Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2014, Article ID 798074, 7 pages
http://dx.doi.org/10.1155/2014/798074

Research Article

On the Cardinality of the T_0-Topologies on a Finite Set

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Received 7 July 2013; Revised 5 February 2014; Accepted 5 February 2014; Published 31 March 2014

Academic Editor: Laszlo A. Szekely

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Let $T_0(n,k)$ be the number of all labeled T_0-topologies having k open sets that we can define on n points, and let $t_0(n,k)$ be the number of those which are nonhomeomorphic. In this paper, we compute these numbers for $k \geq 5 \cdot 2^{n-4}$ and arbitrary $n \geq 4$. The numbers $t_{n0}(n,k)$ of all unlabeled and non-T_0-topologies with k open sets are also given for $k \geq 2^{n-2}$.

1. Introduction

Let E be a finite set with n elements. The total number $T(n)$ of all labelled topologies one can define on E is a long standing open question. No reasonable explicit or recursive counting formula for $T(n)$ is known. Moreover, the number $T_0(n)$ of all labelled T_0-topologies definable on E is also an open problem. Recall that a T_0-topology is a topology satisfying the separate axiom: for all $x \neq y$, there is an open set containing one but not the other.

A classical and well known result is that each topology on E corresponds to a quasi-order on E (reflexive and transitive relation) and each T_0-topology corresponds to a partial order (reflexive, transitive, and antisymmetric relation). This bijective correspondence is due, independently, to Alexandroff [1] and Birkhoff [2]. On the other hand, several authors (Comtet [3], Evans et al. [4], Renteln [5] and others) have observed the following connexion between the $T(n)$ and the $T_0(n)$: $T(n) = \sum_{l=1}^{n} S_{n,l} T_0(l)$, where the $S_{n,l}$ are the Stirling numbers of the second kind. This formula was later refined by Erné [6] to $T^p(n) = \sum_{l=1}^{n} S_{n,l} T_0^p(l)$, where p is any topological property which is invariant under lattice isomorphisms. Numerically, the values $T(n)$ have been found for $n \leq 6$ by several authors (see, e.g., [3]), for $n \leq 7$ by Evans et al. [4], for $n \leq 9$ by Erné [6], for $n \leq 11$ by Das [7], and for $n \leq 14$ by Erné and Stege [8]. Later, the calculations were pursued by Brinkmann and McKay [9, 10] until $n = 18$.

Another approach is the study of the topologies by their number of open sets. For this, let $T(n,k)$ be the number of all labeled topologies with k open sets that we can define on E and let $T_0(n,k)$ be the number of those which are T_0. As for $T(n)$, there is no explicit formula giving the $T(n,k)$ or the $T_0(n,k)$, while some values are known. When k is large, Sharp [11] and Stephen [12] showed that there is no topology with $3 \cdot 2^{n-2} < k < 2^n$ open sets. In 1971, Stanley [13] enumerated the $T(n,k)$ for $k \geq 7 \cdot 2^{n-4}$ and arbitrary n, and in 2007, the author [14] extended the range to $k \geq 3 \cdot 2^{n-3}$. For small values of k, Erné and Stege [8, 15] determined the $T(n,k)$ and $T_0(n,k)$ for all k and $n \leq 12$. Independently, Benoumhani [16] computed by a direct method the $T(n,k)$ for $k \leq 12$ and arbitrary n. Later, the range was extended in [17] until $k \leq 17$.

For the unlabelled case, let $t(n,k)$ be the number of all unlabeled topologies having k open sets, let $t_0(n,k)$ be the number of those which are T_0, and let $t_{n0}(n,k)$ be the number of those which are non-T_0. Unfortunately, all these numbers are missing until now. However, some values are obtained. For example, Stanley [13] computed the $t_0(n,k)$ and the $t_{n0}(n,k)$ for $k \geq 7 \cdot 2^{n-4}$ and $k \geq 3 \cdot 2^{n-3}$, respectively. In the opposite sense, the numbers $t_0(n, n+k)$ are given for $k \leq 12$ by Erné and Stege [15], see also the paper [17] for $k \leq 8$.

Our present paper is a continuation of the previous effort in the field. Here we give the values $T_0(n,k)$ and $t_0(n,k)$ until $k \geq 5 \cdot 2^{n-4}$. For the $t_{n0}(n,k)$, the calculations are given until $k \geq 2^{n-2}$. Our approach is based on the notion of level partitions (Definition 2). Let $T_0^m(n,k)$ denote the number of all labeled T_0-topologies with k open sets and m levels; then, $T_0(n,k) = \sum_{m=1}^{n} T_0^m(n,k)$. In Section 2, we prove that if $k > 3 \cdot 2^{n-4}$; then, $m \leq 4$. The summation is then reduced to $T_0(n,k) = \sum_{m=1}^{4} T_0^m(n,k)$ and this reduction

permits us to enumerate, in Section 3, all the possibilities and find the desired values. The unlabeled case is studied in Section 4. In Lemma 13, we prove that any two T_0-topologies are homeomorphic if and only if their corresponding levels are also homeomorphic. This property enables us to calculate the desired numbers.

2. Basic Results

In this section, we prove that if τ is a T_0-topology with $k > 3 \cdot 2^{n-4}$ open sets, then τ has at most 4 levels.

Definition 1. Let τ be a topology on E. The nonempty open set A is called a minimal open set if, for all O in τ, we have

$$A \cap O = A \quad \text{or} \quad \emptyset. \tag{1}$$

In the lattice terminology, a minimal open set is an atom. Thereby, for any topology τ, we define inductively the decreasing sequence of the subsets E^j and the topologies τ^j on E^j as follows: for $j = 1$, we set $E^1 = E$, $\tau^1 = \tau$, and we denote by $A_1^1, \ldots, A_{n_1}^1$ the minimal opens sets of τ^1. We put $A^1 = \bigcup_{i=1}^{n_1} A_i^1$, $E^2 = E^1 - A^1$, $\tau^2 = \{O - A^1,\ O \in \tau^1\}$, and we denote by $A_1^2, \ldots, A_{n_2}^2$ the minimal open sets of τ^2. Again, $A^2 = \bigcup_{i=1}^{n_2} A_i^2$, $E^3 = E^2 - A^2$, $\tau^3 = \{O - A^2,\ O \in \tau^2\}$, and so on. This process should stop after at most n iterations. Let $U^j = \bigcup_{i=1}^{j} A^i$, and let F^j be the increasing family defined as follows:

$$F^0 = \{\emptyset\}, \qquad F^j = \left\{O \in \tau,\ O \subset U^j\right\}, \quad j \geq 1. \tag{2}$$

We can see that for all j, $U^j \in \tau$, $F^j = \tau|_{U^j}$, and then F^j is a topology on U^j. Now, we recall the notion of level.

Definition 2. Let τ be a topology on the finite set E, and let

$$L^0 = \{\emptyset\}, \qquad L^j = F^j - F^{j-1}, \quad j \geq 1. \tag{3}$$

For all $j \geq 0$, L^j is called the jth level of τ.

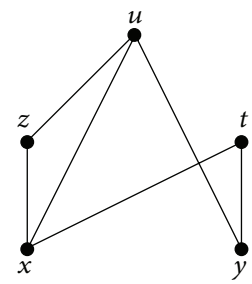
Lemma 3. *There exists a finite index $m \geq 1$, such that $\tau^{m+1} = \{\emptyset\}$, and τ^m is a Boolean algebra.*

The smallest integer m such that $\tau^{m+1} = \{\emptyset\}$ is the number of level of τ. The number m is at most n, and it is reached if and only if τ is the maximal chain. On the other hand, we observe that $\{L^j, j = 0, \ldots, m\}$ is a partition of τ, $\{A_1^j, \ldots, A_{n_j}^j\}$ is a partition of A^j, and $\{A_1^1, \ldots, A_{n_1}^1, A_1^2, \ldots, A_{n_2}^2, \ldots, A_1^m, \ldots, A_{n_m}^m\}$ and $\{A^1, A^2, \ldots, A^m\}$ are both partitions of E. Let us now give an example.

Example 4. Take $E = \{x, y, z, t, u\}$ and

$$\begin{aligned}\tau = \{&\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{x, y, z\}, \{x, y, t\}, \\ &\{x, y, z, t\}, \{x, y, z, u\}, E\}.\end{aligned} \tag{4}$$

We have $\tau^2 = \{\emptyset, \{z\}, \{t\}, \{z, t\}, \{z, u\}, E^2\}$, $\tau^3 = \{\emptyset, E^3\}$, and $\tau^4 = \{\emptyset\}$. Then τ has three levels with $L^1 = \{\{x\}, \{y\}, \{x, y\}\}$, $L^2 = \{\{x, z\}, \{x, y, z\}, \{x, y, t\}, \{x, y, z, t\}\}$, and $L^3 = \{\{x, y, z, u\}, E\}$. We see that $A^1 = \{x, y\}$, with $A_1^1 = \{x\}$ and $A_2^1 = \{y\}$, $A^2 = \{z, t\}$, with $A_1^2 = \{z\}$ and $A_2^2 = \{t\}$, and finally $A^3 = \{u\}$, with $A_1^3 = \{u\}$. The partially ordered set corresponding to τ is as follows:



(5)

Since the elements of a minimal open set cannot be distinguishable, we have the following.

Lemma 5. *If τ is T_0, then every minimal open set of τ is a singleton. Moreover, for all O in τ, the topology $\tau' = \{A - O, A \in \tau\}$ is also T_0.*

A consequence of Lemma 5 is the following.

Theorem 6. *Let τ be a topology with m levels. Then τ is T_0 if and only if τ^j is T_0 for all $j = 1, \ldots, m$.*

It follows again that a topology is T_0 if and only if all its blocks

$$\left\{A_1^1, \ldots, A_{n_1}^1, A_1^2, \ldots, A_{n_2}^2, \ldots, A_1^m, \ldots, A_{n_m}^m\right\} \tag{6}$$

are singletons and then $\sum_{j=1}^{m} n_j = n$. In Example 4, this equality is satisfied and of course τ is T_0.

Lemma 7. *Let τ be a T_0-topology with m levels. Then, τ has at least*

$$2^{n_1} + \cdots + 2^{n_m} - m + 1 \tag{7}$$

open sets and at most

$$2^{n_1-1} + 2^{n_1+n_2-2} + \cdots + 2^{n_1+\cdots+n_{m-1}-(m-1)} + 2^{n_1+\cdots+n_m-m+1} \tag{8}$$

open sets. In addition, these two bounds are optimal.

Proof. The jth level L^j is generated by $\{B_1, \ldots, B_{n_j}\}$, where $B_l = A_l^j \cup O_l$ and O_l is arbitrary from the previous level L^{j-1}. So, L^j contains at least $2^{n_j} - 1$ open sets, and then τ has at least $2^{n_1} + \cdots + 2^{n_m} - m + 1$ open sets. On the other hand, if we take $B_l = A_l^j \cup U^{j-1}$, then L^j contains exactly $2^{n_j} - 1$ open sets and the lower bound is optimal. Let us prove the upper bound. We see that each B_l has the form $A_l^j \bigcup (\bigcup_{i=1}^{j-1} O_i)$, where each O_i is a union of a certain number

of the minimal open sets $A_1^i, \ldots, A_{n_i}^i$. So each B_l generates at most $2^{n_1-1} \cdot 2^{n_2-1} \cdots 2^{n_{j-1}-1}$ open sets, and then L^j contains at most $(2^{n_j}-1)2^{n_1+\cdots+n_{j-1}-(j-1)}$ open sets. All these numbers added give the desired upper bound; in addition, if we take $B_l = A_l^j \bigcup (\bigcup_{i=1}^{j-1} A_1^i)$, then L^j contains exactly $2^{n_1+\cdots+n_j-(j-1)} - 2^{n_1+\cdots+n_{j-1}-(j-1)}$ open sets and the upper bound is reached. □

Lemma 8. *Let τ be a T_0-topology with m levels. Then τ has at most $(m+1)2^{n-m}$ open sets, and this bound is optimal.*

Proof. Since τ is T_0, then $\sum_{j=1}^{m} n_j = n$, and then, for all $j = 1, \ldots, m$,

$$2^{n_1+\cdots+n_j-j} \leq 2^{n-m}. \tag{9}$$

By injecting all these inequalities in the upper bound appearing in Lemma 7, we obtain the desired bound. The optimality is also a consequence of the optimality in Lemma 7. □

It is trivial to see that two T_0-topologies having different number of levels are also different, so we have the following.

Theorem 9. *For all integers $n \geq 1$ and $k \in [2, 2^n]$, we have*

$$T_0(n,k) = \sum_{m=1}^{n} T_0^m(n,k). \tag{10}$$

This last formula remains true in the non-T_0 case. For large values of k it reduces to the following.

Theorem 10. *Let $n \geq 4$. If $k > 3 \cdot 2^{n-4}$, then*

$$T_0(n,k) = \sum_{m=1}^{4} T_0^m(n,k). \tag{11}$$

Proof. As $3 \cdot 2^{n-4} < k \leq (m+1)2^{n-m}$, then m should be less or equal to 4. □

This reduction enables us to enumerate easily all the possibilities and then to find the numbers $T_0(n,k)$. This is the content of the next section.

3. Computing $T_0(n,k)$ until $k \geq 5 \cdot 2^{n-4}$

As mentioned before, the values $T_0(n,k)$ are known for $k \geq 3 \cdot 2^{n-3}$. So, we have to compute them only for k in the interval $[5 \cdot 2^{n-4}, 3 \cdot 2^{n-3})$. Equivalently, we have to compute the numbers $T_0^m(n,k)$, for $m = 2, 3, 4$. Details are given only in the case $m = 2$, the others are similar. Thereby, let τ be a T_0-topology with two levels and let $A^1 = \{x_1, \ldots, x_{n-i}\}$, $A^2 = \{y_1, \ldots, y_i\}$ be the associated partition. The level L^1 is generated by $\{x_1\}, \ldots, \{x_{n-i}\}$, and then it contains $2^{n-i} - 1$ open sets. The level L^2 is generated by $\{U_{y_1}, \ldots, U_{y_i}\}$, where $U_{y_i} = O_i \cup \{y_i\}$ and O_i is arbitrary from L^1. The question is then how to choose the O_i in order to get a topology τ such that $|\tau| \in [5 \cdot 2^{n-4}, 3 \cdot 2^{n-3})$? For this, we consider all the cases $i = 1, \ldots, n-1$ one by one and we use the notation $(n)_i = n(n-1)\cdots(n-i+1)$ if $1 \leq i \leq n$, and 0 otherwise.

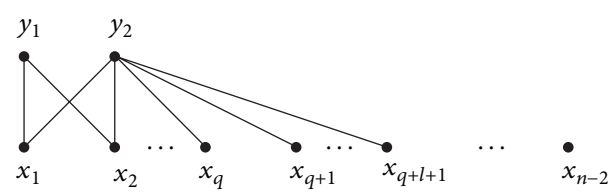
Case 1 ($(n_1, n_2) = (n-1, 1)$). In this case, τ has at least $2^{n-1}+1$ open sets which is outside the interval $[5 \cdot 2^{n-4}, 3 \cdot 2^{n-3})$; so there is no solution in this case.

Case 2 ($(n_1, n_2) = (n-2, 2)$). In this case, we prove that there are only 8 possibilities. For a given partition $A^1 = \{x_1, \ldots, x_{n-2}\}$, $A^2 = \{y_1, y_2\}$ of E, we try to reconstruct all the T_0-topologies τ having this partition and such that $|\tau|$ is in the desired interval. For this, we have to remark first that:

if one of the $y_i, i = 1, 2$ is adjoined to the singleton $\{x_j\}$, then $|\tau| > 2^{n-2} + 2^{n-3} = 3 \cdot 2^{n-3}$, which is outside the desired interval;

if the $y_i, i = 1, 2$ are both adjoined to more than three elements, then $|\tau| \leq 2^{n-2} + 3 \cdot 2^{n-6}$, which is also outside the desired interval.

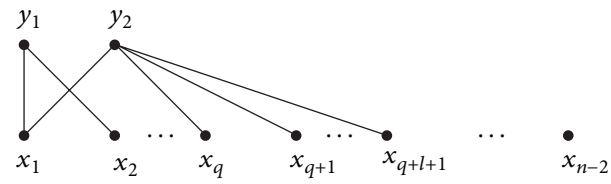
So, at least one of the $y_i, i = 1, 2$ should be adjoined to a set with two or three elements. Without loss of generality, suppose that y_1 is adjoined to $\{x_1, x_2\}$. Here we enumerate 3 possibilities. The first one is when y_2 is adjoined to $\{x_1, x_2, x_q, \ldots, x_{q+l+1}\}$, $l = 0, \ldots, n-6$. The corresponding poset is



(12)

Obviously, the contribution of the open set $U_{y_1} = \{x_1, x_2, y_1\}$ to the topology τ is 2^{n-4} open sets, that of $U_{y_2} = \{x_1, x_2, x_q, \ldots, x_{q+l+1}, y_2\}$ is 2^{n-6-l} open sets, and that of their union is also 2^{n-6-l} open sets. So, τ has $k = 5 \cdot 2^{n-4} + 2^{n-5-l}$ open sets and the number of such topologies is $2\binom{n}{n-2}\binom{n-2}{2}\binom{n-4}{l+2} = (n)_{l+6}/2(l+2)!$

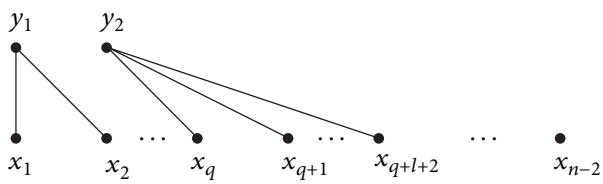
The second possibility is when y_2 is adjoined to $\{x_1, x_q, \ldots, x_{q+l+1}\}$, $l = 0, \ldots, n-6$, as follows:



(13)

Here the contribution of U_{y_1} is 2^{n-4}, that of U_{y_2} is 2^{n-5-l}, and that of their union is 2^{n-6-l}. So, the topology τ has $k = 5 \cdot 2^{n-4} + 3 \cdot 2^{n-6-l}$ open sets and the number of such topologies is $4\binom{n}{n-2}\binom{n-2}{2}\binom{n-4}{l+2} = (n)_{l+6}/(l+2)!$

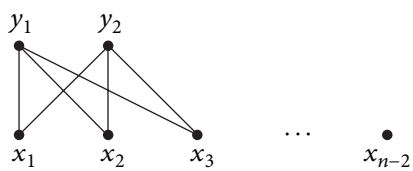
The third possibility is when y_2 is adjoined to $\{x_q, \ldots, x_{q+l+2}\}$, $l = 0, \ldots, n-7$, as follows:



(14)

The contribution of U_{y_1} is 2^{n-4}, that of U_{y_2} is 2^{n-5-l}, and that of their union is 2^{n-l-7}. So, the topology τ has $k = 5 \cdot 2^{n-4} + 5 \cdot 2^{n-7-l}$ open sets and the number of such topologies is $2\binom{n}{n-2}\binom{n-2}{2}\binom{n-4}{l+3} = (n)_{l+7}/2(l+3)!$

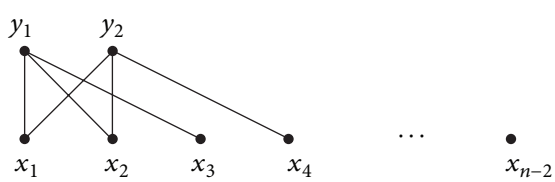
Suppose now that y_1 is adjoined to three elements $\{x_1, x_2, x_3\}$. Remark that if y_2 is adjoined to more than four elements, then $|\tau| \leq 2^{n-2} + 2^{n-5} + 2^{n-6}$, which is excluded. So, y_2 should be adjoined to three or four elements. When y_2 is adjoined to three elements, we find four possibilities. The first one is when y_2 is adjoined to $\{x_1, x_2, x_3\}$. The corresponding poset is



(15)

Here the contribution of U_{y_1} is 2^{n-5}, that of U_{y_2} is 2^{n-5}, and that of their union is also 2^{n-5}. So, τ has $k = 5 \cdot 2^{n-4} + 2^{n-5}$ open sets. The number of such topologies is $\binom{n}{n-2}\binom{n-2}{3} = (n)_5/12$.

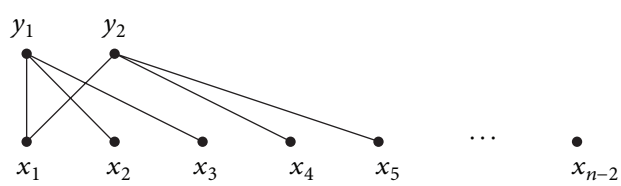
The second possibility is when y_2 is adjoined to $\{x_1, x_2, x_4\}$ as follows:



(16)

Here the contribution of U_{y_1} is 2^{n-5}, that of U_{y_2} is 2^{n-5}, and that of their union is 2^{n-6}. So, τ has $k = 5 \cdot 2^{n-4} + 2^{n-6}$ open sets. The number of such topologies is $12\binom{n}{n-2}\binom{n-2}{4} = (n)_6/4$.

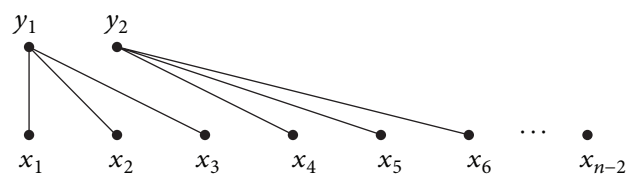
The third possibility is when y_2 is adjoined to $\{x_1, x_4, x_5\}$ as follows:



(17)

Here the contribution of U_{y_1} is 2^{n-5}, that of U_{y_2} is 2^{n-5}, and that of their union is 2^{n-7}. So, τ has $k = 5 \cdot 2^{n-4} + 2^{n-7}$ open sets. The number of such topologies is $30\binom{n}{n-2}\binom{n-2}{5} = (n)_7/8$.

The fourth possibility is when y_2 is adjoined to $\{x_4, x_5, x_6\}$ as follows:



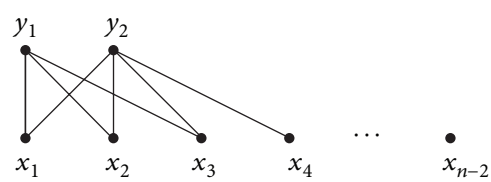
(18)

Here the contribution of U_{y_1} is 2^{n-5}, that of U_{y_2} is 2^{n-5}, and that of their union is 2^{n-8}. So, τ has $k = 5 \cdot 2^{n-4} + 2^{n-8}$ open sets. The number of such topologies is $20\binom{n}{n-2}\binom{n-2}{6} = (n)_8/72$.

When y_2 is adjoined to four elements $\{x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}\}$, we remark that if

$$\left|\{x_1, x_2, x_3\} \bigcap \{x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}\}\right| \leq 2, \tag{19}$$

then $|\tau| \leq 2^{n-2} + 2^{n-5} + 2^{n-6} + 2^{n-7}$, which is outside the desired interval; so the unique possibility is when y_2 is adjoined to $\{x_1, x_2, x_3, x_4\}$ as follows:



(20)

Here the contribution of U_{y_1} is 2^{n-5}, that of U_{y_2} is 2^{n-6}, and that of their union is also 2^{n-6}. So, τ has $k = 5 \cdot 2^{n-4}$ open sets and the number of such topologies is $8\binom{n}{n-2}\binom{n-2}{4} = (n)_6/6$.

Then we have proved that there are only 8 possibilities. We enumerate them in Table 1.

We use the same reasoning for all the rest of the cases $(n-i, i)$, $i = 3, \ldots, n-1$. Note that we obtain 13 possibilities for the case $(n-3, 3)$, also 13 for $(n-4, 4)$, 7 for $(n-5, 5)$, 6 for $(n-6, 6)$, and 5 for all the cases $(n-i, i)$, $i = 7, \ldots, n-1$. Since the machinery is the same as above, we have omitted the tedious details for the reader's convenience. The total number of T_0-topologies with two levels and $5 \cdot 2^{n-4} \leq k < 3 \cdot 2^{n-3}$ open sets is then given by Table 2.

Using the same approach for three levels, we obtain two possibilities for the case $(n-2, 1, 1)$, nine for $(n-3, 2, 1)$, one for $(n-3, 1, 2)$, eight for $(n-4, 3, 1)$, one for $(n-4, 1, 3)$, two for $(n-4, 2, 2)$, two for all the cases $(n-i, i-1, 1)$, $i = 5, \ldots, n-1$, and finally one for all the cases $(n-i, i-2, 2)$, $i = 5, \ldots, n-1$. Omitting again the tedious details, the numbers $T_0^3(n, k)$ with $5 \cdot 2^{n-4} \leq k < 3 \cdot 2^{n-3}$ are given by Table 3.

For four levels, we obtain only one possibility for the case $(n-3, 1, 1, 1)$. The numbers $T_0^4(n, k)$ with $5 \cdot 2^{n-4} \leq k < 3 \cdot 2^{n-3}$ are given by Table 4.

Table 1

Possibility	Number of open sets	Number of such topologies
$U_{y_1} = \{x_1, x_2, y_1\}$, $U_{y_2} = \{x_1, x_2, x_q, \ldots, x_{q+l+1}, y_2\}$	$k = 5 \cdot 2^{n-4} + 2^{n-5-l}$	$\frac{(n)_{l+6}}{2(l+2)!}$, $l = 0, \ldots, n-6$
$U_{y_1} = \{x_1, x_2, y_1\}$, $U_{y_2} = \{x_1, x_q, \ldots, x_{q+l+1}, y_2\}$	$k = 5 \cdot 2^{n-4} + 3 \cdot 2^{n-6-l}$	$\frac{(n)_{l+6}}{(l+2)!}$, $l = 0, \ldots, n-6$
$U_{y_1} = \{x_1, x_2, y_1\}$, $U_{y_2} = \{x_q, \ldots, x_{q+l+2}, y_2\}$	$k = 5 \cdot 2^{n-4} + 5 \cdot 2^{n-7-l}$	$\frac{(n)_{l+7}}{2(l+3)!}$, $l = 0, \ldots, n-7$
$U_{y_1} = \{x_1, x_2, x_3, y_1\}$, $U_{y_2} = \{x_1, x_2, x_3, y_2\}$	$k = 5 \cdot 2^{n-4} + 2^{n-5}$	$\frac{(n)_5}{12}$
$U_{y_1} = \{x_1, x_2, x_3, y_1\}$, $U_{y_2} = \{x_1, x_2, x_4, y_2\}$	$k = 5 \cdot 2^{n-4} + 2^{n-6}$	$\frac{(n)_6}{4}$
$U_{y_1} = \{x_1, x_2, x_3, y_1\}$, $U_{y_2} = \{x_1, x_4, x_5, y_2\}$	$k = 5 \cdot 2^{n-4} + 2^{n-7}$	$\frac{(n)_7}{8}$
$U_{y_1} = \{x_1, x_2, x_3, y_1\}$, $U_{y_2} = \{x_4, x_5, x_6, y_2\}$	$k = 5 \cdot 2^{n-4} + 2^{n-8}$	$\frac{(n)_8}{72}$
$U_{y_1} = \{x_1, x_2, x_3, y_1\}$, $U_{y_2} = \{x_1, x_2, x_3, x_4, y_2\}$	$k = 5 \cdot 2^{n-4}$	$\frac{(n)_6}{6}$

Table 2

k	$T_0^2(n,k)$
$5 \cdot 2^{n-4}$	$\frac{n^2 + 41n - 234}{36}(n)_6$
$5 \cdot 2^{n-4} + 2^{n-8}$	$\frac{n^2 - 11n + 214}{720}(n)_8$
$5 \cdot 2^{n-4} + 2^{n-7}$	$\frac{n^2 - 10n + 66}{120}(n)_7$
$5 \cdot 2^{n-4} + 2^{n-6}$	$\frac{n^2 + 15n - 84}{24}(n)_6$
$5 \cdot 2^{n-4} + 2^{n-5}$	$\frac{n^2 - 2n - 14}{6}(n)_5$
$5 \cdot 2^{n-4} + 2^{n-i}$, $i = 9, \ldots, n$	$\frac{n-3}{(i-2)!}(n)_{i+1}$
$5 \cdot 2^{n-4} + 3 \cdot 2^{n-i}$, $i = 6, \ldots, n$	$\frac{2}{(i-4)!}(n)_i$
$5 \cdot 2^{n-4} + 5 \cdot 2^{n-7}$	$\frac{5}{6}(n)_7$
$5 \cdot 2^{n-4} + 5 \cdot 2^{n-i}$, $i = 8, \ldots, n$	$\frac{2}{(i-4)!}(n)_i$

Table 3

k	$T_0^3(n,k)$
$5 \cdot 2^{n-4}$	$\frac{n^2 + 3n - 22}{4}(n)_5$
$5 \cdot 2^{n-4} + 2^{n-6}$	$\frac{n+3}{3}(n)_6$
$5 \cdot 2^{n-4} + 2^{n-5}$	$(n-1)(n)_5$
$5 \cdot 2^{n-4} + 2^{n-i}$, $i = 7, \ldots, n$	$\frac{2n-6}{(i-3)!}(n)_i$

Table 4

k	$T_0^4(n,k)$
$5 \cdot 2^{n-4}$	$(n)_4$

Adding all these computations, we arrive at the following.

Theorem 11. *Let $n \geq 4$. The number $T_0(n,k)$ of all labeled T_0-topologies on n points and having $5 \cdot 2^{n-4} \leq k < 3 \cdot 2^{n-3}$ open sets is given in Table 5.*

All the other T_0-topologies have either $k < 5 \cdot 2^{n-4}$ or $k \geq 3 \cdot 2^{n-3}$ open sets.

Corollary 12. *According to Erné's formula $T(n,k) = \sum_{l=1}^{n} S_{n,l} T_0(l,k)$, the total number of non-T_0-topologies on arbitrary n points and having k open sets may be known until $k \geq 5 \cdot 2^{n-5}$.*

4. Computing Unlabeled Topologies

As mentioned in the Introduction, the numbers $t_0(n,k)$ and $t_{n0}(n,k)$ are computed by Stanley [13], respectively, for $k \geq 7 \cdot 2^{n-4}$ and $k \geq 3 \cdot 2^{n-3}$. In this section, we extend the range until $k \geq 5 \cdot 2^{n-4}$ for the T_0-topologies and until $k \geq 2^{n-2}$ for the non-T_0-topologies. Then we rediscover Stanley's results. The following observation is the key of our result.

Lemma 13. *Let τ and τ' be two T_0-topologies on n points. Then τ and τ' are homeomorphic if and only if their corresponding levels are, respectively, homeomorphic.*

Proof. It is trivial to see that if the corresponding levels are, respectively, homeomorphic, then the two topologies are homeomorphic. Let us prove the nontrivial sense. Suppose that f is an homeomorphism from (E,τ) into (E,τ'), and let

$$\left\{A_1^1, \ldots, A_{n_1}^1, A_1^2, \ldots, A_{n_2}^2, \ldots, A_1^m, \ldots, A_{n_m}^m\right\}, \quad \left\{B_1^1, \ldots, B_{n_1'}^1, B_1^2, \ldots, B_{n_2'}^2, \ldots, B_1^{m'}, \ldots, B_{n_{m'}'}^{m'}\right\} \tag{21}$$

Table 5

k	$T_0(n,k)$
$5\cdot 2^{n-4}$	$\frac{n^2+41n-234}{36}(n)_6+\frac{n^2+3n-22}{4}(n)_5+(n)_4$
$5\cdot 2^{n-4}+2^{n-8}$	$\frac{n^2+n+178}{720}(n)_8$
$5\cdot 2^{n-4}+2^{n-7}$	$\frac{n^2+36}{120}(n)_7$
$5\cdot 2^{n-4}+2^{n-6}$	$\frac{n^2+23n-60}{24}(n)_6$
$5\cdot 2^{n-4}+2^{n-5}$	$\frac{n^2+4n-20}{6}(n)_5$
$5\cdot 2^{n-4}+2^{n-i},\quad i=9,\dots,n$	$\frac{n-3}{(i-2)!}(n)_{i+1}+\frac{2n-6}{(i-3)!}(n)_i$
$5\cdot 2^{n-4}+3\cdot 2^{n-i},\quad i=6,\dots,n$	$\frac{2}{(i-4)!}(n)_i$
$5\cdot 2^{n-4}+5\cdot 2^{n-7}$	$\frac{5}{6}(n)_7$
$5\cdot 2^{n-4}+5\cdot 2^{n-i},\quad i=8,\dots,n$	$\frac{2}{(i-4)!}(n)_i$

Table 6

k	$t_0(n,k)$
$5\cdot 2^{n-4}$	22
$5\cdot 2^{n-4}+1$	4
$5\cdot 2^{n-4}+2$	8
$5\cdot 2^{n-4}+2^{n-i}, 9\le i\le n-2$	10
$5\cdot 2^{n-4}+2^{n-8}$	16
$5\cdot 2^{n-4}+2^{n-7}$	12
$5\cdot 2^{n-4}+2^{n-6}$	18
$5\cdot 2^{n-4}+2^{n-5}$	16
$5\cdot 2^{n-4}+3\cdot 2^{n-i}, 6\le i\le n$	2
$5\cdot 2^{n-4}+5\cdot 2^{n-7}$	6
$5\cdot 2^{n-4}+5\cdot 2^{n-i}, 8\le i\le n$	4
$3\cdot 2^{n-3}$	9
$3\cdot 2^{n-3}+1$	2
$3\cdot 2^{n-3}+2^{n-j}, 7\le j\le n-1$	4
$3\cdot 2^{n-3}+3\cdot 2^{n-j}, 7\le j\le n-1$	2
$25\cdot 2^{n-6}$	7
$13\cdot 2^{n-5}$	6
$27\cdot 2^{n-6}$	3
$7\cdot 2^{n-4}$	5
$15\cdot 2^{n-5}$	2
2^{n-1}	2
$2^{n-1}+2^{n-j}, 5\le j\le n$	2
$9\cdot 2^{n-4}$	3
$5\cdot 2^{n-3}$	2
$3\cdot 2^{n-2}$	1
2^n	1

be, respectively, their corresponding partitions. Since the first blocks $A_1^1,\dots,A_{n_1}^1$ are open and singletons, then $f(\bigcup_{i=1}^{n_1}A_i^1)=\bigcup_{i=1}^{n_1'}B_i^1$ and so $n_1=n_1'$. But f is also an homeomorphism from (E^2,τ^2) into (E'^2,τ'^2) and for the same reason, we obtain $n_2=n_2'$. We repeat this process to obtain that $m=m'$, and $n_j=n_j'$ for all j. On the other hand, let $A^j=\bigcup_{i=1}^{n_j}A_i^j$, $U^j=\bigcup_{i=1}^{j}A^j$ and $B^j=\bigcup_{i=1}^{n_j}B_i^j$, $Y^j=\bigcup_{i=1}^{j}B^j$. As the above, we have $f(A^j)=B^j$, and then $f(U^j)=Y^j$ for all j. Since U^j and Y^j are open with respect to τ and τ', respectively, f is also an homeomorphism from the subspace $(U^j,\tau|_{U^j})$ into the subspace $(Y^j,\tau'|_{Y^j})$. But $\tau|_{U^j}=\bigcup_{i=0}^{j}L^i$ and $\tau'|_{Y^j}=\bigcup_{i=0}^{j}L'^i$; so for $j=1$, we obtain that L^1 is homeomorphic to L'^1. For $j=2$, we have $L^1\cup L^2$ which is homeomorphic to $L'^1\cup L'^2$ and then L^2 which is homeomorphic to L'^2. We repeat this process to obtain that L^j is homeomorphic to L'^j for all j, and the proof is done. □

Corollary 14. *Lemma 13 holds if τ and τ' are non-T_0.*

It follows that the number of unlabelled T_0-topologies is exactly the number of the possibilities given in the labelled case (see the above section). So, omitting the easy computations for $n\le 9$, we obtain the following.

Theorem 15. *Let $n\ge 10$. The number $t_0(n,k)$ of all unlabeled T_0-topologies, with $k\ge 5\cdot 2^{n-4}$ open sets, is detailed in Table 6, totaling $24n-43$ unlabeled T_0-topologies. All the other unlabelled T_0-topologies have $k<5\cdot 2^{n-4}$ open sets.*

For the non-T_0-topologies, we obtain the following.

Table 7

k	$t_{n0}(n,k)$
2^{n-2}	9
$2^{n-2}+1$	4
$2^{n-2}+2^{n-j}, 6 \le j \le n-1$	6
$2^{n-2}+2^{n-5}$	8
$2^{n-2}+2^{n-4}$	6
$2^{n-2}+2^{n-3}$	3
2^{n-1}	1

Theorem 16. *Let $n \ge 6$. The number $t_{n0}(n,k)$ of all unlabeled and non-T_0-topologies, with $k \ge 2^{n-2}$ open sets, is given in Table 7, totaling $6n-5$ unlabeled non-T_0-topologies. All the other unlabelled and non-T_0-topologies have $k < 2^{n-2}$ open sets.*

5. Conclusion

In this paper, we have computed the numbers $T_0(n,k)$, $t_0(n,k)$, and $t_{n0}(n,k)$ via the notion of level and the numbers $T_0^m(n,k)$. Unfortunately, there is no explicit formula giving the $T_0^m(n,k)$, while some cases are already known. For example, there is one T_0-topology with one level (the discrete topology) and $\sum_{n_1=1}^{n-1}\binom{n}{n_1}(2^{n_1}-1)^{n-n_1}$ T_0-topologies with two levels. In the opposite sense, there are $((n-1)^2/2)n!$ T_0-topologies with $n-1$ levels and $n!$ T_0-topologies with n levels. But for $3 \le m \le n-2$, the calculation of the $T_0^m(n,k)$ seems to be a very hard task. However, for small values of n, these calculations can be programmed and then computed by computer. This is the project of a forthcoming paper.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author is very grateful to the anonymous referee for her/his corrections, valuable suggestions, and comments. This work is supported by the second project of the King Khalid University, under the Project no. 208 (1433).

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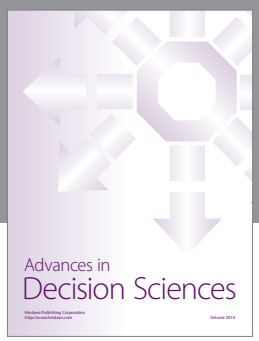

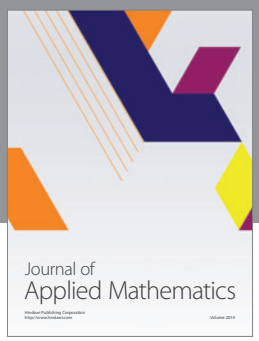

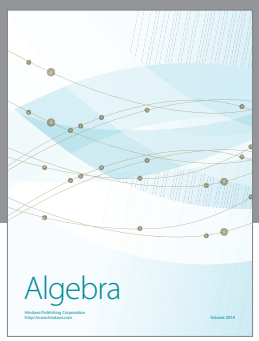

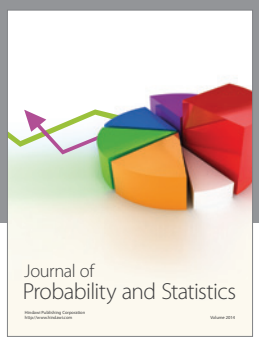

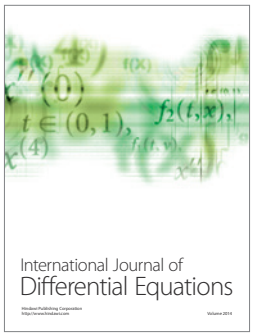

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