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Research Article

ϕ-Prime and ϕ-Primary Elements in Multiplicative Lattices

C. S. Manjarekar[1] and A. V. Bingi[2]

[1] *Department of Mathematics, Shivaji University, Kolhapur 416 004, India*
[2] *Department of Mathematics, St. Xavier's College-Autonomous, Mumbai 400 001, India*

Correspondence should be addressed to A. V. Bingi; ashok.bingi@xaviers.edu

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We investigate ϕ-prime and ϕ-primary elements in a compactly generated multiplicative lattice L. By a counterexample, it is shown that a ϕ-primary element in L need not be primary. Some characterizations of ϕ-primary and ϕ-prime elements in L are obtained. Finally, some results for almost prime and almost primary elements in L with characterizations are obtained.

1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative, and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $a \in L$ is called compact if, for $X \subseteq L$, $a \leqslant \vee X$ implies the existence of a finite number of elements $a_1, a_2, \ldots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. The set of compact elements of L will be denoted by L_*. A multiplicative lattice is said to be compactly generated if every element of it is a join of compact elements. Throughout this paper L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact.

An element $a \in L$ is said to be proper if $a < 1$. A proper element $p \in L$ is called a prime element if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$, where $a, b \in L$, and is called a primary element if $ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$ for some $n \in Z_+$, where $a, b \in L_*$. A proper element $p \in L$ is said to be weakly prime if $0 \neq ab \leqslant p$ implies either $a \leqslant p$ or $b \leqslant p$, where $a, b \in L$, and is called weakly primary if $0 \neq ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$ for some $n \in Z_+$, where $a, b \in L_*$. For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a, \text{ for some } n \in Z_+\} = \wedge\{p \in L \mid a \leqslant p \text{ and } p \text{ is a prime element}\}$. An element $a \in L$ is called semiprimary if $\sqrt{a}$ is a prime element and is called semiprime if $\sqrt{a} = a$. An element $a \in L$ is called join irreducible if $a = a_1 \vee a_2$ implies $a = a_1$ or $a = a_2$. A proper element $m \in L$ is said to be a maximal element if $m \nleqslant a$ for any other proper element $a \in L$. An element $a \in L$ is said to be nilpotent if $a^n = 0$ for some $n \in Z_+$. An element $a \in L$ is called a zero divisor if $ab = 0$ for some $0 \neq b \in L$ and is called an idempotent if $a = a^2$. A multiplicative lattice is said to be a domain if it is without zero divisors and is said to be quasi-local if it contains a unique maximal element. A quasi-local multiplicative lattice L with maximal element m is denoted by (L, m). An element $e \in L$ is called meet principal if $a \wedge be = ((a : e) \wedge b)e$ for all $a, b \in L$. An element $e \in L$ is called join principal if $(ae \vee b) : e = (b : e) \vee a$ for all $a, b \in L$. An element $e \in L$ is called weak meet principal if $a \wedge e = (a : e)e$ for all $a \in L$. An element $e \in L$ is called weak join principal if $(ae : e) = (0 : e) \vee a$ for all $a \in L$. An element $e \in L$ is called principal if e is both meet principal and join principal. An element $e \in L$ is called weak principal if e is both weak meet principal and weak join principal. A multiplicative lattice is a Noether lattice if it is modular, principally generated (every element is a join of principal elements) and satisfies the ascending chain condition. A Noether lattice L is local if it contains precisely one maximal prime. In a Noether lattice L, an element $a \in L$ is said to satisfy restricted cancellation law

if, for any $b, c \in L$, $ab = ac \neq 0$ implies $b = c$ [1]. The reader is referred to [2] for general background and terminology.

2. ϕ-Prime and ϕ-Primary Elements in L

The study of ϕ-prime and ϕ-primary ideals for commutative rings is carried out by Anderson and Bataineh [3], Bataineh and Kuhail [4], and Yousefian Darani [5]. We extend these concepts to compactly generated multiplicative lattices. We introduce the notion of ϕ-prime, ϕ-primary, and ω-primary elements in L.

Definition 1. Let $\phi : L \to L$ be a function. A proper element $p \in L$ is said to be ϕ-prime if, for $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \phi(p)$ implies either $a \leqslant p$ or $b \leqslant p$.

Definition 2. Let $\phi : L \to L$ be a function. A proper element $p \in L$ is said to be ϕ-primary if, for $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \phi(p)$ implies either $a \leqslant p$ or $b \leqslant \sqrt{p}$.

Definition 3. Let $\phi : L \to L$ be a function. A proper element $p \in L$ is said to be ϕ_ω-primary (or briefly ω-primary) if, for $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n = \phi_\omega(p)$ implies either $a \leqslant p$ or $b \leqslant \sqrt{p}$.

Similarly, ϕ_α-primary elements in L are defined by following settings in the above definition for ϕ-primary element. For any proper element $p \in L$,

(i) $\phi_0(p) = 0$ and then p is called weakly primary element;

(ii) $\phi_2(p) = p^2$ and then p is called almost primary element;

(iii) $\phi_n(p) = p^n$ $(n \geqslant 2)$ and then p is called n-almost primary element.

Thus an almost primary element in L is a particular case of a ϕ_α-primary element in L. Similarly an almost prime element in L is a particular case of a ϕ_α-prime element in L. Therefore their definitions are as follows.

Definition 4. A proper element $p \in L$ is said to be almost prime if, for $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant p^2$ implies either $a \leqslant p$ or $b \leqslant p$.

Definition 5. A proper element $p \in L$ is said to be almost primary if, for $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant p^2$ implies either $a \leqslant p$ or $b \leqslant \sqrt{p}$.

The reader can verify the following statements.

(1) $\phi(p) \leqslant p$ for all $p \in L$.

(2) Every ϕ-prime element in L is ϕ-primary.

(Its converse holds for semiprime elements.)

(3) Every prime element in L is ϕ-prime.

(4) Every prime element in L is ϕ-primary.

(5) Every primary element in L is ϕ-primary.

The following example (take $\phi(p) = p^2$ for convenience) shows that

① a ϕ-primary element in L need not be ϕ-prime;

② a ϕ-primary element in L need not be prime.

Example 6. Consider the lattice L of ideals of the ring $R = \langle Z_{24}, +, \cdot \rangle$. Then the only ideals of R are the principal ideals (0), (2), (3), (4), (6), (8), (12), (1). Clearly $L = \{(0), (2), (3), (4), (6), (8), (12), (1)\}$ is a compactly generated multiplicative lattice. Its lattice structure is as shown in Figure 1.

From the multiplication table (see Figure 1), we see that element $c \in L$ is almost primary, while c is not almost prime because $(2)(6) \leqslant (4) = c$ but $(2) \nleqslant (4) = c$ and $(6) \nleqslant (4) = c$. Also, c is not prime.

Theorem 7. *If a proper element $q \in L$ is ϕ-primary such that $\sqrt{\phi(q)} = \phi(\sqrt{q})$, then $\sqrt{q}$ is a ϕ-prime element in L.*

Proof. Let $p = \sqrt{q}$. Assume that $ab \leqslant p$ and $ab \nleqslant \phi(p)$ but $a \nleqslant p$ for some $a, b \in L$. Then there exist $n \in Z_+$ such that $(ab)^n \leqslant q$. If $(ab)^n \leqslant \phi(q)$, then by hypothesis $ab \leqslant \phi(\sqrt{q}) = \phi(p)$, a contradiction. So $(ab)^n \nleqslant \phi(q)$. Since q is ϕ-primary and $a^n \nleqslant q$ for all $n \in Z_+$, we have $b^n \leqslant \sqrt{q}$ and hence $b \leqslant \sqrt{\sqrt{q}} = \sqrt{q}$. This shows that $\sqrt{q}$ is a ϕ-prime element in L. □

Definition 8. Given two functions $\gamma_1, \gamma_2 : L \to L$, we define $\gamma_1 \leqslant \gamma_2$ if $\gamma_1(p) \leqslant \gamma_2(p)$ for each $p \in L$.

Theorem 9. *Let $\gamma_1, \gamma_2 : L \to L$ be functions such that $\gamma_1 \leqslant \gamma_2$. Then every proper γ_1-primary element in L is γ_2-primary.*

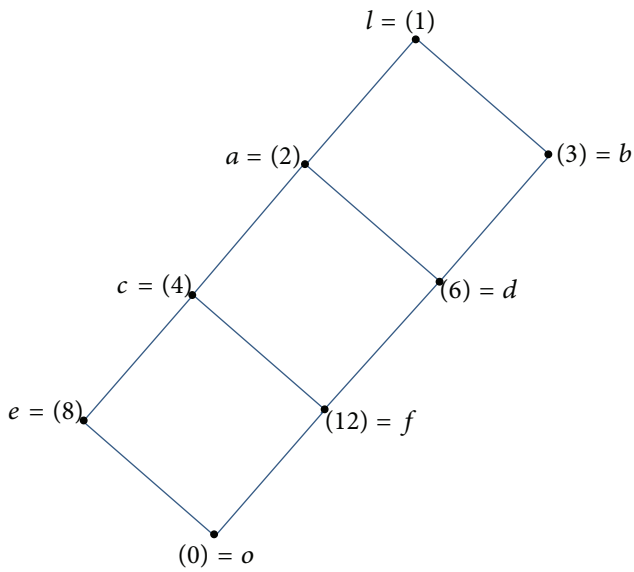
Proof. Let a proper element $p \in L$ be γ_1-primary. Suppose $ab \leqslant p$, $ab \nleqslant \gamma_2(p)$ for some $a, b \in L$. Then as $\gamma_1 \leqslant \gamma_2$ we have $ab \nleqslant \gamma_1(p)$. Since p is γ_1-primary, it follows that either $a \leqslant p$ or $b \leqslant \sqrt{p}$ and hence p is γ_2-primary. □

Theorem 10. *Let $p \in L$ be a proper element. Then p is primary implies p is weakly primary implies p is ω-primary implies p is n-almost primary $(n \geqslant 2)$ implies p is almost primary.*

Proof. Obviously p is primary implies p is weakly primary.

Assume that p is weakly primary but not ω-primary. Then there exist $a, b \in L$ such that $ab \leqslant p$, $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n$, $a \nleqslant p$, and $b \nleqslant \sqrt{p}$. As L is compactly generated, there exist $x, y \in L_*$ such that $x \leqslant a$, $y \leqslant b$, $x \nleqslant p$, and $y \nleqslant \sqrt{p}$. Let $x' \leqslant a$ and $y' \leqslant b$ be any two compact elements of L. Then $(x \vee x'), (y \vee y') \in L_*$ such that $(x \vee x')(y \vee y') \leqslant ab \leqslant p$, $(x \vee x') \nleqslant p$, and $(y \vee y') \nleqslant \sqrt{p}$. Since p is weakly primary, it follows that $(x \vee x')(y \vee y') = 0$. So $x'y' = 0$, which implies $ab = 0 \leqslant \bigwedge_{n=1}^{\infty} p^n$, a contradiction to $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n$. Hence p is ω-primary.

Next we show that p is ω-primary implies p is n-almost primary $(n \geqslant 2)$. Assume p is ω-primary and $n \geqslant 2$. Let $ab \leqslant p$, $ab \nleqslant p^n$ for some $a, b \in L$. Then $ab \leqslant p$, $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n$. Since p is ω-primary, it follows that either $a \leqslant p$ or $b \leqslant \sqrt{p}$. Hence, p is n-almost primary $(n \geqslant 2)$.



$\cdot$	o	a	b	c	d	e	f	l
o	o	o	o	o	o	o	o	o
a	o	c	d	e	f	e	o	a
b	o	d	b	f	d	o	f	b
c	o	e	f	e	o	e	o	c
d	o	f	d	o	f	o	o	d
e	o	e	o	e	o	e	o	e
f	o	o	f	o	o	o	o	f
l	o	a	b	c	d	e	f	l

Figure 1

The last implication is obvious (from $n = 2$). □

From this theorem we get the following characterization of a ω-primary element in L.

Corollary 11. *Let $p \in L$ be a proper element. Then p is ω-primary if and only if p is n-almost primary for every $n \geqslant 2$.*

Proof. Suppose $p \in L$ is n-almost primary for every $n \geqslant 2$. Let $ab \leqslant p$ and $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n$ for some $a, b \in L$. Then $ab \leqslant p$ and $ab \nleqslant p^n$ for some $n \geqslant 2$. Since p is n-almost primary, we have $a \leqslant p$ or $b \leqslant \sqrt{p}$. Hence p is ω-primary. The converse follows from Theorem 10. □

Clearly every primary element in L is ϕ-primary. But the converse is not true as shown in the following example by taking $\phi(p) = p^2$.

Example 12. Consider the lattice L of ideals of the ring $R = \langle Z_{30}, +, \cdot \rangle$. Then the only ideals of R are the principal ideals (0), (2), (3), (5), (6), (10), (15), (1). Clearly $L = \{(0), (2), (3), (5), (6), (10), (15), (1)\}$ is a compactly generated multiplicative lattice. Here the element $(6) \in L$ is almost primary but not primary.

In the following successive three theorems, we show conditions under which a ϕ-primary element in L is primary.

Now we have a characterization of a ϕ-primary element in L.

Theorem 13. *Let L be a Noether lattice. Let $0 \neq a \in L$ be a non-nilpotent proper element satisfying the restricted cancellation law. Then a is ϕ-primary for some $\phi \leqslant \phi_2$ if and only if a is primary.*

Proof. Suppose $a \in L$ is a primary element. Then obviously a is ϕ-primary for every ϕ and hence for $\phi \leqslant \phi_2$. Conversely, let a be ϕ-primary for some $\phi \leqslant \phi_2$. Then, by Theorem 9, a is ϕ_2-primary (almost primary). Let $xy \leqslant a$ for some $x, y \in L_*$. If $xy \nleqslant a^2$, then as a is ϕ_2-primary we have $x \leqslant a$ or $y \leqslant \sqrt{a}$. If $xy \leqslant a^2$, consider $(x \vee a)y = xy \vee ay \leqslant a$. If $(x \vee a)y \nleqslant a^2$, then as a is ϕ_2-primary we have $(x \vee a) \leqslant a$ or $y^m \leqslant a$ for some $m \in Z_+$ and hence $x \leqslant a$ or $y \leqslant \sqrt{a}$. So assume $(x \vee a)y \leqslant a^2$. Then $xy \leqslant a^2$ and $ay \leqslant a^2 \neq 0$. By Lemma 1.11 of [1] we get $y \leqslant a$ which shows that a is primary. □

Definition 14. A proper element $p \in L$ is said to be 2-potent prime if, for $a, b \in L$, $ab \leqslant p^2$ implies either $a \leqslant p$ or $b \leqslant p$.

Definition 15. A proper element $p \in L$ is said to be 2-potent primary if, for $a, b \in L$, $ab \leqslant p^2$ implies either $a \leqslant p$ or $b \leqslant \sqrt{p}$.

Theorem 16. *Let a proper element $q \in L$ be 2-potent primary. If q is ϕ-primary for some $\phi \leqslant \phi_2$, then q is primary.*

Proof. Clearly by Theorem 9, $q \in L$ is ϕ_2-primary (almost primary). Let $xy \leqslant q$ for some $x, y \in L_*$. If $xy \nleqslant q^2$, then as q is ϕ_2-primary we have $x \leqslant q$ or $y \leqslant \sqrt{q}$. If $xy \leqslant q^2$, then as q is 2-potent primary we have $x \leqslant q$ or $y \leqslant \sqrt{q}$. Hence q is primary. □

Theorem 17. *Let a proper element $q \in L$ be ϕ-primary. If $q^2 \nleqslant \phi(q)$, then q is primary.*

Proof. Let $ab \leqslant q$ for some $a, b \in L_*$. If $ab \nleqslant \phi(q)$, then as q is ϕ-primary we have $a \leqslant q$ or $b \leqslant \sqrt{q}$. So assume $ab \leqslant \phi(q)$. First suppose $aq \nleqslant \phi(q)$. Then $ad \nleqslant \phi(q)$ for some $d \leqslant q$ in L. Also $a(b \vee d) = ab \vee ad \leqslant q$ and $a(b \vee d) \nleqslant \phi(q)$. As q is ϕ-primary, either $a \leqslant q$ or $(b \vee d)^k \leqslant q$ for some $k \in Z_+$. Hence $a \leqslant q$ or $b^k \leqslant q$ for some $k \in Z_+$. Similarly if $bq \nleqslant \phi(q)$ we can show that either $a \leqslant q$ or $b^k \leqslant q$ for some $k \in Z_+$. So we can assume that $aq \leqslant \phi(q)$ and $bq \leqslant \phi(q)$. Since $q^2 \nleqslant \phi(q)$, there exist $r, s \leqslant q$ in L such that $rs \nleqslant \phi(q)$.

Then $(a \vee r)(b \vee s) \leqslant q$, but $(a \vee r)(b \vee s) \nleqslant \phi(q)$. As q is ϕ-primary, we have either $(a \vee r) \leqslant q$ or $(b \vee s)^t \leqslant q$ for some $t \in Z_+$. Hence $a \leqslant q$ or $b^t \leqslant q$ for some $t \in Z_+$. Thus q is primary. □

From the above theorem it follows that

① if a proper element $q \in L$ is ϕ-primary but not primary, then $q^2 \leqslant \phi(q)$;

② a ϕ-primary element $q < 1$ in L with $q^2 \nleqslant \phi(q)$ is primary.

Corollary 18. *If a proper element $q \in L$ is ϕ-primary but not primary, then $\sqrt{q} = \sqrt{\phi(q)}$.*

Proof. From Theorem 17 we have $q^2 \leqslant \phi(q)$. So $q \leqslant \sqrt{\phi(q)}$ which gives $\sqrt{q} \leqslant \sqrt{\sqrt{\phi(q)}} = \sqrt{\phi(q)}$. Since $\phi(q) \leqslant q$, we have $\sqrt{\phi(q)} \leqslant \sqrt{q}$. Hence $\sqrt{q} = \sqrt{\phi(q)}$. □

Corollary 19. *If a proper element $q \in L$ is ϕ-primary where $\phi \leqslant \phi_3$, then q is ω-primary.*

Proof. If q is primary, then by Theorem 10, q is ω-primary. So assume that q is not primary. Then by Theorem 17 and hypothesis we get $q^2 \leqslant \phi(q) \leqslant q^3$. Hence $\phi(q) = q^n$ for every $n \geqslant 2$. Consequently q is n-almost primary for every $n \geqslant 2$ and hence it is ω-primary by Corollary 11. □

Corollary 20. *If a proper element $q \in L$ is ϕ_0-primary but not primary, then $q^2 = 0$.*

Proof. The proof is obvious. □

Now we obtain the characterization of a ϕ-primary element in L.

Theorem 21. *Let $q \in L$ be a proper element and let $\phi : L \to L$ be a function. Then the following statements are equivalent:*

① *q is ϕ-primary;*

② *for every $a \in L$ such that $a \nleqslant \sqrt{q}$, either $(q : a) = q$ or $(q : a) = (\phi(q) : a)$;*

③ *for any two elements $r, s \in L_*$, $rs \leqslant q$ and $rs \nleqslant \phi(q)$ implies either $s \leqslant q$ or $r \leqslant \sqrt{q}$.*

Proof. ① ⇒ ②. Suppose ① holds. Let $h \in L$ be compact such that $h \leqslant (q : a)$ and $a \nleqslant \sqrt{q}$. Then $ah \leqslant q$. If $ah \leqslant \phi(q)$, then $h \leqslant (\phi(q) : a)$. If $ah \nleqslant \phi(q)$, then since q is ϕ-primary and $a \nleqslant \sqrt{q}$ it follows that $h \leqslant q$. Hence by Lemma 1 of [6] either $(q : a) \leqslant (\phi(q) : a)$ or $(q : a) \leqslant q$. Consequently either $(q : a) = (\phi(q) : a)$ or $(q : a) = q$.

② ⇒ ③. Suppose ② holds. Let $rs \leqslant q$, $rs \nleqslant \phi(q)$, and $r \nleqslant \sqrt{q}$ for some $r, s \in L_*$. Then by ② we have either $(q : r) = (\phi(q) : r)$ or $(q : r) = q$. If $(q : r) = (\phi(q) : r)$, then as $s \leqslant (q : r)$ it follows that $s \leqslant (\phi(q) : r)$ which contradicts $rs \nleqslant \phi(q)$. If $(q : r) = q$, then $s \leqslant (q : r)$ gives $s \leqslant q$.

③ ⇒ ①. Suppose ③ holds. Let $ab \leqslant q$, $ab \nleqslant \phi(q)$, and $a \nleqslant \sqrt{q}$ for some $a, b \in L$. Then as L is compactly generated, there exist $x, x', y' \in L_*$ such that $x \leqslant a$, $x' \leqslant a$, and $y' \leqslant b$ but $x \nleqslant \sqrt{q}$, $x'y' \nleqslant \phi(q)$. Let $y \leqslant b$ be any compact element of L. Then $(x \vee x'), (y \vee y') \in L_*$ such that $(x \vee x')(y \vee y') \leqslant ab \leqslant q$, $(x \vee x')(y \vee y') \leqslant ab \nleqslant \phi(q)$, and $(x \vee x') \nleqslant \sqrt{q}$. So by ③ $(y \vee y') \leqslant q$ which implies $b \leqslant q$. Therefore, q is ϕ-primary. □

Like ω-primary element in L we introduce the notion of ω-prime element in L.

Definition 22. Let $\phi : L \to L$ be a function. A proper element $p \in L$ is said to be ϕ_ω-prime (or briefly ω-prime) if, for $a, b \in L$, $ab \leqslant p$ and $ab \nleqslant \bigwedge_{n=1}^{\infty} p^n = \phi_\omega(p)$ implies either $a \leqslant p$ or $b \leqslant p$.

Similarly ϕ_α-prime elements in L are defined by following settings in the definition for ϕ-prime element. For any proper element $p \in L$,

(i) $\phi_0(p) = 0$ and then p is called weakly prime element;

(ii) $\phi_2(p) = p^2$ and then p is called almost prime element;

(iii) $\phi_n(p) = p^n$ $(n \geqslant 2)$ and then p is called n-almost prime element.

The analogous theorems and corollaries (obtained for ϕ-primary elements in L) for ϕ-prime elements in L are as follows whose proofs being on similar arguments are omitted.

Theorem 23. *Let $\gamma_1, \gamma_2 : L \to L$ be functions such that $\gamma_1 \leqslant \gamma_2$. Then every proper γ_1-prime element in L is γ_2-prime.*

Theorem 24. *Let $p \in L$ be a proper element. Then p is prime implies p is weakly prime implies p is ω-prime implies p is n-almost prime $(n \geqslant 2)$ implies p is almost prime.*

Now we have the characterization of an ω-prime element in L.

Corollary 25. *Let $p \in L$ be a proper element. Then p is ω-prime if and only if p is n-almost prime for every $n \geqslant 2$.*

Clearly every prime element in L is ϕ-prime. But the converse is not true as shown in the following example by taking $\phi(p) = p^2$.

Example 26. Consider L as in Example 6. Here the element $(0) \in L$ is weakly prime and hence almost prime, while (0) is not prime since $(2)(12) \leqslant (0)$, but neither $(2) \leqslant (0)$ nor $(12) \leqslant (0)$.

In the following successive three theorems we show conditions under which a ϕ-prime element in L is prime.

The next theorem gives a characterization of a ϕ-prime element in L.

Theorem 27. *Let L be a Noether lattice. Let $0 \neq a \in L$ be a non-nilpotent proper element satisfying the restricted cancellation law. Then a is ϕ-prime for some $\phi \leqslant \phi_2$ if and only if a is prime.*

Theorem 28. *Let a proper element $q \in L$ be 2-potent prime. If q is ϕ-prime for some $\phi \leqslant \phi_2$, then q is prime.*

Theorem 29. *Let a proper element $q \in L$ be ϕ-prime. If $q^2 \nleqslant \phi(q)$, then q is prime.*

From the above theorem it follows that

① if a proper element $q \in L$ is ϕ-prime but not prime, then $q^2 \leqslant \phi(q)$;

② a ϕ-prime element $q < 1$ in L with $q^2 \nleqslant \phi(q)$ is prime.

Corollary 30. *If a proper element $q \in L$ is ϕ-prime but not prime, then $\sqrt{q} = \sqrt{\phi(q)}$.*

Corollary 31. *If a proper element $q \in L$ is ϕ-prime where $\phi \leqslant \phi_3$, then q is ω-prime.*

Corollary 32. *If a proper element $q \in L$ is ϕ_0-prime but not prime, then $q^2 = 0$.*

Theorem 33. *Let $q \in L$ be a proper element and let $\phi : L \to L$ be a function. Then the following statements are equivalent:*

① *q is ϕ-prime;*

② *for every $a \in L$ such that $a \nleqslant q$, either $(q : a) = q$ or $(q : a) = (\phi(q) : a)$;*

③ *for any two elements $r, s \in L_*$, $rs \leqslant q$ and $rs \nleqslant \phi(q)$ implies either $s \leqslant q$ or $r \leqslant q$.*

3. Almost Prime and Almost Primary Elements in L

The study of almost prime and almost primary ideals for commutative rings is carried out by Bhatwadekar and Sharma [7] and Bataineh and Kuhail [4], respectively. From the previous section we know that almost prime and almost primary elements in L are particular cases of ϕ_α-prime and ϕ_α-primary elements in L, respectively. In this section, we obtain some results on an almost prime element and on an almost primary element in L with characterizations.

The reader can verify the following statements.

(1) Every almost prime element in L is almost primary.

(Its converse holds for semiprime elements.)

(2) Every prime element in L is almost prime.

(3) Every idempotent element in L is almost prime.

(4) Every prime element in L is almost primary.

(5) Every idempotent element in L is almost primary.

But the converse is not true as shown in the following example.

Example 34. Consider the lattice L of ideals of the ring $R = \langle Z_8, +, \cdot \rangle$. Then the only ideals of R are the principal ideals $(0), (2), (4), (1)$. Clearly $L = \{(0), (2), (4), (1)\}$ is a compactly generated multiplicative lattice. Here the element $(4) \in L$ is almost primary but not idempotent.

(6) Every primary element in L is almost primary.

But the converse is not true, which is clear from Example 12.

In the following successive three theorems we show conditions under which an almost primary element in L is primary.

Theorem 35. *Let a proper element $p \in L$ be 2-potent primary. Then p being almost primary implies that p is primary.*

Proof. The proof is obvious. □

Theorem 36. *Let a proper element $p \in L$ be 2-potent prime. Then*

① *p being almost prime implies that p is prime;*

② *p being almost primary implies that p is primary.*

Proof. The proof is obvious. □

From the following examples it is clear that

① an almost primary element in L need not be 2-potent prime;

② a 2-potent prime element in L which is almost primary need not be prime.

Example 37. Consider L as in Example 12. Here the element $(6) \in L$ is almost primary but not 2-potent prime.

Example 38. Consider L as in Example 34. Here the element $(4) \in L$ is 2-potent prime, almost primary but not prime.

Theorem 39. *Let L be a local Noetherian domain. If a proper element $p \in L$ is n-almost primary ($n \geqslant 2$), then p is primary.*

Proof. Let $ab \leqslant p$ for some $a, b \in L_*$. If $ab \nleqslant p^n$ for all $n \geqslant 2$, then as $p \in L$ is n-almost primary we have $a \leqslant p$ or $b \leqslant \sqrt{p}$. If $ab \leqslant p^n$ for all $n \geqslant 1$, then as L is a local Noetherian domain, from Corollary 3.3 of [8], it follows that $ab \leqslant \bigwedge_{n=1}^{\infty} p^n = \{0\}$. Thus $ab = 0$. Since L is domain, we have either $a = 0$ or $b = 0$ which implies $a \leqslant p$ or $b \leqslant p$ and hence p is primary. □

Theorem 40. *Let (L, m) be a quasi-local Noether lattice. If a proper element $p \in L$ is such that $p^2 = m^2 \leqslant p \leqslant m$, then p is almost primary.*

Proof. Let $xy \leqslant p$ and $xy \nleqslant p^2$ for some $x, y \in L$. If $x \nleqslant m$, then $x = 1$. So $xy \leqslant p$ gives $y \leqslant p$. Similarly $y \nleqslant m$ gives $x \leqslant p$. Now if $x \leqslant m$, then $x^2 \leqslant m^2 = p^2 \leqslant p$ and hence $x \leqslant \sqrt{p}$. Similarly $y \leqslant m$ gives $y \leqslant \sqrt{p}$. Hence in any case p is almost primary. □

The next theorem gives the characterization of an almost primary element in L.

Theorem 41. *Let a proper element $q \in L$ be join irreducible. Then the following statements are equivalent:*

① *q is almost primary;*

② *for every $a \in L$ such that $a \nleqslant \sqrt{q}$, $(q : a) = q \vee (q^2 : a)$;*

③ *for every $a \in L$ such that $a \nleqslant \sqrt{q}$, either $(q : a) = q$ or $(q : a) = (q^2 : a)$.*

Proof. ① ⇒ ②. Suppose ① holds. Let $h \in L$ be compact such that $h \leqslant (q : a)$ and $a \nleqslant \sqrt{q}$. Then $ah \leqslant q$. If $ah \leqslant q^2$, then $h \leqslant (q^2 : a)$. If $ah \nleqslant q^2$, then since q is almost primary and $a \nleqslant \sqrt{q}$ it follows that $h \leqslant q$. Hence by Lemma 1 of [6] either $(q : a) \leqslant (q^2 : a) \leqslant q \vee (q^2 : a)$ or $(q : a) \leqslant q \leqslant q \vee (q^2 : a)$. But as $q \vee (q^2 : a) \leqslant (q : a)$, we have $(q : a) = q \vee (q^2 : a)$.

② ⇒ ③. Suppose ② holds. Let $a \nleqslant \sqrt{q}$ for some $a \in L$. Then by ② $(q : a) = q \vee (q^2 : a)$. Now as q is join irreducible, we have either $(q : a) = q$ or $(q : a) = (q^2 : a)$.

③ ⇒ ①. Suppose ③ holds. Let $rs \leqslant q$, $rs \nleqslant q^2$, and $r \nleqslant \sqrt{q}$ for some $r, s \in L$. Then by ③ we have either $(q : r) = (q^2 : r)$ or $(q : r) = q$. If $(q : r) = (q^2 : r)$, then as $s \leqslant (q : r)$ it follows that $s \leqslant (q^2 : r)$ which contradicts $rs \nleqslant q^2$. If $(q : r) = q$, then $s \leqslant (q : r)$ gives $s \leqslant q$. Therefore q is almost primary. □

In view of Proposition 2 of [9] and the above Theorems 21 and 41 we have the following result.

Corollary 42. *Let (L, m) be a quasi-local Noether lattice. Let a proper element $q \in L$ be weak meet principal and let $f \leqslant q$ be a nonzero weak join principal element in L. Then the following statements are equivalent:*

① *q is almost primary;*

② *for every $a \in L$ such that $a \nleqslant \sqrt{q}$, $(q : a) = q \vee (q^2 : a)$;*

③ *for every $a \in L$ such that $a \nleqslant \sqrt{q}$, either $(q : a) = q$ or $(q : a) = (q^2 : a)$;*

④ *for any two elements $r, s \in L_*$, $rs \leqslant q$ and $rs \nleqslant q^2$ implies either $s \leqslant q$ or $r \leqslant \sqrt{q}$.*

Proof. The proof is obvious. □

The following theorem gives the similar characterization of an almost primary element in L.

Theorem 43. *A proper element $q \in L$ is almost primary if and only if for every $a \in L$ such that $a \nleqslant q$ either $(q : a) \leqslant \sqrt{q}$ or $(q : a) = (q^2 : a)$.*

Proof. Assume that the proper element $q \in L$ is almost primary. Let $h \in L$ be compact such that $h \leqslant (q : a)$ and $a \nleqslant q$. Then $ah \leqslant q$. If $ah \leqslant q^2$, then $h \leqslant (q^2 : a)$. If $ah \nleqslant q^2$, then since q is almost primary and $a \nleqslant q$ it follows that $h \leqslant \sqrt{q}$. Hence by Lemma 1 of [6] either $(q : a) \leqslant (q^2 : a)$ or $(q : a) \leqslant \sqrt{q}$. But as $(q^2 : a) \leqslant (q : a)$, we have $(q : a) \leqslant \sqrt{q}$ or $(q : a) = (q^2 : a)$. Conversely assume that for every $a \in L$ such that $a \nleqslant q$ either $(q : a) \leqslant \sqrt{q}$ or $(q : a) = (q^2 : a)$. Let $rs \leqslant q$, $rs \nleqslant q^2$, and $r \nleqslant q$ for some $r, s \in L$. Then either $(q : r) = (q^2 : r)$ or $(q : r) \leqslant \sqrt{q}$. If $(q : r) = (q^2 : r)$, then as $s \leqslant (q : r)$ it follows that $s \leqslant (q^2 : r)$ which contradicts $rs \nleqslant q^2$. If $(q : r) \leqslant \sqrt{q}$, then $s \leqslant (q : r)$ gives $s \leqslant \sqrt{q}$. Therefore q is almost primary. □

The analogous theorems and corollaries (obtained for almost primary elements in L) for almost prime elements in L are as follows whose proofs being on similar arguments are omitted.

Theorem 44. *Let L be a local Noetherian domain. If a proper element $p \in L$ is n-almost prime ($n \geqslant 2$), then p is prime.*

Theorem 45. *Let (L, m) be a quasi-local Noether lattice. If a proper element $p \in L$ is such that $p^2 = m^2 \leqslant p \leqslant m$, then p is almost prime.*

The next theorem gives the characterization of an almost prime element in L.

Theorem 46. *Let a proper element $q \in L$ be join irreducible. Then the following statements are equivalent:*

① *q is almost prime;*

② *for every $a \in L$ such that $a \nleqslant q$, $(q : a) = q \vee (q^2 : a)$;*

③ *for every $a \in L$ such that $a \nleqslant q$, either $(q : a) = q$ or $(q : a) = (q^2 : a)$.*

Corollary 47. *Let (L, m) be a quasi-local Noether lattice. Let a proper element $q \in L$ be weak meet principal and let $f \leqslant q$ be a nonzero weak join principal element in L. Then the following statements are equivalent:*

① *q is almost prime;*

② *for every $a \in L$ such that $a \nleqslant q$, $(q : a) = q \vee (q^2 : a)$;*

③ *for every $a \in L$ such that $a \nleqslant q$, either $(q : a) = q$ or $(q : a) = (q^2 : a)$;*

④ *for any two elements $r, s \in L_*$, $rs \leqslant q$ and $rs \nleqslant q^2$ implies either $s \leqslant q$ or $r \leqslant q$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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