Since the 1950s, mathematicians have successfully interpreted the traditional Eulerian numbers and \( q \)-Eulerian numbers combinatorially. In this paper, the authors give a combinatorial interpretation to the general Eulerian numbers defined on general arithmetic progressions \( \{a, a+d, a+2d, \ldots \} \).

### 1. Introduction

**Definition 1.** Given a positive integer \( n \), define \( \Omega_n \) as the set of all permutations of \( [n] = \{1, 2, 3, \ldots, n\} \). For a permutation \( \pi = p_1p_2p_3 \ldots p_n \in \Omega_n \), \( i \) is called an ascent of \( \pi \) if \( p_i < p_{i+1} \); \( i \) is called a weak exceedance of \( \pi \) if \( p_i \geq i \).

It is well known that a traditional Eulerian number \( A_{n, k} \) is the number of permutations \( \pi \in \Omega_n \) that have \( k \) weak exceedances [1, page 215]. And \( A_{n,k} \) satisfies the recurrence: 

\[
A_{n,1} = 1, \quad (n \geq 1), \quad A_{n,k} = 0 \quad (k > n),
\]

\[
A_{n,k} = kA_{n-1,k} + (n + 1 - k)A_{n-1,k-1} \quad (1 \leq k \leq n)
\]  

(1)

Besides the recursive formula (1), \( A_{n,k} \) can be calculated directly by the following analytic formula [2, page 8]:

\[
A_{n,k} = \sum_{i=0}^{k-1} (-1)^i (k-i)^n \binom{n+1}{i} \quad (1 \leq k \leq n).
\]

(2)

**Definition 2.** Given a permutation \( \pi = p_1p_2p_3 \ldots p_n \in \Omega_n \), define functions

\[
\text{maj} \pi = \sum_{p_i > p_{i+1}} j,
\]

\[
a(n,k,i) = \# \{ \pi \mid \text{maj} \pi = i \& \pi \text{ has } k \text{ ascents} \}.
\]

(3)

Under Carlitz’s definition, the \( q \)-Eulerian numbers \( A_{n,k}(q) \) are given by

\[
A_{n,k}(q) = q^{(m-k+1)(m-k)/2} \sum_{i=0}^{k(n-k-1)} a(n,n-k,i) q^i.
\]

(4)

where functions \( a(n,k,i) \) are as defined in Definition 2.

In [5], instead of studying \( q \)-sequences, the authors have generalized Eulerian numbers to any general arithmetic progression \( \{a, a+d, a+2d, \ldots \} \).

Under the new definition, and given an arithmetic progression as defined in (5), the general Eulerian numbers \( A_{n,k}(a,d) \) can be calculated directly by the following equation [5, Lemma 2.6]:

\[
A_{n,k}(a,d) = \sum_{i=0}^{k} (-1)^i [(k + 1 - i)d - a]^n \binom{n+1}{i}.
\]

(6)

Interested readers can find more results about the general Eulerian numbers and even general Eulerian polynomials in [5].

### 2. Combinatorial Interpretation of General Eulerian Numbers

The following concepts and properties will be heavily used in this section.
Definition 3. Let \( W_{n,k} \) be the set of \( n \)-permutations with \( k \) weak exceedances. Then \( |W_{n,k}| = A_{n,k} \). Furthermore, given a permutation \( \pi = p_1 p_2 p_3 \ldots p_n \), let \( Q_\theta(\pi) = i \), where \( p_i = n \).

Given a permutation \( \pi \in \Omega_n \), it is known that \( \pi \) can be written as a one-line form like \( \pi = p_1 p_2 p_3 \ldots p_n \), or \( \pi \) can be written in a disjoint union of distinct cycles. For \( \pi \) written in a cycle form, we can use a standard representation by writing (a) each cycle starting with its largest element and (b) the cycles in increasing order of their largest element. Moreover, given a permutation \( \pi \) written in a standard representation, define a function \( f \) as \( f(\pi) \) to be the permutation obtained from \( \pi \) by erasing the parentheses. Then \( f \) is known as the fundamental bijection from \( \Omega_n \) to itself [6, page 30]. Indeed, the inverse map \( f^{-1} \) of the fundamental bijection function \( f \) is also famous in illustrating the relation between the ascents and weak exceedances as follows [2, page 98].

Proposition 4. The function \( f^{-1} \) gives a bijection between the set of permutations on \( [n] \) with \( k \) ascents and the set \( W_{n,k+1} \).

Example 5. The standard representation of permutation \( \pi = 5 2 4 3 7 1 6 \) is \( 2(45)(7615) \in \Omega_7 \), and \( f(\pi) = 2437615 \). The standard representation \( \pi \) written as a one-line form like \( \pi = p_1 p_2 p_3 \ldots p_n \) has 3 + 1 = 4 weak exceedances because \( p_1 = 6 > 1, p_2 = 4 > 2, p_3 = 5 > 3, \) and \( p_6 = 7 > 6 \).

Now suppose we want to construct a sequence consisting of \( k \) vertical bars and the first \( n \) positive integers. Then the \( k \) vertical bars divide these \( n \) numbers into \( k+1 \) compartments. In each compartment, there is either no number or all the numbers are listed in a decreasing order. The following definition is analogous to the definition of [2, page 8].

Definition 6. A bar in the above construction is called extraneous if either

(a) it is immediately followed by another bar; or

(b) each of the rest compartment is either empty or consists of integers in a decreasing order if this bar is removed.

Example 7. Suppose \( n = 7, k = 4 \); then in the following arrangement

\[
32\,11\,7654
\]

(7)

the 1st, 2nd, and 4th bars are extraneous.

Now we are ready to give combinatorial interpretations to the general Eulerian numbers \( A_{n,k}(a,d) \). First note that (6) implies that \( A_{n,k}(a,d) \) is a homogeneous polynomial of degree \( n \) with respect to \( a \) and \( d \). Indeed,

\[
A_{n,k}(a,d) = \sum_{i=0}^{k} (-1)^i [(k + 1 - i)(d - a) + (k - i)a]^{(n+1)} \left( \begin{array}{c} n+1 \\ i \end{array} \right)
\]

(8)

where

\[
c_{n,k} (j) = \sum_{i=0}^{k} (-1)^i (k + 1 - i)^{n-j}(k - i)^j \left( \begin{array}{c} n+1 \\ i \end{array} \right), \quad 0 \leq j \leq n.
\]

The following theorem gives combinatorial interpretations to the coefficients \( c_{n,k}(j) \), \( 0 \leq j \leq n \).

Theorem 8. Let the general Eulerian numbers \( A_{n,k}(a,d) \) be written as in (8). Then

\[
c_{n,k} (j) = \# \{ \pi \in W_{n,k+1}, j < Q_\theta(\pi) \leq n \}
\]

(10)

\[
+ \# \{ \pi \in W_{n,k}, 1 \leq Q_\theta(\pi) \leq j \}.
\]

Proof. We can check the result in (10) for two special values \( j = 0 \) and \( j = n \) quickly by (2),

\[
\text{when } j = 0, \quad c_{n,k}(0) = \sum_{i=0}^{k} (-1)^i (k + 1 - i)(n+1) = A_{n+1,k+1};
\]

\[
\text{when } j = n, \quad c_{n,k}(n) = \sum_{i=0}^{k} (-1)^i (k - i)^n = A_{n,k}.
\]

Therefore, (10) is true for \( j = 0 \) and \( j = n \).

Generally, for \( 1 \leq j \leq n - 1 \), we write down \( k \) bars with \( k+1 \) compartments in between. Place each element of \( [n] \) in a compartment. If none of the \( k \) bars is extraneous, then the arrangement corresponds to a permutation with \( k \) ascents. Let \( B \) be the set of arrangements with at most one extraneous bar at the end and none of integers \( \{1, 2, \ldots, j\} \) locating in the last compartment. We will show that \( c_{n,k}(j) = |B| \).

To achieve that goal, we use the Principle of Inclusion and Exclusion. There are \( (k+1)^{n-j}k^j \) ways to put \( n \) numbers into \( k+1 \) compartments with elements \( \{1, 2, \ldots, j\} \) avoiding the last compartments.

Let \( B_j \) be the number of arrangements with the following features:

1. none of \( \{1, 2, \ldots, j\} \) sits in the last compartment;
2. each arrangement in \( B_j \) has at least \( i \) extraneous bars.
3. in each arrangement in \( B_j \), any two extraneous bars are not located right next to each other.
Then the Principle of Inclusion and Exclusion shows that
\[ |B| = (k + 1)^{n-i}k! - B_1 + B_2 + \cdots - (-1)^k B_k. \]  \hspace{1cm} (11)

Now we consider the value of \( B_i \), where \( 1 \leq i \leq k \). Suppose that we have \( k+1-i \) compartments with \( k-i \) bars in between. There are \( (k+1-i)^{n-i}(k-i)! \) ways to insert \( n \) numbers into these \( k+1-i \) compartments with first \( j \) integers avoiding the last compartment and list integers in each component in a decreasing order. Then insert \( i \) separating extraneous bars into \( n+1 \) positions. So we get
\[ B_i = (k + 1 - i)^{n-i}(k-i)! \left( \frac{n+1}{i} \right). \]  \hspace{1cm} (12)

Plug formula (12) into (11); we have \( c_{n,k}(j) = |B| \).

Given an arrangement \( \pi \in B \), if we remove the bars, then we obtain a permutation \( \pi \in \Omega_n \). So without confusion, we just use the same notation \( \pi \) to represent both an arrangement in set \( B \) and a permutation on \( [n] \). Now for each \( \pi \in B, \pi \) either

(1) has no extraneous bar and none of \( [1,2,\ldots,j] \) locates in the last compartment or

(2) has only one extraneous bar at the end.

If \( \pi \) is in case 1, then \( \pi \) has \( k \) ascents since each bar is non-extraneous. And the last compartment of \( \pi \) is nonempty. Therefore the last cycle of \( f^{-1}(\pi) \) has to be \( (n \ldots p_1) \). In other words, \( Q_n(f^{-1}(\pi)) = p_g > j \) since none of \( [1,2,\ldots,j] \) locates in the last compartment. And by Proposition 4, \( f^{-1}(\pi) \in W_{n,k+1} \).

If \( \pi \) is in case 2, then \( \pi \) has \( k-1 \) ascents since only the last bar is extraneous. Note that in this case, the arrangement with no elements of \( [1,2,\ldots,j] \) in the compartment second to the last or the last nonempty compartment has been removed by the Principle of Inclusion and Exclusion. Equivalently, at least one number of \( [1,2,\ldots,j] \) has to be in the compartment second to the last. So the last cycle of \( f^{-1}(\pi) \) has to be \( (n \ldots p_1) \), and \( Q_n(f^{-1}(\pi)) = p_1 \leq j \). Also by Proposition 4, \( f^{-1}(\pi) \in W_{n,k} \).

Combing all the results above, statement (10) is correct.

The next Theorem describes some interesting properties of the coefficients \( c_{n,k} \).

**Theorem 9.** Let the coefficients \( c_{n,k} \) be as described in Theorem 8. Then,

1. \( \sum_{i=0}^{n} c_{n,k}(j) = n! \) for any \( 0 \leq j \leq n \);
2. \( c_{n,k}(j) = c_{n,n-k}(n-j) \) for all \( 0 \leq j, k \leq n \).

Before we can prove Theorem 9, we need the following lemma which is also interesting by itself.

**Lemma 10.** Given a positive integer \( n \), then
\[ \# \left\{ \pi \in W_{n,k} \cap Q_n(\pi) = j \right\} \]
\[ = \# \left\{ \pi \in W_{n,n+1-k} \cap Q_n(\pi) = n+1-j \right\} \]  \hspace{1cm} (13)

for any \( 1 \leq k, j \leq n \).

**Proof.** First of all, given a positive integer \( n \), we define a function \( g : \Omega_n \to \Omega_n \), as follows:
\[ g(\pi) = (n+1-p_1)(n+1-p_2)\ldots(n+1-p_n). \]  \hspace{1cm} (14)

For instance, for \( \pi = 53214 \in \Omega_5 \), \( g(\pi) = 13452 \) is obviously a bijection of \( \Omega_n \) to itself.

Now for some fixed \( 1 \leq k, j \leq n \), suppose \( S_{k,j} = \left\{ \pi \in W_{n,k} \cap Q_n(\pi) = j \right\} \), and \( T_{k,j} = \left\{ \pi \in W_{n,n+1-k} \cap Q_n(\pi) = n+1-j \right\} \). For any \( \pi \in S_{k,j} \), we write \( \pi \) in the standard representation cycle form. So \( \pi = (p_u \ldots) \ldots (n \ldots j) \) and \( f(\pi) = p_u \ldots n \ldots j \) has \( k-1 \) ascents by Proposition 4.

Now we compose \( f(\pi) \) with the bijection function \( g \) as just defined. Then \( g(f(\pi)) = n+1-p_u \ldots n+1-j \) has \( n-k \) ascents, which implies that \( f^{-1}(g(f(\pi))) \) has \( n+1-k \) weak excedances. So \( f^{-1}(g(f(\pi))) \in W_{n,n+1-k} \). Note that the last cycle of \( f^{-1}(g(f(\pi))) \) has to be \( (n \ldots n+1-j) \). Therefore, \( f^{-1}(g(f(\pi))) \in T_{k,j} \).

Since both \( f \) and \( g \) are bijection functions, \( f^{-1}gf \) gives a bijection between \( S_{k,j} \) and \( T_{k,j} \).

Now we are ready to prove Theorem 9.

**Proof of Theorem 9.** For part 1, by Theorem 8,
\[ \sum_{k=0}^{n} c_{n,k}(j) = \sum_{k=0}^{n} \left[ \# \{ \pi \in W_{n,k+1}, j < Q_n(\pi) \leq n \} \right] \]
\[ + \sum_{k=0}^{n} \left[ \# \{ \pi \in W_{n,k}, 1 \leq Q_n(\pi) \leq j \} \right] \]  \hspace{1cm} (15)

For part 2, also by Theorem 8,
\[ c_{n,k}(j) = \sum_{i=j+1}^{n} \left[ \# \{ \pi \in W_{n,k+1}, Q_n(\pi) = i \} \right] \]
\[ + \sum_{m=1}^{j} \left[ \# \{ \pi \in W_{n,k}, Q_n(\pi) = m \} \right] \]
\[ = n \left[ \# \{ \pi \in W_{n,n-k}, Q_n(\pi) = n+1-i \} \right] \]
\[ + \sum_{m=1}^{j} \left[ \# \{ \pi \in W_{n,n+1-k}, Q_n(\pi) = n+1-m \} \right] \]  \hspace{1cm} (16)

by Lemma 10.
Remark 11. Using the analytic formula of \( c_{n,k}(j) \) as in (9), part 2 of Theorem 9 implies the following identity:

\[
\sum_{i=0}^{k} (-1)^i (n + 1 - i)^{n-j}(k - i)\binom{n+1}{i} = \sum_{i=0}^{n-k} (-1)^i(n + 1 - k - i)^{n-k-1}(k - i)\binom{n+1}{i},
\]

(17)

where \( n \) is a positive integer, and \( 0 \leq j, k \leq n \).

3. Another Combinatorial Interpretation of \( c_{n,k}(1) \) and \( c_{n,k}(n-1) \)

In pursuing the combinatorial meanings of the coefficients \( c_{n,k} \), the authors have found some other interesting properties about permutations. The results in this section will reveal close connections between the traditional Eulerian numbers \( A_{n,k} \) and \( c_{n,k}(j) \), where \( j = 1 \) or \( j = n - 1 \).

One fundamental concept of permutation combinatorics is inversion. A pair \( (p_i, p_j) \) is called an inversion of the permutation \( \pi = p_1 p_2 \ldots p_n \) if \( i < j \) and \( p_i > p_j \) [6, page 36]. The following definition provides the main concepts of this section.

Definition 12. For a fixed positive integer \( n \), let \( AW_{n,k} = \{ \pi = p_1 p_2 \ldots p_n \mid \pi \in W_{n,k} \text{ and } p_i < p_j \} \) or (\( p_i, p_j \) is not an inversion) and \( BW_{n,k} = W_{n,k} \setminus AW_{n,k} \) or (\( p_i, p_j \) is an inversion).

It is obvious that \( |AW_{n,k}| + |BW_{n,k}| = A_{n,k} \). The following theorem interprets coefficients \( c_{n,k}(1) \) and \( c_{n,k}(n-1) \) in terms of \( AW_{n,k} \) and \( BW_{n,k} \).

Theorem 13. Let the coefficients \( c_{n,k} \) of the general Eulerian numbers be written as in (9). \( AW_{n,k} \) and \( BW_{n,k} \) are as defined in Definition 12. Then

1. \( c_{n,k}(1) = 2|AW_{n,k+1}| \).
2. \( c_{n,k}(n-1) = 2|BW_{n,k}| \).

Proof. For part (1), by Theorem 8, \( c_{n,k}(1) = |S_1| + |S_2| \), where \( S_1 = \{ \pi = p_1 p_2 \ldots p_n \mid \pi \in W_{n+1,k} \text{ and } p_i \neq j \} \), \( S_2 = \{ \pi = p_1 p_2 \ldots p_n \mid \pi \in W_{n,k} \text{ and } p_i = j \} \). Given a permutation \( \pi = p_1 p_2 \ldots p_n \in S_1 \), then both \( p_i, p_2, \ldots, p_n \) and \( p_1, p_2, \ldots, p_n \) belong to \( S_1 \), so one of them has to be in \( AW_{n+1,k} \).

If \( \pi = p_1 p_2 \ldots p_n \in S_1 \), then \( \pi \in AW_{n,k+1} \), but \( p_1, p_2, \ldots, p_n \in S_2 \). Therefore, (1/2)\( c_{n,k}(1) = |AW_{n+1,k}| \).

Part (2) can be proved using exactly the same method. So we leave it to the readers as an exercise.

\( |AW_{n,k}| \) and \( |BW_{n,k}| \) are interesting combinatorial concepts by themselves. Note that generally speaking, \( |AW_{n,k}| \neq |BW_{n,k}| \). Indeed, \( |AW_{n,k}| = |BW_{n+1-k,k}| \).

Theorem 14. For any positive integer \( n \geq 2 \), the sets \( AW_{n,k} \) and \( BW_{n,k} \) are defined in Definition 12. Then \( |AW_{n,k}| = |BW_{n+1-k,k}| \) for \( 1 \leq k \leq n \).

Proof. It is an obvious result of part 2 of Theorems 9 and 13.

Our last result of this paper is the following theorem which reveals that both \( |AW_{n,k}| \) and \( |BW_{n,k}| \) take exactly the same recursive formula as the traditional Eulerian numbers \( A_{n,k} \) as shown in (1).

Theorem 15. For a fixed positive integer \( n \), let \( AW_{n,k} \) and \( BW_{n,k} \) be as defined in Definition 12; then

\[
k |AW_{n-1,k} + (n + 1 - k) |AW_{n-1,k-1}| = |AW_{n,k}|, \tag{18}
\]

\[
k |BW_{n-1,k} + (n + 1 - k) |BW_{n-1,k-1}| = |BW_{n,k}|. \tag{19}
\]

Proof. A computational proof can be obtained straightforward by using (9) and Theorem 13. But here we provide a proof in a flavor of combinatorics.

Idea of the Proof. For (18), given a permutation \( A_1 = p_1 p_2 \ldots p_n \in AW_{n-1,k} \), for each position \( i \) with \( p_i > i \), we insert \( n \) into a certain place of \( A_1 \), such that the new permutation \( A'_1 \) is in \( AW_{n,k} \). There are \( k \) such positions, so we can get \( k \) new permutations in \( AW_{n,k} \). Similarly, if \( A_2 = p_1 p_2 \ldots p_n \in AW_{n-1,k-1} \), for each position \( i \) with \( p_i < i \), and the position at the end of \( A_2 \), we insert \( n \) into a specific position of \( A_2 \) and the resulting new permutation \( A'_2 \) is in \( AW_{n,k} \). There are \( n + 1 - k \) such positions, so we can get \( n + 1 - k \) new permutations in \( AW_{n,k} \). We will show that all the permutations obtained from the above constructions are distinct, and they have exhausted all the permutations in \( AW_{n,k} \).

For any fixed \( A' = \pi_1 \pi_2 \ldots \pi_n \in AW_{n,k} \), then \( \pi_1 < \pi_n \).

We classify \( A' \) into the following disjoint cases:

Case a. Consider that \( \pi_i = n \) with \( i < n \). So \( A' = \pi_1 \pi_2 \ldots \pi_{i-1} n \pi_{i+1} \ldots \pi_n n \).

(a1) \( \pi_1 < \pi_{n-1} \) and \( \pi_{n+1} \geq i \);

(a2) \( \pi_1 < \pi_{n-1} \) and \( \pi_{n+1} < i \);

(a3) \( \pi_1 > \pi_{n-1} \), \( \pi_n < n - 1 \), and \( \pi_{n+1} \geq i \);

(a4) \( \pi_1 > \pi_{n-1} \), \( \pi_n < n - 1 \), and \( \pi_{n+1} < i \);

(a5) \( \pi_1 > \pi_{n-1} \) and \( \pi_n = n - 1 \).

Case b. Consider that \( \pi_n = n \). So \( \pi_i = n - 1 \) for some \( i < n \) and \( A' = \pi_1 \pi_2 \ldots \pi_{i-1} n - 1 \pi_{i+1} \ldots \pi_n n \).

(b1) \( \pi_1 < \pi_{n-1} \);

(b2) \( \pi_{n+1} < \pi_1 < n - 1 \), and \( \pi_{n+1} \geq i \);

(b3) \( \pi_{n+1} < \pi_1 < n - 1 \), and \( \pi_{n+1} < i \);

(b4) \( \pi_1 = n - 1 \).

Based on the classifications listed above, we can construct a map \( f : \{ AW_{n-1,k} \} \rightarrow AW_{n,k} \) by applying the idea of the proof we have illustrated at the beginning of the proof. To save space, the map \( f \) is demonstrated in Table 1. From Table 1 we can see that in each case, the positions of inserting \( n \) are all different. So all the images obtained in a certain case are different. Since all the cases are disjoint, all the images \( A' \in AW_{n,k} \) are distinct.
Table 1: The map \( f: \{ AW_{n-1,k}, AW_{n-1,k-1} \} \rightarrow AW_{n,k} \).

<table>
<thead>
<tr>
<th>( A = p_1p_2\ldots p_{n-1} )</th>
<th>Position ( i )</th>
<th>Condition</th>
<th>( A' \in AW_{n,k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \in AW_{n-1,k} )</td>
<td></td>
<td>( p_i &gt; p_1 )</td>
<td>( A' = p_1p_2\ldots p_{n-1}p_{i+1}\ldots p_{n-1}p_i )</td>
</tr>
<tr>
<td>( 1 &lt; i \leq n - 1 )</td>
<td></td>
<td>( p_i &lt; p_1 ) and ( p_1 \leq i )</td>
<td>( A' = p_1p_2\ldots p_{i+1}p_{i+1}\ldots p_{n-1}p_{n-1}p_i )</td>
</tr>
<tr>
<td>( A \in AW_{n-1,k-1} )</td>
<td>( i = 1 )</td>
<td>( p_{n-1} = n - 1 )</td>
<td>( A' = n-1p_2\ldots p_{n-3}p_{n-1} )</td>
</tr>
</tbody>
</table>

Case (c5)

| \( B \in BW_{n-1,k} \)         |                | \( p_i > p_1 \) | \( B' = p_1p_2\ldots p_{n-1}p_{i+1}\ldots p_{n-1}p_i \) |
| \( 1 < i \leq n - 1 \)         |                | \( p_i < p_1 \) and \( p_1 \leq i \) | \( B' = p_1p_2\ldots p_{i+1}p_{i+1}\ldots p_{n-1}p_{n-1}p_i \) |
| \( A \in AW_{n-1,k-1} \)       | \( i = 1 \)    | \( p_{n-1} = n - 1 \) | \( B' = n-1p_2\ldots p_{n-3}p_{n-1} \) |

Case (a4)

| \( B \in BW_{n-1,k} \)         | \( i = n \)    | \( p_i > p_1 \) | \( B' = p_1p_2\ldots p_{n-1}p_{n-1}p_i \) |
| \( 1 < i < n - 1 \)            |                | \( p_i < p_1 \) and \( p_1 > i \) | \( B' = p_1p_2\ldots p_{i+1}p_{i+1}\ldots p_{n-1}p_{n-1}p_i \) |
| \( A \in AW_{n-1,k-1} \)       | \( i = n \)    | \( p_i > p_1 \) | \( B' = p_1p_2\ldots p_{n-1}p_{n-1}p_i \) |

Case (b2)

Table 2: The map \( g: \{ BW_{n-1,k}, BW_{n-1,k-1} \} \rightarrow BW_{n,k} \).

<table>
<thead>
<tr>
<th>( B = p_1p_2\ldots p_{n-1} )</th>
<th>Position ( i )</th>
<th>Condition</th>
<th>( B' \in BW_{n,k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B \in BW_{n-1,k} )</td>
<td>( i = 1 )</td>
<td>( p_i &gt; p_1 )</td>
<td>( B' = p_1p_2\ldots p_{n-1}p_{i+1}\ldots p_{n-1}p_i )</td>
</tr>
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<td>( 1 &lt; i &lt; n - 1 )</td>
<td></td>
<td>( p_i &lt; p_1 ) and ( p_1 &gt; i )</td>
<td>( B' = p_1p_2\ldots p_{i+1}p_{i+1}\ldots p_{n-1}p_{n-1}p_i )</td>
</tr>
</tbody>
</table>

Case (c3)

| \( B \in BW_{n-1,k-1} \)       | \( i = n - 1 \) | \( p_{n-1} = n - 1 \) | \( B' = p_1p_2\ldots p_{n-1}p_{n-1}p_i \) |
| \( 1 < i < n - 1 \)            |                | \( p_{n-1} > p_1 \) | \( B' = p_1p_2\ldots p_{i+1}p_{i+1}\ldots p_{n-1}p_{n-1}p_i \) |

Case (c4)

| \( B \in BW_{n-1,k-1} \)       | \( i = n - 1 \) | \( p_i > p_1 \) | \( B' = p_1p_2\ldots p_{n-1}p_{n-1}p_i \) |
| \( 1 < i < n - 1 \)            |                | \( p_i < p_1 \) and \( p_1 < i \) | \( B' = p_1p_2\ldots p_{i+1}p_{i+1}\ldots p_{n-1}p_{n-1}p_i \) |

Case (c5)
Similarly, for each $B' = \pi_1\pi_2\pi_3\ldots\pi_n \in BW_{n,k}$, then $\pi_1 > \pi_n$. We classify $B'$ into the following disjoint cases.

**Case c.** Consider that $\pi_i = n$ with $1 < i \leq n - 1$. So $B' = \pi_1\pi_2\ldots \pi_{i-1}n\pi_{i+1}\ldots \pi_{n-1}\pi_n$:

- (c1) $\pi_1 > \pi_{n-1}$, and $\pi_n \geq i$;
- (c2) $\pi_1 > \pi_{n-1}$, and $\pi_n < i$;
- (c3) $\pi_1 < \pi_{n-1} < n-1$, $\pi_{n-1} \geq i$;
- (c4) $\pi_1 < \pi_{n-1} < n-1$, $\pi_{n-1} < i$;
- (c5) $\pi_{n-1} = n - 1$;
- (c6) $\pi_{n-1} = n$.

**Case d.** Consider that $\pi_1 = n$. So $B' = n\pi_2\ldots \pi_{n-2}\pi_{n-1}$:

- (d1) $\pi_{n-2} < \pi_{n-1}$;
- (d2) $\pi_{n-2} > \pi_{n-1}$.

To prove (19), we use a similar idea of proof as shown above. If $B_1 = p_1p_2p_3\ldots p_{n-1} \in BW_{n-1,k}$, for each position $i$ with $p_i \geq i$, we insert $n$ into a certain place of $B_1$ to get $B'_1 \in AW_{n,k}$. If $B_2 = p_1p_2p_3\ldots p_{n-1} \in BW_{n-1,k-1}$, for each position $i$ with $p_i < i$, and the position $i$ where $p_i = n-1$, we insert $n$ into a specific position of $B_2$ to obtain $B'_2 \in AW_{n,k}$. Such a map $g : \{BW_{n-1,k}, BW_{n-1,k-1}\} \rightarrow BW_{n,k}$ is illustrated in Table 2. And the distinct images under $g$ exhaust all the permutations in $BW_{n,k}$.

Here is a concrete example for the constructions illustrated in Table 2.

**Example 16.** Suppose $n = 4$, $k = 2$. We want to obtain $BW_{4,2} = \{3142, 3412, 3421, 4132, 4213, 4312, 4321\}$ from $BW_{3,2} = \{321, 231\}$ and $BW_{3,1} = \{312\}$. For $321 \in BW_{3,2}$, $p_1 = 3 \geq 1$, then it corresponds to $B' = 4213$ which is case (d1) in Table 2; $p_2 = 2 \geq 2$, then it corresponds to $B' = 3412$ which is case (c1) in Table 2. Similarly, we can construct $\{4312, 4321\}$ from $231 \in BW_{3,2}$ and $\{3421, 3412, 4132\}$ from $312 \in BW_{3,1}$ using Table 2.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


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