Research Article

A Single Sweep AGE Algorithm on a Variable Mesh Based on Off-Step Discretization for the Solution of Nonlinear Burgers' Equation

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We discuss a new single sweep alternating group explicit iteration method, along with a third-order numerical method based on off-step discretization on a variable mesh to solve the nonlinear ordinary differential equation $y'' = f(x, y, y')$ subject to given natural boundary conditions. Using the proposed method, we have solved Burgers' equation both in singular and nonsingular cases, which is the main attraction of our work. The convergence of the proposed method is discussed in detail. We compared the results of the proposed iteration method with the results of the corresponding double sweep alternating group explicit iteration methods to demonstrate computationally the efficiency of the proposed method.

1. Introduction

Consider the general nonlinear ordinary differential equation

$$y'' = f(x, y, y'), \quad a < x < b \quad (1)$$

subject to essential boundary conditions

$$y(a) = A, \quad y(b) = B, \quad (2)$$

where $-\infty < a < b < \infty$, A, B are finite constants.

We assume that for $x \in [a, b]$, $-\infty < y, z < \infty$

- (i) $f(x, y, z)$ is continuous,
- (ii) $\partial f / \partial y$ and $\partial f / \partial z$ exist and are continuous,
- (iii) $\partial f / \partial y > 0$ and $|\partial f / \partial z| < W$ for some positive constant W .

These conditions ensure that the boundary value problem (1) and (2) possesses a unique solution (see Keller [1]).

With the advent of parallel computers, scientists are focusing on developing finite difference methods with the property

of parallelism. Working on this, in the early 1980s, Evans [2, 3] introduced the Group Explicit methods for large linear system of equations. Further he discussed the Alternating Group Explicit (AGE) method to solve periodic parabolic equations in a coupled manner. Mohanty and Evans applied AGE method along with various high order methods [4, 5] for the solution of two-point boundary value problems. Later, Sukon and Evans [6] introduced a Two-parameter Alternating Group Explicit (TAGE) method for the two-point boundary value problem with a lower order accuracy scheme. In 2003 Mohanty et al. [7] discussed the application of TAGE method for nonlinear singular two point boundary value problems using a fourth-order difference scheme. In 1990, Evans introduced the Coupled Alternating Group Explicit method [8] and applied it to periodic parabolic equations. Many scientists are applying these parallel algorithms to solve ordinary and partial differential equations [9–11].

Recently, Mohanty [12] has proposed a high order variable mesh method for nonlinear two-point boundary value problem. Mohanty and Khosla [13, 14] also devised a new third-order accurate arithmetic average variable mesh method for

the solution of the boundary value problem (1) and (2), using three grid points, which is applicable to both singular and nonsingular problems. No special technique is required to handle singular problems. They also discussed the application of two-parameter double sweep alternating group explicit methods. Here, we discuss a new single sweep group explicit iteration method along with a third-order accurate variable mesh method based on two extra off-step grid points for the solution of the boundary value problem (1) and (2).

2. Off-Step Discretization

Consider the solution interval $[a, b]$ with a nonuniform mesh such that $a = x_0 < x_1 < \dots < x_N < x_{N+1} = b$. Let $h_k = x_k - x_{k-1}$, $k = 1(1)N+1$, be the mesh size and let $\sigma_k = h_{k+1}/h_k > 0$, $k = 1(1)N$, be the mesh ratio. Grid points are given by $x_i = x_0 + \sum_{k=1}^i h_k$, $i = 1(1)N+1$. Let $Y_k = y(x_k)$ be the exact solution of y at the grid point x_k and approximated by y_k . Let $x_{k+1/2} = x_k + (\sigma_k h_k/2)$ and $x_{k-1/2} = x_k - (h_k/2)$.

The new third-order method is described as follows.

For $y'' = f$, we first obtain the off-step discretization

$$\begin{aligned} Y_{k+1} - (1 + \sigma_k) Y_k + \sigma_k Y_{k-1} \\ = \frac{\sigma_k h_k^2}{3} \left[\sigma_k f_{k+1/2} + \frac{(1 + \sigma_k)}{2} f_k + f_{k-1/2} \right] + T_k, \end{aligned} \quad (3)$$

where $f_{k+1/2} = y''(x_{k+1/2})$, $f_{k-1/2} = y''(x_{k-1/2})$, and $T_k = O(h_k^5)$.

Now, let

$$\bar{Y}_{k+1/2} = \frac{1}{2} (Y_{k+1} + Y_k), \quad (4a)$$

$$\bar{Y}_{k-1/2} = \frac{1}{2} (Y_{k-1} + Y_k), \quad (4b)$$

$$\bar{Y}'_{k+1/2} = \frac{1}{\sigma_k h_k} (Y_{k+1} - Y_k), \quad (4c)$$

$$\bar{Y}'_{k-1/2} = \frac{1}{h_k} (Y_k - Y_{k-1}), \quad (4d)$$

$$\bar{Y}'_k = \frac{(Y_{k+1} - (1 - \sigma_k^2) Y_k - \sigma_k^2 Y_{k-1})}{(h_k \sigma_k (\sigma_k + 1))}, \quad (4e)$$

$$\bar{f}_{k+1/2} = f(x_{k+1/2}, \bar{Y}_{k+1/2}, \bar{Y}'_{k+1/2}), \quad (4f)$$

$$\bar{f}_{k-1/2} = f(x_{k-1/2}, \bar{Y}_{k-1/2}, \bar{Y}'_{k-1/2}), \quad (4g)$$

$$\hat{Y}_k = Y_k - (1 - \sigma_k + \sigma_k^2) \frac{h_k^2}{8} (\bar{f}_{k+1/2} + \bar{f}_{k-1/2}), \quad (4h)$$

$$\hat{Y}'_k = \bar{Y}'_k - \frac{(1 + \sigma_k + \sigma_k^2) h_k}{6(1 + \sigma_k)} (\bar{f}_{k+1/2} - \bar{f}_{k-1/2}), \quad (4i)$$

$$\hat{f}_k = f(x_k, \hat{Y}_k, \hat{Y}'_k). \quad (4j)$$

Then, at each interior grid point x_k , $k = 1(1)N$, the proposed differential equation (1) is discretized by

$$\begin{aligned} Y_{k+1} - (1 + \sigma_k) Y_k + \sigma_k Y_{k-1} \\ = \frac{\sigma_k h_k^2}{3} \left[\sigma_k \bar{f}_{k+1/2} + \left(\frac{1 + \sigma_k}{2} \right) \hat{f}_k + \bar{f}_{k-1/2} \right] + \bar{T}_k, \end{aligned} \quad \sigma_k \neq 1. \quad (5)$$

With the help of (3), from (5) it is easy to verify that $\bar{T}_k = O(h_k^5)$, provided $\sigma_k \neq 1$. However, for uniform mesh $\sigma_k = 1$, the local truncation error associated with (5) becomes $\bar{T}_k = O(h_k^6)$ (see Chawla and Shivakumar [15]).

Note that the proposed method (5) is applicable to both singular and nonsingular problems. We do not require any special technique to handle singular problems (see Mohanty [13]). Further note that, the boundary values are given by $y_0 = Y_0 = A$ and $y_{N+1} = Y_{N+1} = B$. If the differential equation (1) is linear we use the proposed single sweep AGE iterative method and in nonlinear case, we use the Newton-AGE iterative method to obtain the solution.

3. Application to Singular Problems

3.1. Linear Singular Problems. We now discuss the application of the proposed numerical method (5) to the linear differential equation with variable coefficients

$$y'' = D(x) y' + g(x), \quad 0 < x < 1, \quad (6)$$

where $g(x)$ represents a forcing function.

For $D(x) = -\alpha/x$ and for $\alpha = 1$ and 2 , the equation above represents singular equation in cylindrical and spherical symmetry, respectively.

Let us denote

$$D_k = D(x_k), \quad D_{k\pm 1/2} = D(x_{k\pm 1/2}),$$

$$g_k = g(x_k), \quad g_{k\pm 1/2} = g(x_{k\pm 1/2}), \quad \sigma_k = \frac{h_{k+1}}{h_k}, \quad (7)$$

$$R_k = \frac{(1 + \sigma_k + \sigma_k^2)}{12}.$$

Therefore applying the method (5) to the differential equation (6) and neglecting error term, we obtain a linear difference equation of the form

$$a_k y_{k-1} + 2b_k y_k + c_k y_{k+1} = J_k, \quad k = 1(1)N, \quad (8)$$

where

$$\begin{aligned} a_k &= -\sigma_k \left[1 + \frac{h_k}{3} (1 + D_k R_k h_k) D_{k-1/2} + \frac{\sigma_k h_k}{6} D_k \right] \\ 2b_k &= \left[1 + \sigma_k - \frac{h_k}{3} (\sigma_k - D_k R_k h_k) D_{k+1/2} \right. \\ &\quad \left. + \frac{\sigma_k h_k}{3} (1 + D_k R_k h_k) D_{k-1/2} \right. \\ &\quad \left. - \frac{h_k}{6} (1 - \sigma_k^2) D_k \right] \end{aligned}$$

$$c_k = \left[-1 + \frac{h_k}{3} (\sigma_k - D_k R_k h_k) D_{k+1/2} + \frac{h_k}{6} D_k \right]$$

$$J_k = -\frac{\sigma_k h_k^2}{3} \left[\left(\frac{1 + \sigma_k}{2} \right) g_k + (\sigma_k - D_k R_k h_k) g_{k+1/2} + (1 + D_k R_k h_k) g_{k-1/2} \right]. \quad (9)$$

The linear difference scheme (8) has a local truncation error of $O(h_k^5)$ and is free from the terms $1/x_{k\pm 1}$ and therefore can very easily be solved for $k = 1(1)N$ in the region $x \in (0, 1)$.

3.2. Nonlinear Singular Problems. Now we consider the application to nonlinear differential equation (1). Neglecting the error term, we may rewrite the nonlinear difference equation (5) as

$$y_{k+1} - (1 + \sigma_k) y_k + \sigma_k y_{k-1} = \frac{\sigma_k h_k^2}{3} \left[\sigma_k \bar{f}_{k+1/2} + \left(\frac{1 + \sigma_k}{2} \right) \hat{f}_k + \bar{f}_{k-1/2} \right]. \quad (10)$$

Let

$$\begin{aligned} \phi_k &\equiv -y_{k+1} + (1 + \sigma_k) y_k - \sigma_k y_{k-1} \\ &+ \frac{\sigma_k h_k^2}{3} \left[\sigma_k \bar{f}_{k+1/2} + \left(\frac{1 + \sigma_k}{2} \right) \hat{f}_k + \bar{f}_{k-1/2} \right] \\ &\equiv \phi_k(y_{k-1}, y_k, y_{k+1}), \quad k = 1(1)N, \quad \sigma_k \neq 1. \end{aligned} \quad (11)$$

We denote

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}_{N \times 1}, \quad \boldsymbol{\phi}(\mathbf{y}) = \begin{bmatrix} \phi_1(\mathbf{y}) \\ \phi_2(\mathbf{y}) \\ \vdots \\ \phi_N(\mathbf{y}) \end{bmatrix}_{N \times 1}, \quad (12)$$

$$a_k(\mathbf{y}) = \frac{\partial \phi_k}{\partial y_{k-1}}, \quad k = 2(1)N$$

$$2b_k(\mathbf{y}) = \frac{\partial \phi_k}{\partial y_k}, \quad k = 1(1)N$$

$$c_k(\mathbf{y}) = \frac{\partial \phi_k}{\partial y_{k+1}}, \quad k = 1(1)N-1.$$

The Jacobian $\boldsymbol{\phi}(\mathbf{y})$ may be represented as

$$\mathbf{T} \equiv \frac{\partial \boldsymbol{\phi}(\mathbf{y})}{\partial \mathbf{y}} = \begin{bmatrix} 2b_1(\mathbf{y}) & c_1(\mathbf{y}) & & & 0 \\ a_2(\mathbf{y}) & 2b_2(\mathbf{y}) & c_2(\mathbf{y}) & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{N-1}(\mathbf{y}) \\ 0 & & & a_N(\mathbf{y}) & 2b_N(\mathbf{y}) \end{bmatrix}_{N \times N}. \quad (13)$$

4. Single Sweep AGE Method

4.1. Description of the Method. In this section we discuss the single sweep AGE iteration method. Using the boundary conditions $y_0 = A$, $y_{N+1} = B$, the linear difference equation (8) may be written in the matrix form as

$$\mathbf{A}\mathbf{y} = \mathbf{R}\mathbf{H}, \quad (14)$$

where

$$\mathbf{A} = \begin{bmatrix} 2b_1 & c_1 & & & 0 \\ a_2 & 2b_2 & c_2 & & \\ & & \ddots & & \\ & & & a_{N-1} & 2b_{N-1} & c_{N-1} \\ 0 & & & & a_N & 2b_N \end{bmatrix}_{N \times N},$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}_{N \times 1}, \quad (15)$$

$$\mathbf{R}\mathbf{H} = \begin{bmatrix} J_1 - a_1 y_0 \\ J_2 \\ \vdots \\ J_N - c_N y_{N+1} \end{bmatrix}_{N \times 1} = \begin{bmatrix} RH_1 \\ RH_2 \\ \vdots \\ RH_N \end{bmatrix}_{N \times 1} \quad (\text{say}).$$

To implement the single sweep AGE iterative method, we split the coefficient matrix $\mathbf{A}$ into two submatrices $\mathbf{A} = \mathbf{G}_1 + \mathbf{G}_2$, where $\mathbf{G}_1$ and $\mathbf{G}_2$ satisfy the following conditions.

- (i) $\mathbf{G}_1 + \omega_1 \mathbf{I}$ and $\mathbf{G}_2 + \omega_2 \mathbf{I}$ are nonsingular for suitable choice of $\omega_1 > 0$ and $\omega_2 > 0$.
- (ii) For any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and $\omega_1 > 0$, $\omega_2 > 0$, it is "convenient" to solve the system explicitly; that is, $\mathbf{z}_1 = (\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1} \mathbf{v}_1$ and $\mathbf{z}_2 = (\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \mathbf{v}_2$ for vectors $\mathbf{z}_1$ and $\mathbf{z}_2$, respectively.

We will be concerned here with the situation where $\mathbf{G}_1$ and $\mathbf{G}_2$ are small (2×2) block systems.

Now we discuss the case when N is even (with $x_0 = 0$, $x_{N+1} = 1$).

Let

$$\mathbf{G}_1 = \begin{bmatrix} \boxed{b_1 \ c_1} & & & & 0 \\ a_2 \ b_2 & & & & \\ & \boxed{b_3 \ c_3} & & & \\ a_4 \ b_4 & & \ddots & & \\ & & & \boxed{b_{N-1} \ c_{N-1}} & \\ 0 & & & a_N \ b_N \end{bmatrix}_{N \times N},$$

$$\mathbf{G}_2 = \begin{bmatrix} \boxed{b_1} & & & & \mathbf{0} \\ & \boxed{\begin{matrix} b_2 & c_2 \\ a_3 & b_3 \end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} b_{N-2} & c_{N-2} \\ a_{N-1} & b_{N-1} \end{matrix}} & \\ \mathbf{0} & & & & \boxed{b_N} \end{bmatrix}_{N \times N}. \quad (16)$$

So that the system (14) can be rewritten as

$$(\mathbf{G}_1 + \mathbf{G}_2) \mathbf{y} = \mathbf{RH}. \quad (17)$$

Then a two-parameter AGE method for solving the afore mentioned system may be written as

$$\begin{aligned} (\mathbf{G}_1 + \omega_1 \mathbf{I}) \mathbf{z}^{(s)} \\ = \mathbf{RH} - (\mathbf{G}_2 - \omega_1 \mathbf{I}) \mathbf{y}^{(s)}, \quad s = 0, 1, 2, \dots, \end{aligned} \quad (18)$$

$$\begin{aligned} (\mathbf{G}_2 + \omega_2 \mathbf{I}) \mathbf{y}^{(s+1)} \\ = \mathbf{RH} - (\mathbf{G}_1 - \omega_2 \mathbf{I}) \mathbf{z}^{(s)}, \quad s = 0, 1, 2, \dots, \end{aligned} \quad (19)$$

where $\mathbf{z}^{(s)}$ is an intermediate vector.

Eliminating $\mathbf{z}^{(s)}$ and combining (18) and (19), we obtain the iterative method

$$\begin{aligned} (\mathbf{G}_2 + \omega_2 \mathbf{I}) \mathbf{y}^{(s+1)} \\ = [\mathbf{I} - (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}] (\mathbf{G}_2 - \omega_1 \mathbf{I}) \mathbf{y}^{(s)} \\ + (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1} \mathbf{RH}, \quad s = 0, 1, 2, \dots, \end{aligned} \quad (20)$$

or

$$\mathbf{u}^{(s+1)} = \mathbf{T}_w \mathbf{y}^{(s)} + \mathbf{RH}_w, \quad s = 0, 1, 2, \dots, \quad (21)$$

where

$$\begin{aligned} \mathbf{T}_w &= (\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \\ &\times [(\mathbf{G}_2 - \omega_1 \mathbf{I}) \\ &- (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}(\mathbf{G}_2 - \omega_1 \mathbf{I})], \end{aligned} \quad (22)$$

$$\mathbf{RH}_w = (\omega_1 + \omega_2)(\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1}(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1} \mathbf{RH}.$$

The new iterative method (20) or (21) is called the two-parameter single sweep AGE iterative method and the matrix $\mathbf{T}_w$ is called the iteration matrix.

Now we discuss the single sweep AGE algorithm, when N is even.

For simplicity, we denote

$$\begin{aligned} p_k &= b_k + \omega_1, & q_k &= b_k - \omega_1, & r_k &= b_k + \omega_2 \\ \text{for } k &= 1(1)N, & \text{for } (p_k p_{k+1} - c_k a_{k+1}) &\neq 0, \end{aligned} \quad (23)$$

and we define $d_k = 1/(p_k p_{k+1} - c_k a_{k+1})$ for $k = 1(1)N - 1$.

By carrying out the necessary algebra in (20), we obtain the following algorithm.

For $k = 1$,

$$\begin{aligned} y_1^{(s+1)} &= (q_1 y_1^{(s)} - (\omega_1 + \omega_2) d_1 \\ &\times [-c_1 q_2 y_2^{(s)} - c_1 c_2 y_3^{(s)} + p_2 q_1 y_1^{(s)} \\ &+ c_1 R H_2 - p_2 R H_1]) \\ &\times (r_1)^{-1}, \quad s = 0, 1, 2, \dots \end{aligned} \quad (24)$$

For $k = 2(2)N - 2$.

Let

$$\begin{aligned} \Delta &= r_k r_{k+1} - c_k a_{k+1} \neq 0, \\ S_1 &= q_k y_k^{(s)} + c_k y_{k+1}^{(s)} - (\omega_1 + \omega_2) d_{k-1} \\ &\times [-a_{k-1} a_k y_{k-2}^{(s)} - a_k q_{k-1} y_{k-1}^{(s)} + p_{k-1} q_k y_k^{(s)} \\ &+ p_{k-1} c_k y_{k+1}^{(s)} + a_k R H_{k-1} - p_{k-1} R H_k], \\ S_2 &= a_{k+1} y_k^{(s)} + q_{k+1} y_{k+1}^{(s)} - (\omega_1 + \omega_2) d_{k+1} \\ &\times [p_{k+2} a_{k+1} y_k^{(s)} + p_{k+2} q_{k+1} y_{k+1}^{(s)} \\ &- c_{k+1} q_{k+2} y_{k+2}^{(s)} - c_{k+1} c_{k+2} y_{k+3}^{(s)} \\ &- p_{k+2} R H_{k+1} + c_{k+1} R H_{k+2}], \end{aligned} \quad (25)$$

and then

$$\begin{aligned} y_k^{(s+1)} &= \frac{(S_1 r_{k+1} - S_2 c_k)}{\Delta}, \quad s = 0, 1, 2, \dots \\ y_{k+1}^{(s+1)} &= \frac{(S_2 r_k - S_1 a_{k+1})}{\Delta}, \quad s = 0, 1, 2, \dots \end{aligned} \quad (26)$$

Finally, for $k = N$,

$$\begin{aligned} y_N^{(s+1)} &= (q_N y_N^{(s)} - (\omega_1 + \omega_2) d_{N-1} \\ &\times [-a_{N-1} a_N y_{N-2}^{(s)} - a_N q_{N-1} y_{N-1}^{(s)} + p_{N-1} q_N y_N^{(s)} \\ &+ a_N R H_{N-1} - p_{N-1} R H_N]) \\ &\times (r_N)^{-1}, \quad s = 0, 1, 2, \dots \end{aligned} \quad (27)$$

Similarly, we can write the single sweep AGE algorithm when N is odd.

4.2. Convergence Analysis. The single sweep AGE iteration method is given by

$$\begin{aligned} (\mathbf{G}_2 + \omega_2 \mathbf{I}) \mathbf{y}^{(s+1)} \\ = [\mathbf{I} - (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}] (\mathbf{G}_2 - \omega_1 \mathbf{I}) \mathbf{y}^{(s)} \\ + (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1} \mathbf{RH}, \quad s = 0, 1, 2, \dots, \end{aligned} \quad (28)$$

or

$$\mathbf{y}^{(s+1)} = \mathbf{T}_w \mathbf{y}^{(s)} + \mathbf{R} \mathbf{H}_w, \quad s = 0, 1, 2, \dots, \quad (29)$$

where

$$\begin{aligned} \mathbf{T}_w &= (\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \\ &\times [(\mathbf{G}_2 - \omega_1 \mathbf{I}) \\ &- (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}(\mathbf{G}_2 - \omega_1 \mathbf{I})], \end{aligned} \quad (30)$$

$$\mathbf{R} \mathbf{H}_w = (\omega_1 + \omega_2)(\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1}(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1} \mathbf{R} \mathbf{H}.$$

The matrix $\mathbf{T}_w$ is called the iteration matrix.

To prove the convergence of the method, we need to prove that $S(\mathbf{T}_w) \leq 1$, where $S(\mathbf{T}_w)$ denotes the spectral radius of $\mathbf{T}_w$.

Lemma 1. *Let h_k be sufficiently small. Then, the eigenvalues of $\mathbf{G}_1$ and $\mathbf{G}_2$ are all real.*

Proof. Consider

$$\begin{aligned} a_k &= -\sigma_k \left[1 + \frac{h_k}{3} (1 + D_k R_k h_k) D_{k-1/2} + \frac{\sigma_k h_k}{6} D_k \right] \\ &= -\sigma_k - O(h_k) < 0, \quad \text{for sufficiently small } h_k, \\ c_k &= \left[-1 + \frac{h_k}{3} (\sigma_k - D_k R_k h_k) D_{k+1/2} + \frac{h_k}{6} D_k \right] \\ &= -1 + O(h_k) < 0, \quad \text{for sufficiently small } h_k. \end{aligned} \quad (31)$$

Therefore, we have $a_{k+1} c_k > 0$, for $k = 1(2)N - 1$.

Let λ_i , $i = 1(1)N$, be the eigenvalues of $\mathbf{G}_1$. Then λ_i 's are the roots of the quadratic equation

$$\lambda_i^2 - (b_i + b_{i+1}) \lambda_i + (b_i b_{i+1} - a_{i+1} c_i) = 0. \quad (32)$$

Simplifying, we get

$$\lambda_i = \frac{1}{2} \left[(b_i + b_{i+1}) \pm \sqrt{(b_i - b_{i+1})^2 + 4a_{i+1} c_i} \right]. \quad (33)$$

The discriminants of the quadratic equations are

$$(b_i - b_{i+1})^2 + 4a_{i+1} c_i > 0, \quad k = 1(2)N - 1. \quad (34)$$

Hence, the eigenvalues of $\mathbf{G}_1$ are real.

In a similar manner, we can show that the eigenvalues of $\mathbf{G}_2$ are real. $\square$

Now we give the sufficient condition for the convergence of the method.

Theorem 2. *Let λ_i and μ_i , $i = 1(1)N$, be the eigenvalues of $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. If*

$$\omega_1 > \max \{0, -\lambda_1, \dots, -\lambda_N\}, \quad (35)$$

$$\omega_2 > \max \{0, -\mu_1, \dots, -\mu_N\}, \quad (36)$$

$$\omega_2 - 2 \min_i \lambda_i < \omega_1 < \omega_2 + 2 \min_i \mu_i, \quad (37)$$

then the iterative method is convergent for the system (14).

Proof. Let

$$\begin{aligned} \mathbf{D} &= \text{diag} \left(1, \frac{c_1}{a_2}, \frac{c_1 c_2}{a_2 a_3}, \dots, \frac{c_1 c_2 \cdots c_{N-1}}{a_2 a_3 \cdots a_N} \right) \\ &\equiv \text{diag} (d_1, d_2, d_3, \dots, d_N). \end{aligned} \quad (38)$$

Since the off-diagonal entries of $\mathbf{A}$ are negative, therefore $a_{k+1} c_k > 0$, $k = 1, \dots, N - 1$. Therefore, the diagonal entries of $\mathbf{D}$ are positive.

The iteration matrix is given by

$$\begin{aligned} \mathbf{T}_w &= (\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \\ &\times [(\mathbf{G}_2 - \omega_1 \mathbf{I}) \\ &- (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}(\mathbf{G}_2 - \omega_1 \mathbf{I})] \\ &= (\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \\ &\times [\mathbf{I} - (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}](\mathbf{G}_2 - \omega_1 \mathbf{I}). \end{aligned} \quad (39)$$

Define

$$\begin{aligned} \mathbf{T}_w^* &= (\mathbf{G}_2 + \omega_2 \mathbf{I}) \mathbf{T}_w (\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \\ &= [\mathbf{I} - (\omega_1 + \omega_2)(\mathbf{G}_1 + \omega_1 \mathbf{I})^{-1}] \\ &\quad \times (\mathbf{G}_2 - \omega_1 \mathbf{I})(\mathbf{G}_2 + \omega_2 \mathbf{I})^{-1} \\ \mathbf{S}(\mathbf{T}_w) &= \mathbf{S}(\mathbf{T}_w^*) = \mathbf{S}(\mathbf{D}^{1/2} \mathbf{T}_w^* \mathbf{D}^{-1/2}) \\ \mathbf{D}^{1/2} \mathbf{T}_w^* \mathbf{D}^{-1/2} &= [\mathbf{I} - (\omega_1 + \omega_2)(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}] \\ &\quad \times (\bar{\mathbf{G}}_2 - \omega_1 \mathbf{I})(\bar{\mathbf{G}}_2 + \omega_2 \mathbf{I})^{-1}, \end{aligned} \quad (40)$$

where $\bar{\mathbf{G}}_1 = \mathbf{D}^{1/2} \mathbf{G}_1 \mathbf{D}^{-1/2}$ and $\bar{\mathbf{G}}_2 = \mathbf{D}^{1/2} \mathbf{G}_2 \mathbf{D}^{-1/2}$, and

$$\begin{aligned} \mathbf{S}(\mathbf{T}_w) &= \mathbf{S}(\mathbf{T}_w^*) = \mathbf{S}(\mathbf{D}^{1/2} \mathbf{T}_w^* \mathbf{D}^{-1/2}) \leq \|\mathbf{D}^{1/2} \mathbf{T}_w^* \mathbf{D}^{-1/2}\|_2 \\ &\leq \left\| [\mathbf{I} - (\omega_1 + \omega_2)(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}] \right\|_2 \\ &\quad \times \left\| (\bar{\mathbf{G}}_2 - \omega_1 \mathbf{I})(\bar{\mathbf{G}}_2 + \omega_2 \mathbf{I})^{-1} \right\|_2. \end{aligned} \quad (41)$$

It is easy to verify that $\bar{\mathbf{G}}_1$ and $\bar{\mathbf{G}}_2$ are symmetric. Therefore, the matrices $(\bar{\mathbf{G}}_2 - \omega_1 \mathbf{I})(\bar{\mathbf{G}}_2 + \omega_2 \mathbf{I})^{-1}$ and $[\mathbf{I} - (\omega_1 + \omega_2)(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}]$ are also symmetric. Hence,

$$\begin{aligned} \left\| (\bar{\mathbf{G}}_2 - \omega_1 \mathbf{I})(\bar{\mathbf{G}}_2 + \omega_2 \mathbf{I})^{-1} \right\|_2 &= \max_{\mu_i \in \sigma(\bar{\mathbf{G}}_2)} \left| \frac{(\mu_i - \omega_1)}{(\mu_i + \omega_2)} \right| \\ &= \left\| [\mathbf{I} - (\omega_1 + \omega_2)(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}] \right\|_2 \\ &= \left\| [(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1} \right. \\ &\quad \left. - (\omega_1 + \omega_2)(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}] \right\|_2 \\ &= \left\| [(\bar{\mathbf{G}}_1 - \omega_2 \mathbf{I})][(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}] \right\|_2. \end{aligned} \quad (42)$$

TABLE 1: Example 1.

N	TAGE			Single sweep AGE			MAE	RMS errors
	ω_1	ω_2	Iter	ω_1	ω_2	Iter		
$\alpha = 1, \sigma = 0.7$								
10	0.441	0.44	38	0.47	0.34	20	0.4062 (−03)	0.2435 (−03)
20	0.286	0.286	59	0.33	0.25	33	0.3604 (−03)	0.1558 (−03)
30	0.25	0.247	77	0.32	0.23	44	0.3592 (−03)	0.1269 (−03)
40	0.218	0.218	75	0.24	0.20	47	0.3592 (−03)	0.1099 (−03)
50	0.203	0.203	86	0.26	0.21	54	0.3592 (−03)	0.9829 (−04)
60	0.202	0.199	96	0.26	0.20	62	0.3592 (−03)	0.8973 (−04)
70	0.194	0.194	100	0.26	0.20	65	0.3592 (−03)	0.8307 (−04)
80	0.149	0.149	79	0.25	0.17	62	0.3592 (−03)	0.7771 (−04)
$\alpha = 2, \sigma = 1.1$								
10	0.529	0.524	39	0.51	0.40	23	0.2990 (−02)	0.2910 (−02)
20	0.298	0.297	78	0.34	0.26	44	0.9774 (−03)	0.9580 (−03)
30	0.213	0.213	113	0.29	0.20	63	0.6774 (−03)	0.6672 (−03)
40	0.169	0.168	146	0.16	0.15	81	0.5925 (−03)	0.5855 (−03)
50	0.142	0.141	178	0.14	0.13	101	0.5633 (−03)	0.5579 (−03)
60	0.124	0.124	180	0.14	0.12	113	0.5525 (−03)	0.5481 (−03)
70	0.112	0.112	201	0.14	0.12	118	0.5484 (−03)	0.5446 (−03)
80	0.103	0.103	221	0.13	0.10	128	0.5469 (−03)	0.5435 (−03)
$\alpha = 1, \sigma = 1.0$								
10	0.57	0.37	47	0.46	0.34	26	0.4258 (−03)	0.3739 (−03)
20	0.31	0.23	115	0.25	0.239	75	0.3371 (−04)	0.2934 (−04)
30	0.20	0.19	204	0.203	0.183	128	0.7197 (−05)	0.6256 (−05)
40	0.17	0.15	309	0.19	0.17	225	0.2367 (−05)	0.2058 (−05)
50	0.14	0.14	445	0.1749	0.173	362	0.9928 (−06)	0.8634 (−06)
60	0.13	0.13	623	0.179	0.164	503	0.4864 (−06)	0.4232 (−06)
70	0.125	0.124	842	0.177	0.163	681	0.2654 (−06)	0.2310 (−06)
80	0.124	0.123	1117	0.176	0.163	888	0.1567 (−06)	0.1365 (−06)
$\alpha = 2, \sigma = 1.0$								
10	0.59	0.42	40	0.45	0.39	24	0.5725 (−03)	0.5344 (−03)
20	0.32	0.25	86	0.34	0.25	45	0.4488 (−04)	0.4152 (−04)
30	0.20	0.19	131	0.21	0.18	73	0.9532 (−05)	0.8792 (−05)
40	0.16	0.14	180	0.16	0.14	99	0.3125 (−05)	0.2877 (−05)
50	0.12	0.12	228	0.11	0.10	126	0.1307 (−05)	0.1202 (−05)
60	0.1	0.1	274	0.10	0.09	151	0.6392 (−06)	0.5876 (−06)
70	0.08	0.08	332	0.09	0.08	176	0.3485 (−06)	0.1890 (−06)
80	0.07	0.07	382	0.08	0.07	202	0.2058 (−06)	0.1890 (−06)

Also, $(\bar{\mathbf{G}}_1 - \omega_2 \mathbf{I})(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}$ is symmetric, and therefore

$$\left\| [(\bar{\mathbf{G}}_1 - \omega_2 \mathbf{I})] [(\bar{\mathbf{G}}_1 + \omega_1 \mathbf{I})^{-1}] \right\|_2 = \max_{\lambda_i \in \sigma(\bar{\mathbf{G}}_1)} \left| \frac{(\lambda_i - \omega_2)}{(\lambda_i + \omega_1)} \right|. \quad (43)$$

Therefore, we have

$$\mathbf{S}(\mathbf{T}_w) \leq \max_{\lambda_i \in \sigma(\bar{\mathbf{G}}_1)} \left| \frac{(\lambda_i - \omega_2)}{(\lambda_i + \omega_1)} \right| \max_{\mu_i \in \sigma(\bar{\mathbf{G}}_2)} \left| \frac{(\mu_i - \omega_1)}{(\mu_i + \omega_2)} \right|. \quad (44)$$

From (35) and (36), we have $\omega_1, \omega_2 > 0$ and $\lambda_i + \omega_1 > 0$ for $k = 1, \dots, N$.

Hence,

$$\frac{\lambda_i - \omega_2}{\lambda_i + \omega_1} < 1, \quad k = 1(1)N. \quad (45)$$

Also, from (37), we have

$$\omega_2 < \omega_1 + 2 \min_i \lambda_i < \omega_1 + 2\lambda_i, \quad i = 1(1)N$$

$$-1 < \frac{\lambda_i - \omega_2}{\lambda_i + \omega_1}, \quad k = 1(1)N. \quad (46)$$

Hence, we conclude that

$$\left| \frac{\lambda_i - \omega_2}{\lambda_i + \omega_1} \right| < 1, \quad k = 1(1)N. \quad (47)$$

TABLE 2: Example 2.

N	TAGE			Single sweep AGE			MAE	RMS errors
	ω_1	ω_2	Iter	ω_1	ω_2	Iter		
$R_e = 10, \sigma = 1.4$								
10	0.0578	0.0585	6	0.0568	0.0568	6	0.1067(−03)	0.6084(−04)
20	0.0368	0.0374	7	0.0281	0.0285	7	0.8511(−04)	0.3726(−04)
30	0.0318	0.0321	8	0.0320	0.0315	7	0.8442(−04)	0.3028(−04)
40	0.0282	0.0286	9	0.0170	0.0170	8	0.8440(−04)	0.2622(−04)
50	0.0236	0.0239	10	0.0160	0.0166	8	0.8440(−04)	0.2345(−04)
60	0.0161	0.0162	10	0.0161	0.0166	8	0.8440(−04)	0.2141(−04)
$R_e = 50, \sigma = 1.1$								
10	0.0152	0.0155	5	0.016	0.016	4	0.8156(−03)	0.3181(−03)
20	0.0093	0.0096	6	0.009	0.0089	5	0.3228(−04)	0.1508(−04)
30	0.0063	0.0068	7	0.006	0.0059	7	0.8624(−05)	0.4435(−05)
40	0.0044	0.0045	10	0.0032	0.0042	9	0.5159(−05)	0.2771(−05)
50	0.0036	0.0035	12	0.002	0.0031	12	0.4238(−05)	0.2232(−05)
60	0.0029	0.0029	10	0.0027	0.00278	15	0.3917(−05)	0.1968(−05)
$R_e = 50, \sigma = 1.0$								
10	0.006	0.014	5	0.005	0.014	4	0.3335(−02)	0.1197(−02)
20	0.012	0.012	5	0.008	0.0089	4	0.3326(−03)	0.9523(−04)
30	0.008	0.008	6	0.008	0.008	4	0.7487(−04)	0.2147(−04)
40	0.0061	0.0061	7	0.0051	0.0053	5	0.2502(−04)	0.7150(−05)
50	0.0049	0.0049	8	0.0058	0.0049	5	0.1055(−04)	0.3012(−05)
60	0.0041	0.0041	9	0.0043	0.0041	5	0.5186(−05)	0.1478(−05)

Thus,

$$\max_{\lambda_i \in \sigma(\bar{G}_1)} \left| \frac{(\lambda_i - \omega_2)}{(\lambda_i + \omega_1)} \right| < 1. \quad (48)$$

Similarly, we can prove that

$$\max_{\mu_i \in \sigma(\bar{G}_2)} \left| \frac{(\mu_i - \omega_1)}{(\mu_i + \omega_2)} \right| < 1. \quad (49)$$

Using (44), (48), and (49), we get $S(T_w) < 1$. $\square$

Hence, the convergence of the method (20) follows.

4.3. Application of Single Sweep Newton-AGE Method. Now we discuss the single sweep Newton-AGE iterative method for the nonlinear difference equation (11). We follow the approaches given by Evans [8, 16].

Let us define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}_{N \times 1}, \quad \boldsymbol{\varphi}(\mathbf{y}) = \begin{bmatrix} \phi_1(\mathbf{y}) \\ \phi_2(\mathbf{y}) \\ \vdots \\ \phi_N(\mathbf{y}) \end{bmatrix}_{N \times 1},$$

$$a_k(\mathbf{y}) = \frac{\partial \phi_k}{\partial y_{k-1}}, \quad k = 2(1)N$$

$$2b_k(\mathbf{y}) = \frac{\partial \phi_k}{\partial y_k}, \quad k = 1(1)N$$

$$c_k(\mathbf{y}) = \frac{\partial \phi_k}{\partial y_{k+1}}, \quad k = 1(1)N-1. \quad (50)$$

Then the Jacobian of $\boldsymbol{\varphi}(\mathbf{y})$ can be written as the N th-order tridiagonal matrix

$$\mathbf{T} = \frac{\partial \boldsymbol{\varphi}(\mathbf{y})}{\partial \mathbf{y}} = \begin{bmatrix} 2b_1(\mathbf{y}) & c_1(\mathbf{y}) & & & \mathbf{0} \\ a_2(\mathbf{y}) & 2b_2(\mathbf{y}) & c_2(\mathbf{y}) & & \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & a_N(\mathbf{y}) & 2b_N(\mathbf{y}) & \end{bmatrix}_{N \times N}. \quad (51)$$

Now with any initial vector $\mathbf{y}^{(0)}$, we define

$$\mathbf{y}^{(s+1)} = \mathbf{y}^{(s)} + \Delta \mathbf{y}^{(s)}, \quad s = 0, 1, 2, \dots, \quad (52)$$

where $\Delta \mathbf{y}^{(s)}$ is the solution of the nonlinear system

$$\mathbf{T} \Delta \mathbf{y}^{(s)} = -\boldsymbol{\varphi}(\mathbf{y}^{(s)}), \quad s = 0, 1, 2, \dots \quad (53)$$

For the single sweep Newton-AGE method, we consider the case when N is even. We split the matrix $\mathbf{T}$ as $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, where

$$\mathbf{T}_1 = \begin{bmatrix} \boxed{\begin{matrix} b_1 & c_1 \\ a_2 & b_2 \end{matrix}} & & & & \mathbf{0} \\ & \boxed{\begin{matrix} b_3 & c_3 \\ a_4 & b_4 \end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} b_{N-1} & c_{N-1} \\ a_N & b_N \end{matrix}} & \\ \mathbf{0} & & & & \end{bmatrix}_{N \times N}, \quad (54)$$

$$\mathbf{T}_2 = \begin{bmatrix} \boxed{b_1} & & & & \mathbf{0} \\ & \boxed{\begin{matrix} b_2 & c_2 \\ a_3 & b_3 \end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} b_{N-2} & c_{N-2} \\ a_{N-1} & b_{N-1} \end{matrix}} & \\ \mathbf{0} & & & & \boxed{b_N} \end{bmatrix}_{N \times N}$$

and then we write single sweep Newton-AGE method as

$$\begin{aligned} & (\mathbf{T}_2 + \omega_2 \mathbf{I}) \Delta \mathbf{y}^{(s+1)} \\ &= [\mathbf{I} - (\omega_1 + \omega_2)(\mathbf{T}_1 + \omega_1 \mathbf{I})^{-1}] (\mathbf{T}_2 - \omega_1 \mathbf{I}) \Delta \mathbf{y}^{(s)} \\ & \quad - (\omega_1 + \omega_2)(\mathbf{T}_1 + \omega_1 \mathbf{I})^{-1} \boldsymbol{\varphi}(\mathbf{y}^{(s)}), \\ & \quad s = 0, 1, 2, \dots, \end{aligned} \quad (55)$$

where $\omega_1 > 0, \omega_2 > 0$ are relaxation parameters and $(\mathbf{T}_1 + \omega_1 \mathbf{I})$ and $(\mathbf{T}_2 + \omega_2 \mathbf{I})$ are nonsingular.

Since $(\mathbf{T}_1 + \omega_1 \mathbf{I})$ and $(\mathbf{T}_2 + \omega_2 \mathbf{I})$ consist of (2×2) submatrices, they can be easily inverted as

$$(\mathbf{T}_1 + \omega_1 \mathbf{I})^{-1} = \begin{bmatrix} \boxed{\frac{\begin{pmatrix} p_2 & -c_1 \\ -a_2 & p_1 \end{pmatrix}}{\Delta_1}} & & & & \mathbf{0} \\ & \ddots & & & \\ & & \ddots & & \\ & & & \boxed{\frac{\begin{pmatrix} p_N & -c_{N-1} \\ -a_N & p_{N-1} \end{pmatrix}}{\Delta_{N-1}}} & \\ \mathbf{0} & & & & \end{bmatrix}_{N \times N}, \quad (56a)$$

$$(\mathbf{T}_2 + \omega_2 \mathbf{I})^{-1} = \begin{bmatrix} \boxed{\frac{1}{r_1}} & & & & \mathbf{0} \\ & \boxed{\frac{\begin{pmatrix} r_3 & -c_2 \\ -a_3 & r_2 \end{pmatrix}}{\Delta_2}} & & & \\ & & \ddots & & \\ & & & \boxed{\frac{1}{r_N}} & \\ \mathbf{0} & & & & \end{bmatrix}_{N \times N} \quad (56b)$$

with $p_k = b_k + \omega_1, k = 1(1)N, \Delta_k = p_k p_{k+1} - c_k a_{k+1}, k = 1(2)N - 1, r_k = b_k + \omega_2, k = 1(1)N$, and $\Delta_k = r_k r_{k+1} - c_k a_{k+1}, k = 2(2)N - 2$.

The matrices $(\mathbf{T}_2 + \omega_2 \mathbf{I})^{-1}(\mathbf{T}_1 + \omega_1 \mathbf{I})^{-1}(\mathbf{T}_2 - \omega_1 \mathbf{I})$ and $(\mathbf{T}_2 + \omega_2 \mathbf{I})^{-1}(\mathbf{T}_1 + \omega_1 \mathbf{I})^{-1}$ can be evaluated in a manner suitable for parallel computing. In order for this method to converge, it is sufficient that the initial vector $\mathbf{y}^{(0)}$ be close to the solution.

Similarly, we can write the single sweep Newton-AGE algorithm when N is odd.

5. Results and Observations

We have applied the methods to the following three examples, whose exact solutions are known to us, and have compared the results with the corresponding double sweep AGE and Newton-AGE method [14]. For single sweep Newton-AGE method, we use the technique given by Evans [16]. The right hand side function and boundary conditions may be obtained using the exact solutions. Here, we have taken $\sigma_k = \sigma = a$ constant, $k = 1(1)N + 1$. The value of the first mesh spacing on the left is given by

$$h_1 = \frac{(b-a)(1-\sigma)}{(1-\sigma^{N+1})}, \quad \sigma \neq 1. \quad (57)$$

Therefore, given the value of N and σ , we can calculate h_1 from the above relation and the remaining mesh points are determined by $h_{k+1} = \sigma h_k, k = 1(1)N$. The initial vector $\mathbf{y}^{(0)} = \mathbf{0}$ is used in all iterative methods and the iterations were stopped when the absolute error tolerance $|\mathbf{y}^{(s+1)} - \mathbf{y}^{(s)}| \leq 10^{-10}$ was achieved.

Example 1 (linear singular problem). Consider

$$y'' + \frac{\alpha}{x} y' = f(x), \quad 0 < x < 1. \quad (58)$$

The exact solution is $y(x) = e^{x^4}$. The root mean square errors (RMSE), the maximum absolute errors (MAE), and number of iterations both for single and double sweep AGE methods are tabulated in Table 1 for $\alpha = 1, \sigma = 0.7, \alpha = 2, \sigma = 1.1, \alpha = 1, \sigma = 1.0$, and $\alpha = 2, \sigma = 1.0$. The graph of Example 1 of exact solution versus numerical solution is plotted in Figure 1.

Example 2 (Burgers' equation). Consider

$$v y'' = \left(y - \frac{1}{2}\right) y', \quad 0 < x < 1. \quad (59)$$

The exact solution is given by $y(x) = (1/2)[1 - \tanh(x/4v)]$, where $v = R_e^{-1}$. The RMSE, MAE, and the number of iterations for both single and double sweep AGE methods are tabulated in Table 2 for $R_e = 50, \sigma = 1.1, R_e = 10, \sigma = 1.4$, and $R_e = 50, \sigma = 1.0$. The graph of Example 2 of exact solution versus numerical solution for $N = 60, R_e = 50, \sigma = 1.0$ is plotted in Figure 2.

Example 3 (Burgers' equation in polar coordinates). Consider

$$\frac{1}{R_e} \left(y'' + \frac{\alpha}{x} y' - \frac{\alpha}{x^2} y \right) = y y' + f(x), \quad 0 < x < 1. \quad (60)$$

TABLE 3: Example 3.

N	TAGE			Single sweep AGE			MAE	RMS errors
	ω_1	ω_2	Iter	ω_1	ω_2	Iter		
$\alpha = 1, R_e = 10, \sigma = 1.3$								
10	0.0368	0.0373	8	0.050	0.052	5	0.8036(−03)	0.2692(−03)
20	0.0329	0.0334	9	0.034	0.046	6	0.7645(−03)	0.1770(−03)
30	0.0297	0.0311	10	0.017	0.034	7	0.7605(−03)	0.1436(−03)
40	0.0283	0.0287	10	0.023	0.0242	8	0.7602(−03)	0.1243(−03)
50	0.0261	0.0269	10	0.025	0.027	8	0.7602(−03)	0.1112(−03)
60	0.0242	0.0251	10	0.0211	0.0218	9	0.7602(−03)	0.1015(−03)
$\alpha = 2, R_e = 10, \sigma = 1.2$								
10	0.0554	0.0562	7	0.0555	0.0559	5	0.6954(−03)	0.2261(−03)
20	0.0457	0.0464	9	0.035	0.037	7	0.4545(−03)	0.1034(−04)
30	0.0403	0.0412	10	0.027	0.029	8	0.4245(−03)	0.7896(−04)
40	0.0381	0.0393	10	0.025	0.030	9	0.4198(−03)	0.6765(−04)
50	0.0359	0.0364	10	0.03	0.033	9	0.4191(−03)	0.6040(−04)
60	0.0326	0.0335	10	0.022	0.029	10	0.4190(−03)	0.5512(−04)
$\alpha = 1, R_e = 50, \sigma = 0.9$								
10	0.0204	0.0211	8	0.016	0.020	7	0.8035(−03)	0.3961(−03)
20	0.0183	0.0189	8	0.011	0.014	6	0.5906(−03)	0.1492(−03)
30	0.0079	0.0084	9	0.007	0.0072	7	0.5252(−03)	0.1101(−03)
40	0.0056	0.0067	11	0.0051	0.0052	9	0.5044(−03)	0.9227(−04)
50	0.0036	0.0041	15	0.0043	0.0043	12	0.4973(−03)	0.8159(−04)
60	0.0034	0.0035	17	0.0032	0.0034	16	0.4949(−03)	0.7419(−04)
$\alpha = 2, R_e = 50, \sigma = 0.9$								
10	0.0198	0.0211	8	0.016	0.017	7	0.4542(−03)	0.3755(−03)
20	0.0172	0.0181	8	0.011	0.018	6	0.6976(−04)	0.4929(−04)
30	0.0069	0.0072	9	0.007	0.008	7	0.4018(−04)	0.2422(−04)
40	0.0064	0.0064	10	0.0065	0.0057	8	0.3333(−04)	0.1789(−04)
50	0.0045	0.0043	13	0.0036	0.0041	12	0.3124(−04)	0.1525(−04)
60	0.0035	0.0034	17	0.0043	0.0033	15	0.3054(−04)	0.1372(−04)
$\alpha = 1, R_e = 50, \sigma = 1.0$								
10	0.0174	0.0177	10	0.01	0.027	8	0.1217(−02)	0.5654(−03)
20	0.009	0.011	12	0.0113	0.0115	9	0.1044(−03)	0.4913(−04)
30	0.0063	0.0065	14	0.0072	0.0077	10	0.4547(−04)	0.1367(−04)
40	0.0063	0.0069	16	0.0054	0.0056	11	0.2617(−04)	0.6119(−05)
50	0.0042	0.0046	18	0.0044	0.0045	12	0.1697(−04)	0.3416(−05)
60	0.0034	0.0039	21	0.0032	0.0041	13	0.1188(−04)	0.2152(−05)
$\alpha = 2, R_e = 50, \sigma = 1.0$								
10	0.017	0.025	9	0.015	0.026	7	0.1234(−02)	0.5767(−03)
20	0.0101	0.011	11	0.0118	0.0118	9	0.1058(−03)	0.4505(−04)
30	0.0071	0.0083	13	0.0075	0.0075	10	0.2644(−04)	0.1025(−04)
40	0.0054	0.0063	15	0.0049	0.0058	11	0.1502(−04)	0.3813(−05)
50	0.0043	0.0049	17	0.0038	0.0049	12	0.9686(−05)	0.1870(−05)
60	0.0035	0.004	20	0.0042	0.0044	11	0.6762(−05)	0.1084(−05)

The exact solution is given by $y(x) = x^2 \cosh(x)$. The RMSE, MAE, and the number of iterations for both single and double sweep AGE methods are tabulated in Table 3 for various values of α , R_e , and σ . The graph of Example 3 of exact solution versus numerical solution for $N = 60$, $R_e = 50$, $\alpha = 2$, $\sigma = 0.9$ is plotted in Figure 3.

6. Discussion and Conclusion

We have discussed a new single sweep AGE iterative method and three-point off-step method of accuracy $O(h_k^3)$ on a variable mesh for the solution of nonlinear two point boundary value problems. To demonstrate the efficiency of the method,

FIGURE 1: Graph of exact solution and the numerical solution for $\alpha = 2.0$, $\sigma = 1.1$, and $N = 80$ for Example 1.

FIGURE 3: Graph of the exact solution and the numerical solution for $\alpha = 2.0$, $R = 50$, $\sigma = 0.9$, and $N = 60$ for Example 3.

FIGURE 2: Graph of the exact solution and the numerical solution for $R = 50$, $\sigma = 1.0$, and $N = 60$ for Example 2.

three examples including two nonlinear and singular cases are presented. The results obtained are compared with the corresponding double sweep AGE method and show superiority over the latter. The double sweep AGE method requires two sweeps to solve a problem, whereas the single sweep AGE method requires only one sweep to solve the problem. Experimentally, as compared to the double sweep method the corresponding single sweep method requires much less number of iterations as it uses less intermediate variables. The method can be extended to solve multidimensional problems and is suitable for use on parallel computers.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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