}{d - c} \right) f(a, d) \\ & + \left(\frac{(1 - \alpha)(b - a) - \beta t}{b - a} \right) \left(\frac{\lambda(d - c) + \mu s}{d - c} \right) f(b, c) \\ & + \left(\frac{(1 - \alpha)(b - a) - \beta t}{b - a} \right) \left(\frac{(1 - \lambda)(d - c) - \mu s}{d - c} \right) f(b, d) \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{(1-\alpha)(b-a)-\beta t}{b-a} \right) \left(\frac{(1-\lambda)(d-c)-\mu s}{d-c} \right) \\
& \times f(b, d), \\
I_2 = & f \left[\frac{\alpha(b-a)+\beta t}{b-a} a + \frac{(1-\alpha)(b-a)-\beta t}{b-a} b, \right. \\
& \left. \frac{\lambda(d-c)-(1-\mu)s}{d-c} c + \frac{(1-\lambda)(d-c)+(1-\mu)s}{d-c} d \right] \\
\leq & \left(\frac{\alpha(b-a)+\beta t}{b-a} \right) \left(\frac{\lambda(d-c)-(1-\mu)s}{d-c} \right) f(a, c) \\
& + \left(\frac{\alpha(b-a)+\beta t}{b-a} \right) \left(\frac{(1-\lambda)(d-c)+(1-\mu)s}{d-c} \right) f(a, d) \\
& + \left(\frac{(1-\alpha)(b-a)-\beta t}{b-a} \right) \left(\frac{\lambda(d-c)-(1-\mu)s}{d-c} \right) f(b, c) \\
& + \left(\frac{(1-\alpha)(b-a)-\beta t}{b-a} \right) \left(\frac{(1-\lambda)(d-c)+(1-\mu)s}{d-c} \right) \\
& \times f(b, d),
\end{aligned}$$

$$\begin{aligned}
I_3 = & f \left[\frac{\alpha(b-a)-(1-\beta)t}{b-a} a \right. \\
& + \frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} b, \\
& \left. \frac{\lambda(d-c)+\mu s}{d-c} c + \frac{(1-\lambda)(d-c)-\mu s}{d-c} d \right] \\
\leq & \left(\frac{\alpha(b-a)-(1-\beta)t}{b-a} \right) \left(\frac{\lambda(d-c)+\mu s}{d-c} \right) f(a, c) \\
& + \left(\frac{\alpha(b-a)-(1-\beta)t}{b-a} \right) \left(\frac{(1-\lambda)(d-c)-\mu s}{d-c} \right) f(a, d) \\
& + \left(\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} \right) \left(\frac{\lambda(d-c)+\mu s}{d-c} \right) f(b, c) \\
& + \left(\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} \right) \left(\frac{(1-\lambda)(d-c)-\mu s}{d-c} \right) \\
& \times f(b, d),
\end{aligned}$$

$$\begin{aligned}
I_4 = & f \left[\frac{\alpha(b-a)-(1-\beta)t}{b-a} a \right. \\
& + \frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} b, \\
& \frac{\lambda(d-c)-(1-\mu)s}{d-c} c \\
& \left. + \frac{(1-\lambda)(d-c)+(1-\mu)s}{d-c} d \right]
\end{aligned}$$

$$\begin{aligned}
& \leq \left(\frac{\alpha(b-a)-(1-\beta)t}{b-a} \right) \left(\frac{\lambda(d-c)-(1-\mu)s}{d-c} \right) f(a, c) \\
& + \left(\frac{\alpha(b-a)-(1-\beta)t}{b-a} \right) \\
& \times \left(\frac{(1-\lambda)(d-c)+(1-\mu)s}{d-c} \right) f(a, d) \\
& + \left(\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} \right) \left(\frac{\lambda(d-c)-(1-\mu)s}{d-c} \right) \\
& \times f(b, c) \\
& + \left(\frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} \right) \\
& \times \left(\frac{(1-\lambda)(d-c)+(1-\mu)s}{d-c} \right) f(b, d).
\end{aligned} \tag{24}$$

We note that $0 < \alpha \leq (\alpha(b-a)+\beta t)/(b-a) \leq 1$, $0 \leq ((1-\alpha)(b-a)-\beta t)/(b-a) \leq 1$, $0 \leq (\alpha(b-a)-(1-\beta)t)/(b-a) \leq \alpha \leq 1$, $0 < 1-\alpha \leq ((1-\alpha)(b-a)+(1-\beta)t)/(b-a) \leq 1$, $0 < \lambda \leq (\lambda(d-c)+\mu s)/(d-c) \leq 1$, $0 \leq ((1-\lambda)(d-c)-\mu s)/(d-c) \leq 1$, $0 \leq (\lambda(d-c)-(1-\mu)s)/(d-c) \leq \lambda \leq 1$, $0 < 1-\lambda \leq ((1-\lambda)(d-c)+(1-\mu)s)/(d-c) \leq 1$. So

$$\begin{aligned}
h(t, s) = & (1-\beta)(1-\mu)I_1 \\
& + (1-\beta)\mu I_2 + \beta(1-\mu)I_3 + \beta\mu I_4 \\
\leq & \alpha\lambda f(a, c) + (1-\alpha)\lambda f(b, c) \\
& + \alpha(1-\lambda)f(a, d) + (1-\alpha)(1-\lambda)f(b, d),
\end{aligned} \tag{25}$$

which completes the proof. $\square$

Theorem 12. Let f , A , B , α , β , λ , μ , u_0 , and v_0 be defined as in Lemma 11, and let $g : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be nonnegative and integrable and

$$g(A - \beta u, s) = g(A + (1-\beta)u, s),$$

$$(u, s) \in [0, u_0] \times [c, d],$$

$$g(t, B - \mu s) = g(t, B + (1-\mu)s), \quad (t, s) \in [a, b] \times [0, v_0]. \tag{26}$$

Then

$$\begin{aligned}
 & f(\alpha a + (1-\alpha)b, \lambda c + (1-\lambda)d) \\
 & \times \int_{A-\beta u}^{A+(1-\beta)u} \int_{B-\mu v}^{B+(1-\mu)v} g(t, s) dt ds \\
 & \leq \frac{(1-\beta)(1-\mu)}{\beta\mu} \int_{A-\beta u}^A \int_{B-\mu v}^B f(t, s) g(t, s) dt ds \\
 & + \frac{(1-\beta)\mu}{\beta(1-\mu)} \int_{A-\beta u}^A \int_B^{B+(1-\mu)v} f(t, s) g(t, s) dt ds \\
 & + \frac{\beta(1-\mu)}{(1-\beta)\mu} \int_A^{A+(1-\beta)u} \int_{B-\mu v}^B f(t, s) g(t, s) dt ds \\
 & + \frac{\beta\mu}{(1-\beta)(1-\mu)} \int_A^{A+(1-\beta)u} \int_B^{B+(1-\mu)v} f(t, s) g(t, s) dt ds \\
 & \leq [\alpha\lambda f(a, c) + (1-\alpha)\lambda f(b, c) + \alpha(1-\lambda)f(a, d) \\
 & + (1-\alpha)(1-\lambda)f(b, d)] \\
 & \times \int_{A-\beta u}^{A+(1-\beta)u} \int_{B-\mu v}^{B+(1-\mu)v} g(t, s) dt ds.
 \end{aligned} \tag{27}$$

Proof. For every $(u, v) \in [0, u_0] \times [0, v_0]$, we have the identity

$$\begin{aligned}
 & \int_{A-\beta u}^{A+(1-\beta)u} \int_{B-\mu v}^{B+(1-\mu)v} g(t, s) dt ds \\
 & = \int_{B-\mu v}^{B+(1-\mu)v} \left(\int_{A-\beta u}^{A+(1-\beta)u} g(t, s) dt \right) ds \\
 & = \int_{B-\mu v}^{B+(1-\mu)v} \left(\int_{A-\beta u}^A g(t, s) dt + \int_A^{A+(1-\beta)u} g(t, s) dt \right) ds \\
 & = \int_{B-\mu v}^{B+(1-\mu)v} \left(\beta \int_0^u g(A-\beta t, s) dt \right. \\
 & \quad \left. + (1-\beta) \int_0^u g(A-\beta t, s) dt \right) ds \\
 & = \int_{B-\mu v}^{B+(1-\mu)v} \left(\int_0^u g(A-\beta t, s) dt \right) ds \\
 & = \int_0^u \left(\int_{B-\mu v}^{B+(1-\mu)v} g(A-\beta t, s) ds \right) dt \\
 & = \int_0^u \int_0^v g(A-\beta t, B-\mu s) dt ds.
 \end{aligned} \tag{28}$$

Since g is nonnegative, multiplying (21) by $g(A-\beta t, B-\mu s)$, integrating the resulting inequalities over $[0, u] \times [0, v]$, and using (26), we have

$$\begin{aligned}
 & g(A-\beta t, B-\mu s) = g(A-\beta t, B+(1-\mu)s) \\
 & = g(A+(1-\beta)t, B-\mu s) \\
 & = g(A+(1-\beta)t, B+(1-\mu)s), \\
 & f(\alpha a + (1-\alpha)b, \lambda c + (1-\lambda)d) \\
 & \times \int_0^u \int_0^v g(A-\beta t, B-\mu s) dt ds \\
 & \leq (1-\beta)(1-\mu) \\
 & \times \int_0^u \int_0^v f(A-\beta t, B-\mu s) g(A-\beta t, B-\mu s) dt ds \\
 & + (1-\beta)\mu \int_0^u \int_0^v f(A-\beta t, B+(1-\mu)s) \\
 & \quad \times g(A-\beta t, B+(1-\mu)s) dt ds \\
 & + \beta(1-\mu) \int_0^u \int_0^v f(A+(1-\beta)t, B-\mu s) \\
 & \quad \times g(A+(1-\beta)t, B-\mu s) dt ds \\
 & + \beta\mu \int_0^u \int_0^v f(A+(1-\beta)t, B+(1-\mu)s) \\
 & \quad \times g(A+(1-\beta)t, B+(1-\mu)s) dt ds \\
 & = \frac{(1-\beta)(1-\mu)}{\beta\mu} \int_{A-\beta u}^A \int_{B-\mu v}^B f(t, s) g(t, s) dt ds \\
 & + \frac{(1-\beta)\mu}{\beta(1-\mu)} \int_{A-\beta u}^A \int_B^{B+(1-\mu)v} f(t, s) g(t, s) dt ds \\
 & + \frac{\beta(1-\mu)}{(1-\beta)\mu} \int_A^{A+(1-\beta)u} \int_{B-\mu v}^B f(t, s) g(t, s) dt ds \\
 & + \frac{\beta\mu}{(1-\beta)(1-\mu)} \int_A^{A+(1-\beta)u} \int_B^{B+(1-\mu)v} f(t, s) g(t, s) dt ds \\
 & \leq [\alpha\lambda f(a, c) + (1-\alpha)\lambda f(b, c) + \alpha(1-\lambda)f(a, d) \\
 & + (1-\alpha)(1-\lambda)f(b, d)] \\
 & \times \int_0^u \int_0^v g(A-\beta t, B-\mu s) dt ds.
 \end{aligned} \tag{29}$$

Thus, inequalities (27) follow by using the identity (28). $\square$

Remark 13. If we choose $\alpha = \beta = \lambda = \mu = 1/2$, $u = u_0 = b-a$, $v = v_0 = d-c$ in Theorem 12, then inequalities (27) reduce to inequalities (14).

Remark 14. If we choose $\alpha = \beta = \lambda = \mu = 1/2$, $u = u_0 = b - a$, $v = v_0 = d - c$, and $g(x, y) \equiv 1$ in Theorem 12, then inequalities (27) reduce to inequalities (9).

Theorem 15. Let f, A, B, u_0 , and v_0 be defined as in Lemma 11, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \leq 1$, $0 < \lambda < 1$, $0 < \mu < 1$, $\lambda + \mu \leq 1$, and let H be defined on $[0, 1]^2$ by

$$\begin{aligned} H(t, s) &= \frac{(1-\beta)(1-\mu)}{\alpha\lambda(b-a)(d-c)} \\ &\times \int_0^{\alpha(b-a)/(1-\beta)} \int_0^{\lambda(d-c)/(1-\mu)} [(1-\beta)(1-\mu)f(A-\beta tx, B-\mu sy) + (1-\beta)\mu f(A-\beta tx, B+(1-\mu)sy) \\ &\quad + \beta(1-\mu)f(A+(1-\beta)tx, B-\mu sy) + \beta\mu f(A+(1-\beta)tx, B+(1-\mu)sy)] dy dx. \end{aligned} \quad (30)$$

Then, H is coordinated convex and coordinated monotonic nondecreasing on $[0, 1]^2$, and

$$\begin{aligned} &f(\alpha a + (1-\alpha)b, \lambda c + (1-\lambda)d) \\ &= H(0, 0) \leq H(t, s) \leq H(1, 1) \\ &= \frac{(1-\beta)(1-\mu)}{\alpha\lambda(b-a)(d-c)} \\ &\times \int_0^{\alpha(b-a)/(1-\beta)} \int_0^{\lambda(d-c)/(1-\mu)} [(1-\beta)(1-\mu)f(A-\beta x, B-\mu y) + (1-\beta)\mu f(A-\beta x, B+(1-\mu)y) \\ &\quad + \beta(1-\mu)f(A+(1-\beta)x, B-\mu y) + \beta\mu f(A+(1-\beta)x, B+(1-\mu)y)] dy dx \\ &\leq \alpha\lambda f(a, c) + (1-\alpha)\lambda f(b, c) + \alpha(1-\lambda)f(a, d) + (1-\alpha)(1-\lambda)f(b, d). \end{aligned} \quad (31)$$

Proof. That H is coordinated convex follows immediately from the coordinated convexity of f . Next, the conditions $\alpha + \beta \leq 1$ and $\lambda + \mu \leq 1$ imply that $u_0 = \alpha(b-a)/(1-\beta)$ and $v_0 = \lambda(d-c)/(1-\mu)$, respectively. It follows from Lemma 11 that $h(t, s) = (1-\beta)(1-\mu)f(A-\beta t, B-\mu s) + (1-\beta)\mu f(A-\beta t, B+(1-\mu)s) + \beta(1-\mu)f(A+(1-\beta)t, B-\mu s) + \beta\mu f(A+(1-\beta)t, B+(1-\mu)s)$ is coordinated monotonic nondecreasing on $[0, u_0] \times [0, v_0]$ and hence $H(t, s)$ is coordinated monotonic

nondecreasing on $[0, 1]^2$. Finally, the last inequality of (31) follows from (21), and the proof is completed. $\square$

Similarly, we have the following theorem.

Theorem 16. Let f, A, B, u_0 , and v_0 be defined as in Lemma 11, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \leq 1$, $0 < \lambda < 1$, $0 < \mu < 1$, $\lambda + \mu \leq 1$, and let G be defined on $[0, 1]^2$ by

$$\begin{aligned} &G(t, s) \\ &= \frac{(1-\beta)(1-\mu)}{\alpha\lambda(b-a)(d-c)} \\ &\times \int_0^{\alpha(b-a)/(1-\beta)} \int_0^{\lambda(d-c)/(1-\mu)} \left\{ (1-\beta)(1-\mu)f\left(A-\beta\left(\frac{\alpha(b-a)}{1-\beta}-x(1-t)\right), B-\mu\left(\frac{\lambda(d-c)}{1-\mu}-y(1-s)\right)\right) \right. \\ &\quad + (1-\beta)\mu f\left(A-\beta\left(\frac{\alpha(b-a)}{1-\beta}-x(1-t)\right), B+(1-\mu)\left(\frac{\lambda(d-c)}{1-\mu}-y(1-s)\right)\right) \\ &\quad + \beta(1-\mu)f\left(A+(1-\beta)\left(\frac{\alpha(b-a)}{1-\beta}-x(1-t)\right), B-\mu\left(\frac{\lambda(d-c)}{1-\mu}-y(1-s)\right)\right) \\ &\quad \left. + \beta\mu f\left(A+(1-\beta)\left(\frac{\alpha(b-a)}{1-\beta}-x(1-t)\right), B+(1-\mu)\left(\frac{\lambda(d-c)}{1-\mu}-y(1-s)\right)\right) \right\} dy dx. \end{aligned} \quad (32)$$

Then, G is coordinated convex and coordinated monotonic nondecreasing on $[0, 1]^2$, and

$$\begin{aligned}
 & \frac{(1-\beta)^2(1-\mu)^2}{\beta\mu\alpha\lambda(b-a)(d-c)} \\
 & \times \int_{b-(\alpha(b-a)/(1-\beta))}^A \int_{d-(\lambda(d-c)/(1-\mu))}^B f(x, y) dy dx \\
 & + \frac{(1-\beta)^2\mu}{\beta\alpha\lambda(b-a)(d-c)} \int_{b-(\alpha(b-a)/(1-\beta))}^A \int_B^d f(x, y) dy dx \\
 & + \frac{\beta(1-\mu)^2}{\mu\alpha\lambda(b-a)(d-c)} \int_A^b \int_{d-(\lambda(d-c)/(1-\mu))}^B f(x, y) dy dx \\
 & + \frac{\beta\mu}{\alpha\lambda(b-a)(d-c)} \int_A^b \int_B^d f(x, y) dy dx \\
 & = G(0, 0) \leq G(t, s) \leq G(1, 1) \\
 & = (1-\beta)(1-\mu) f\left(b - \frac{\alpha(b-a)}{(1-\beta)}, d - \frac{\lambda(d-c)}{1-\mu}\right) \\
 & + (1-\beta)\mu f\left(b - \frac{\alpha(b-a)}{(1-\beta)}, d\right) \\
 & + \beta(1-\mu) f\left(b, d - \frac{\lambda(d-c)}{1-\mu}\right) + \beta\mu f(b, d) \\
 & \leq \alpha\lambda f(a, c) + (1-\alpha)\lambda f(b, c) \\
 & + \alpha(1-\lambda) f(a, d) + (1-\alpha)(1-\lambda) f(b, d). \tag{33}
 \end{aligned}$$

Proof. The proof is similar to that of Theorem 15, so we omit the details. $\square$

Remark 17. Identity (10) is a special case of (30) if we choose $\alpha = \beta = \lambda = \mu = 1/2$, $u = u_0 = b - a$, $v = v_0 = d - c$.

Remark 18. Identity (12) is a special case of (32) if we choose $\alpha = \beta = \lambda = \mu = 1/2$, $u = u_0 = b - a$, $v = v_0 = d - c$.

Theorem 19. Let f , A , B , u_0 , and v_0 be defined as in Lemma 11, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \leq 1$, $0 < \lambda < 1$, $0 < \mu < 1$, $\lambda + \mu \leq 1$, and let g be defined as in Theorem 12. Let G be defined on $[0, 1] \times [0, 1]$ by

$$\begin{aligned}
 & P(t, s) \\
 & = \int_0^u \int_0^v [(1-\beta)(1-\mu) f(A - \beta tx, B - \mu sy) \\
 & \quad \times g(A - \beta x, B - \mu y) \\
 & \quad + (1-\beta)\mu f(A - \beta tx, B + (1-\mu) sy) \\
 & \quad \times g(A - \beta x, B + (1-\mu) y) \\
 & \quad + \beta(1-\mu) f(A + (1-\beta) tx, B - \mu sy)
 \end{aligned}$$

$$\begin{aligned}
 & \times g(A + (1-\beta) x, B - \mu y) \\
 & + \beta\mu f(A + (1-\beta) tx, B + (1-\mu) sy) \\
 & \times g(A + (1-\beta) x, B + (1-\mu) y)] dx dy, \tag{34}
 \end{aligned}$$

for some $u \in [0, u_0]$, $v \in [0, v_0]$. Then $P(t, s)$ is coordinated convex and coordinated monotonic nondecreasing on $[0, 1]^2$, and

$$\begin{aligned}
 & f(\alpha a + (1-\alpha)b, \lambda c + (1-\lambda)d) \\
 & \times \int_{A-\beta u}^{A+(1-\beta)u} \int_{B-\mu v}^{B+(1-\mu)v} g(x, y) dy dx \\
 & = P(0, 0) \leq P(t, s) \leq P(1, 1) \\
 & = (1-\beta)(1-\mu) \\
 & \times \int_0^u \int_0^v f(A - \beta t, B - \mu s) g(A - \beta t, B - \mu s) dt ds \\
 & + (1-\beta)\mu \\
 & \times \int_0^u \int_0^v f(A - \beta t, B + (1-\mu)s) \\
 & \quad \times g(A - \beta t, B + (1-\mu)s) dt ds \\
 & + \beta(1-\mu) \\
 & \times \int_0^u \int_0^v f(A + (1-\beta)t, B - \mu s) \\
 & \quad \times g(A + (1-\beta)t, B - \mu s) dt ds \\
 & + \beta\mu \int_0^u \int_0^v f(A + (1-\beta)t, B + (1-\mu)s) \\
 & \quad \times g(A + (1-\beta)t, B + (1-\mu)s) dt ds \\
 & = \frac{(1-\beta)(1-\mu)}{\beta\mu} \int_{A-\beta u}^A \int_{B-\mu v}^B f(x, y) g(x, y) dx dy \\
 & + \frac{(1-\beta)\mu}{\beta(1-\mu)} \int_{A-\beta u}^A \int_B^{B+(1-\mu)v} f(x, y) g(x, y) dx dy \\
 & + \frac{\beta(1-\mu)}{(1-\beta)\mu} \int_A^{A+(1-\beta)u} \int_{B-\mu v}^B f(x, y) g(x, y) dx dy \\
 & + \frac{\beta\mu}{(1-\beta)(1-\mu)} \\
 & \times \int_A^{A+(1-\beta)u} \int_B^{B+(1-\mu)v} f(x, y) g(x, y) dx dy. \tag{35}
 \end{aligned}$$

Proof. Since f is coordinated convex and g is nonnegative, we see that $P(t, s)$ is coordinated convex on $[0, 1]^2$. Next, for each $(x, y) \in [0, u] \times [0, v]$, where $u \in [0, u_0]$, $v \in [0, v_0]$, it follows from Lemma 11 that $h(tx, ty) = (1-\beta)(1-\mu)f(A - \beta tx, B - \mu sy) + (1-\beta)\mu f(A - \beta tx, B + (1-\mu) sy) + \beta(1-\mu)f(A + (1-\beta) tx, B - \mu sy) + \beta\mu f(A + (1-\beta) tx, B + (1-\mu) sy)$ is

coordinated monotonic nondecreasing on $[0, 1]^2$. Using (26) we see that $P(t, s)$ is coordinated monotonic nondecreasing on $[0, 1]^2$, which completes the proof. $\square$

Theorem 20. Let f , A , B , u_0 , and v_0 be defined as in Lemma 11, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \leq 1$, $0 < \lambda < 1$, $0 < \mu < 1$, $\lambda + \mu \leq 1$, and let g be defined as in Theorem 12. Let $Q(t, s)$ be defined on $[0, 1]^2$ by

$$\begin{aligned} Q(t, s) &= \int_0^u \int_0^v [(1-\beta)(1-\mu) \\ &\quad \times f(A-\beta u + \beta x(1-t), B-\mu v + \mu(1-s)y) \\ &\quad \times g(A-\beta(u-x), B-\mu(v-y)) \\ &\quad + (1-\beta)\mu f(A-\beta u + \beta x(1-t), B+(1-\mu)v \\ &\quad \quad - (1-\mu)(1-s)y) \\ &\quad \times g(A-\beta(u-x), B+(1-\mu)(v-y)) \\ &\quad + \beta(1-\mu)f(A+(1-\beta)u - (1-\beta)x(1-t), \\ &\quad \quad B-\mu v + \mu(1-s)y) \\ &\quad \times g(A+(1-\beta)(u-x), B-\mu(v-y)) \\ &\quad + \beta\mu f(A+(1-\beta)u - (1-\beta)x(1-t), \\ &\quad \quad B+(1-\mu)v - (1-\mu)y(1-s)) \\ &\quad \times g(A+(1-\beta)(u-x), \\ &\quad \quad B+(1-\mu)(v-y))] dy dx, \end{aligned} \quad (36)$$

for some $u \in [0, u_0]$, $v \in [0, v_0]$. Then $Q(t, s)$ is coordinated convex and coordinated monotonic nondecreasing on $[0, 1]^2$, and

$$\begin{aligned} &\frac{(1-\beta)(1-\mu)}{\beta\mu} \int_{A-\beta u}^A \int_{B-\mu v}^B f(x, y) g(x, y) dx dy \\ &+ \frac{(1-\beta)\mu}{\beta(1-\mu)} \int_{A-\beta u}^A \int_B^{B+(1-\mu)v} f(x, y) g(x, y) dx dy \\ &+ \frac{\beta(1-\mu)}{(1-\beta)\mu} \int_A^{A+(1-\beta)u} \int_{B-\mu v}^B f(x, y) g(x, y) dx dy \\ &+ \frac{\beta\mu}{(1-\beta)(1-\mu)} \\ &\quad \times \int_A^{A+(1-\beta)u} \int_B^{B+(1-\mu)v} f(x, y) g(x, y) dx dy \\ &= Q(0, 0) \leq Q(t, s) \leq Q(1, 1) \\ &= [(1-\beta)(1-\mu)f(A-\beta u, B-\mu v) \\ &\quad + (1-\beta)\mu f(A-\beta u, B+(1-\mu)v) \\ &\quad + \beta(1-\mu)f(A+(1-\beta)u, B-\mu v) \\ &\quad + \beta\mu f(A+(1-\beta)u, B+(1-\mu)v)] \\ &\quad \times \int_{A-\beta u}^{A+(1-\beta)u} \int_{B-\mu v}^{B+(1-\mu)v} g(x, y) dx dy \\ &\leq [\alpha\lambda f(a, c) + (1-\alpha)\lambda f(b, c) + \alpha(1-\lambda)f(a, d) \\ &\quad + (1-\alpha)(1-\lambda)f(b, d)] \\ &\quad \times \int_{A-\beta u}^{A+(1-\beta)u} \int_{B-\mu v}^{B+(1-\mu)v} g(x, y) dx dy. \end{aligned} \quad (37)$$

Proof. That H is coordinated convex follows immediately from the coordinated convexity of f . Next, for each $(x, y) \in [0, u] \times [0, v]$, where $u \in [0, u_0]$, $v \in [0, v_0]$, it follows from Lemma 11 that $h(t, s) = (1-\beta)(1-\mu)f(A-\beta t, B-\mu s) + (1-\beta)\mu f(A-\beta t, B+(1-\mu)s) + \beta(1-\mu)f(A+(1-\beta)t, B-\mu s) + \beta\mu f(A+(1-\beta)t, B+(1-\mu)s)$ is coordinated monotonic nondecreasing on $[u, v] \in [0, u_0] \times [0, v_0]$. Since $k_1(t) = u - (1-t)x$ and $k_2(s) = v - (1-s)y$ are increasing on $[0, 1]$, respectively, we get that $h(k_1(t), k_2(s)) = (1-\beta)(1-\mu)f(A-\beta u + \beta x(1-t), B-\mu v + \mu(1-s)y) + (1-\beta)\mu f(A-\beta u + \beta x(1-t), B+(1-\mu)v - (1-\mu)(1-s)y) + \beta(1-\mu)f(A+(1-\beta)u - (1-\beta)x(1-t), B-\mu v + \mu(1-s)y) + \beta\mu f(A+(1-\beta)u - (1-\beta)x(1-t), B+(1-\mu)v - (1-\mu)(1-s)y)$ is coordinated monotonic nondecreasing on $[0, 1]^2$. Since g is nonnegative and satisfies (26), it follows that $Q(t, s)$ is coordinated monotonic nondecreasing on $[0, 1]^2$. Finally, the last inequality of (37) follows from (21), and the proof is completed. $\square$

Remark 21. Identity (15) is a special case of (34) if we choose $\alpha = \beta = \lambda = \mu = 1/2$, $u = u_0 = b - a$, $v = v_0 = d - c$.

Remark 22. Identity (17) is a special case of (36) if we choose $\alpha = \beta = \lambda = \mu = 1/2$, $u = u_0 = b - a$, $v = v_0 = d - c$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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