Research Article

The Graph of Equivalence Classes of Zero Divisors

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We introduce a graph $G_E(L)$ of equivalence classes of zero divisors of a meet semilattice L with 0. The set of vertices of $G_E(L)$ are the equivalence classes of nonzero zero divisors of L and two vertices $[x]$ and $[y]$ are adjacent if and only if $[x] \wedge [y] = [0]$. It is proved that $G_E(L)$ is connected and either it contains a cycle of length 3 or $G_E(L) \cong K_2$. It is known that two Boolean lattices L_1 and L_2 have isomorphic zero divisor graphs if and only if $L_1 \cong L_2$. This result is extended to the class of SSC meet semilattices. Finally, we show that Beck's Conjecture is true for $G_E(L)$.

1. Introduction

The idea of a zero divisor graph was introduced by Beck in [1] to investigate the interplay between ring theoretic properties and graph theoretic properties. The concept of zero divisor graph is also well studied in ordered structures; see Alizadeh et al. [2, 3], Halaš and Jukl [4], Halaš and Länger [5], Joshi [6], Joshi et al. [7–11], and Lu and Wu [12].

The graph of equivalence classes of zero divisors of commutative rings is well studied in Allen et al. [13], Anderson and LaGrange [14], and Spiroff and Wickham [15].

In this paper, we extend this concept to a meet semilattice L with 0. The main aspect of the graph $G_E(L)$ of equivalence classes of zero divisors of L is the connection to the associated primes of L . In general, every vertex of $G_E(L)$ either corresponds to an associated prime or is adjacent to an associated prime of L . Further, we consider the equivalence relation $\sim$ on a meet semilattice L with 0, $x \sim y$ if and only if $\text{ann}(x) = \text{ann}(y)$ for $x, y \in L$ and construct the meet semilattice $L' = \{[x] \mid x \in Z_{\{0\}}(L)^*\}$. It is proved that L' is an SSC meet semilattice with the property that $G_E(L) \cong G_{\{0\}}(L')$, the zero divisor graph of L' . Hence it is clear that the study of $G_E(L)$ is nothing, but the study of zero divisor graph of SSC meet semilattices. This observation helps us to prove that for bounded SSC meet semilattices L_1 and L_2 , $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ if and only if $L_1 \cong L_2$. This extends the result of LaGrange [16] for Boolean algebra. It is proved that for an SSC meet semilattice L , $G_{\{0\}}(L)$ is connected and either it contains

a cycle of length 3 or $G_{\{0\}}(L) \cong K_2$. In the class of a finite semicomplemented meet semilattice L , $G_E(L)$ is isomorphic to the zero divisor graph $G_{\{0\}}(L)$ if and only if L is SSC.

2. Properties of the Graph $G_E(L)$

We begin with the necessary definitions and terminology.

A nonempty subset I of a meet semilattice L is said to be *semi-ideal*, if $y \leq x \in I$ implies that $y \in I$. A semi-ideal I of a meet semi-lattice L is said to be an *ideal*, if $a, b \in I$ implies that $a \vee b \in I$. Dually we have the concept of a filter. A proper semi-ideal (ideal) is said to be *prime*, if $a \wedge b \in I$ implies either $a \in I$ or $b \in I$.

A prime semi-ideal (ideal) P of a meet semilattice L is said to be *minimal prime* semi-ideal (ideal), if there is no prime ideal Q such that $Q \subsetneq P$.

The concept of SSC lattices is introduced by Janowitz [17]. We write this definition for a meet semilattice with 0.

A meet semilattice L with 0 is said to be *semicomplemented* if, for $a \neq 1$, there exists $b \neq 0$ such that $a \wedge b = 0$ and L is said to be *section semicomplemented* (in brief SSC) if for $a, b \in L$ with $a < b$, there exists nonzero c such that $0 < c \leq b$ and $a \wedge c = 0$. It is clear that every SSC lattice is a semicomplemented lattice, but not conversely. More details about SSC poset can be found in Thakare et al. [18, 19], Joshi [20].

Let L be a meet semilattice with 0. The set of all *zero divisors* of L is denoted by $Z_{\{0\}}(L) = \{x \in L \mid x \wedge y = 0\}$

FIGURE 1: The graph $G_E(L)$ of equivalence classes of zero divisors.

for some $y \in L \setminus \{0\}$ and the set of all nonzero zero divisors is denoted by $Z_{\{0\}}(L)^*$. Clearly, $Z_{\{0\}}(L)^* \cup \{0\} = Z_{\{0\}}(L)$.

Let G be a graph, and let x, y be distinct vertices in G . We denote by $d(x, y)$ the length of a shortest path from x to y if it exists and put $d(x, y) = \infty$ if no such path exists. The *diameter* of G , denoted by $\text{diam}(G)$, is zero if G is the graph on one vertex and is $\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\}$ otherwise. A *cycle* in a graph G is a path that begins and ends at the same vertex. The *girth* of G , written $\text{girth}(G)$, is the length of the shortest cycle in G (and $\text{girth}(G) = \infty$ if G has no cycles). The chromatic number of G is denoted by $\chi(G)$. Thus, $\chi(G)$ is the minimum number of colors which can be assigned to the elements of L such that adjacent elements receive different colors. If this number is not finite, write $\chi(G) = \infty$. A subset C of G is a *clique* if any two distinct vertices of C are adjacent. If G contains a clique with n elements and every clique has at most n elements then the clique number of G is $\text{Clique}(G) = n$. If the sizes of the cliques are not bounded, then $\text{Clique}(G) = \infty$. We always have $\chi(G) \geq \text{Clique}(G)$.

Joshi [6] introduced the concept of the zero divisor graph of a poset P having the smallest element 0 with respect to an ideal I of P . We mention below this definition when the corresponding poset is a meet semilattice and $I = \{0\}$. Note that this definition coincides with the definition given by Lu and Wu [12] when $I = \{0\}$.

Definition 1 (Joshi [6], Lu and Wu [12]). Let L be a semilattice with 0. We associate a simple undirected graph with L , called the *zero divisor graph of L with respect to $\{0\}$* , denoted by $G_{\{0\}}(L)$, with the set of vertices is $V(G_{\{0\}}(L)) = Z_{\{0\}}(L)^* = \{x \in L \setminus \{0\} \mid x \wedge y = 0 \text{ for some } y \in L \setminus \{0\}\}$ and two distinct vertices x, y are adjacent if and only if $x \wedge y = 0$.

For any $x, y \in L$, we say that $x \sim y$ if and only if $\text{ann}(x) = \text{ann}(y)$, where $\text{ann}(a) = \{b \mid a \wedge b = 0\}$. Note that $\sim$ is an equivalence relation on L . Furthermore, if $x_1 \sim x_2$ and $x_1 \wedge y = 0$, then $y \in \text{ann}(x_1) = \text{ann}(x_2)$ and hence $x_2 \wedge y = 0$. We define $[x]$, the equivalence class of x , as follows: $[x] = \{z \in L \mid \text{ann}(z) = \text{ann}(x)\}$.

Lemma 2. Let L be a meet semilattice with 0 and $L' = \{[x] \mid x \in L\}$, where $[x] = \{z \in L \mid \text{ann}(z) = \text{ann}(x)\}$. Then L' is a meet semilattice under the partial order $[x] \leq [y]$ if and only if $\text{ann}(y) \subseteq \text{ann}(x)$.

Proof. Since $\text{ann}(x), \text{ann}(y) \subseteq \text{ann}(x \wedge y)$, we have $[x \wedge y] \leq [x], [y]$. We claim that $[x] \wedge [y] = [x \wedge y]$. Let $[t]$ be another lower bound of $[x]$ and $[y]$. Then $\text{ann}(x), \text{ann}(y) \subseteq \text{ann}(t)$.

FIGURE 2: The graph $G_E(L)$ is finite, whereas $G_E(L_1)$ is infinite.

Now, we claim that $\text{ann}(x \wedge y) \subseteq \text{ann}(t)$. Let $z \in \text{ann}(x \wedge y)$. Then $z \wedge x \in \text{ann}(y) \subseteq \text{ann}(t)$. This gives $x \wedge z \wedge t = 0$; that is, $z \wedge t \in \text{ann}(x) \subseteq \text{ann}(t)$. Hence $z \wedge t = 0$; that is, $z \in \text{ann}(t)$. This proves that $\text{ann}(x \wedge y) \subseteq \text{ann}(t)$. Thus $[t] \leq [x \wedge y]$ and this yields $[x] \wedge [y] = [x \wedge y]$. Clearly, if L is a finite lattice, then L' is also a lattice. $\square$

Lemma 3. L' is an SSC meet semilattice.

Proof. Let $0 < [x] \not\leq [y]$. Then $\text{ann}(y) \not\subseteq \text{ann}(x)$. Hence there exists $z \in \text{ann}(x)$ such that $z \notin \text{ann}(y)$. Thus $0 \neq z \wedge y = t$. But then $[0] < [t] \leq [y]$ and $[t] \wedge [x] = [0]$. Thus L' is an SSC meet semilattice. $\square$

Now, we introduce the graph $G_E(L)$ of equivalence classes of zero divisors of a meet semilattice L with 0, denoted by $G_E(L)$.

Definition 4. Let L be a meet semilattice with 0. We associate a simple undirected graph with L whose vertices are the equivalence classes of elements in $Z_{\{0\}}(L)^*$ and with each pair of distinct classes $[x]$ and $[y]$ joined by an edge if and only if $[x] \wedge [y] = \{0\} = [0]$.

We illustrate the concept of the graph of equivalence classes of zero divisor with an example, see Figure 1.

Consider the following two infinite lattices L and L_1 .

We can see that the zero divisor graph of lattices $L, G_{\{0\}}(L)$ is infinite, but the graph of equivalence classes of zero divisors of the lattice L , namely, $G_E(L)$, is finite. Also for the lattices $L_1, G_{\{0\}}(L_1)$, and $G_E(L_1)$ both are infinite, see Figure 2.

Corollary 5. Let L be a meet semilattice with 0. Then there exists an SSC meet semilattice L' (as constructed in Lemma 2) such that $G_E(L) = G_{\{0\}}(L')$.

Proof. From the construction of L' , it is clear that $G_E(L) = G_{\{0\}}(L')$. $\square$

Hence from Corollary 5, it is clear that the study of the graph of equivalence classes of zero divisors of L is nothing, but the study of zero divisor graphs of SSC meet semilattices.

Remark 6. Recall that a prime semi-ideal P of a meet semilattice L is an associated prime if it is the annihilator of some elements of L ; that is, $P = \text{ann}(x)$ for some nonzero $x \in L$. The set of associated primes is denoted by $\text{Ass}(L)$. If $\text{Clique}(G_{\{0\}}(L)) < \infty$, then $\text{Ass}(L)$ is nonempty and finite. Further, any maximal element of the family of annihilator semi-ideals is prime; see Halaš and Jukl [4, Lemma 2.4, 2.6]. Note that every zero divisor is contained in an annihilator ideal and maximal annihilators are associated primes; we have $Z_{\{0\}}(L) \subseteq \bigcup_{P \in \text{Ass}(L)} P$. Clearly, the reverse inclusion is trivial. Hence $Z_{\{0\}}(L) = \bigcup_{P \in \text{Ass}(L)} P$; that is, the set of zero-divisors of L equals the union of all associated primes of L . It is obvious to prove that a map $\phi : \text{Ass}(L) \rightarrow V(G_E(L))$ defined by $\phi(P) = [x]$, where $P = \text{ann}(x)$ is an injective map. We adopt the conventions of Spiroff and Wickham [15], and by a slight abuse of terminology we will refer to the vertex $[y]$ as an associated prime if $\text{ann}(y) \in \text{Ass}(L)$. It will be clear from context whether $[y]$ refers to an equivalence class, a vertex, or a specific annihilator.

Theorem 7. Let L be a meet semilattice with 0, and let $\text{Clique}(G_{\{0\}}(L)) < \infty$. If $\text{ann}(x)$ and $\text{ann}(y)$ are distinct prime semi-ideals of L , then $[x]$ and $[y]$ are adjacent in $G_E(L)$. Furthermore, every vertex of $G_E(L)$ is either an associated prime semi-ideal or adjacent to a maximal semi-ideal in $\mathcal{L} = \{\text{ann}(x) \mid (0 \neq) x \in L\}$.

Proof. Since $\text{ann}(x)$ and $\text{ann}(y)$ are distinct, we have $x \wedge y = 0$. This further gives that $[x] \wedge [y] = [x \wedge y] = [0]$. Thus $[x]$ and $[y]$ are adjacent. Furthermore, suppose that $[x] \in V(G_E(L))$ is not an associated prime semi-ideal. Then there exists $[z] \in V(G_E(L))$ such that $[x] \wedge [z] = [0] = \{0\}$. Hence $x \in \text{ann}(z)$. We claim that $\text{ann}(z)$ is maximal. Otherwise, since $\text{Clique}(G_{\{0\}}(L)) < \infty$, $\mathcal{L}$ has a maximal element, say $\text{ann}(z')$ ($z' \neq 0$) containing $\text{ann}(z)$, then $x \in \text{ann}(z) \subseteq \text{ann}(z')$. It is clear that every maximal element of $\mathcal{L}$ is prime. Therefore $x \wedge z' = 0$. Thus there is an edge between $[x]$ and $[z']$. This completes the proof. $\square$

Remark 8. For a Boolean lattice, $[a] = \{a\}$ for every $a \in Z_{\{0\}}(L)$, and hence in this case, $G_E(L)$ and $G_{\{0\}}(L)$ are isomorphic. This fact is illustrated in Figure 3.

In view of Remark 8, we raise the following problem.

Question 1. Find a class of meet semilattices $\mathcal{L}$ such that $G_{\{0\}}(L) \cong G_E(L)$ for $L \in \mathcal{L}$.

Before we answer this question, we need the following result which follows from the definition of SSC meet semilattice.

FIGURE 3: The isomorphism of $G_E(L)$ and $G_{\{0\}}(L)$.

Lemma 9. Let L be a meet semilattice with 0. Then L is SSC if and only if for $a, b \in L$, $\text{ann}(a) = \text{ann}(b)$ implies $a = b$.

Proof. Suppose that L is SSC. Further, assume that there exist $a, b \in L$ such that $\text{ann}(a) = \text{ann}(b)$ and $a \neq b$. Since L is SSC, there exist $c \in L$ such that $0 < c \leq a$ and $c \wedge b = 0$. This gives $c \in \text{ann}(b) = \text{ann}(a)$, a contradiction to the fact that $c \neq 0$. Hence $a = b$. $\square$

Theorem 10. Let L be a finite semicomplemented meet semilattice. Then $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism if and only if L is SSC.

Proof. Let L be a SSC meet semilattice. Then $G_E(L) \cong G_{\{0\}}(L)$ follows from Lemma 9.

Conversely, assume that $G_E(L) \cong G_{\{0\}}(L)$. First, we prove that $[a] = \{a\}$, $\forall a \in Z_{\{0\}}(L)^*$. Let $b \in [a]$ for $a \neq b \in Z_{\{0\}}(L)^*$. Then we have $[a] = [b]$ for $a \neq b$, a contradiction to $|V(G_E(L))| = |V(G_{\{0\}}(L))|$. Now, let $a < b$ ($a \neq 0$). Since L is semicomplemented, $a, b \in Z_{\{0\}}(L)^*$. This gives $\text{ann}(b) \subsetneq \text{ann}(a)$. Otherwise, if $\text{ann}(a) = \text{ann}(b)$, then $b \in [a] = \{a\}$, a contradiction. From the proof of Lemma 9, it is clear that L is SSC. $\square$

If we consider that the $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism defined by $\phi(a) = [a]$, then we do not need even the finiteness of L .

Theorem 11. Let L be a semicomplemented meet semilattice. Then $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism defined by $\phi(a) = [a]$ if and only if L is SSC.

Proof. Let L be a SSC meet semilattice. Then $G_E(L) \cong G_{\{0\}}(L)$ follows from Lemma 9.

Conversely, assume that $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism defined by $\phi(a) = [a]$. This gives $[a] = \{a\}$, for all $a \in Z_{\{0\}}(L)^*$. Let $a < b$ ($a \neq 0$). Since L is semicomplemented, $a, b \in Z_{\{0\}}(L)^*$. This gives $\text{ann}(b) \subsetneq \text{ann}(a)$. From the proof of Lemma 9, it is clear that L is SSC. $\square$

From Figure 4, it is clear that for the posets P_1, P_2 , $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \not\Rightarrow P_1 \cong P_2$. Hence, it is worth to study the following problem.

Problem 12. Find a class of posets $\mathcal{P}$ such that $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$ for the posets $P_1, P_2 \in \mathcal{P}$.

FIGURE 4: Nonisomorphic lattices with isomorphic zero divisor graphs.

In [8], Joshi and Khiste answered this problem in the case of Boolean posets. More details about Boolean posets can be found in Waphare and Joshi [21].

Theorem 13 (Joshi and Khiste [8, Theorem 2.11]). *Let P_1 and P_2 be Boolean posets. Then $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$.*

It is well known that there is a 1 – 1 correspondence between Boolean algebras and Boolean rings.

A consequence of the above theorem is the following result.

Corollary 14 (LaGrange [16, Theorem 4.1], Mohammadian [22, Theorem 7]). *Let P_1 and P_2 be Boolean algebras. Then $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$.*

Now, we are ready to extend this result to the class of SSC meet semilattices. Note that every Boolean lattice is SSC but not conversely. Consider the lattice L_1 depicted in Figure 2 is SSC but not a Boolean lattice.

Theorem 15. *Let L_1 and L_2 be bounded SSC meet semilattices. Then the following statements are equivalent:*

- (1) $G_E(L_1) \cong G_E(L_2)$;
- (2) $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$;
- (3) $L_1 \cong L_2$.

Proof. The equivalence of statements (1) and (2) follows from Lemma 9.

(2) $\Rightarrow$ (3) : Since L_1 and L_2 are semicomplemented, $V(G_{\{0\}}(L_i)) = L_i \setminus \{0_{L_i}, 1_{L_i}\}$ for $i = 1, 2$. Let $\phi : V(G_{\{0\}}(L_1)) \rightarrow V(G_{\{0\}}(L_2))$ be a graph isomorphism. Define a map $f : L_1 \rightarrow L_2$ such that $f(a) = \phi(a)$, for all $a \in V(G_{\{0\}}(L_1))$ along with $f(0_{L_1}) = 0_{L_2}$ and $f(1_{L_1}) = 1_{L_2}$, where $0_{L_i}(1_{L_i})$ is the smallest (largest) element of L_i . Since ϕ is bijective, f is bijective. We claim that f is a biorder preserving map.

Let $a \leq b$. First, we show that $f(a) \leq f(b)$. If any of a, b is either 0_{L_1} or 1_{L_1} , then we are through. Hence assume that $a, b \notin \{0_{L_1}, 1_{L_1}\}$. Suppose that $f(a) \not\leq f(b)$; that is, $\phi(a) \not\leq \phi(b)$. Since L_2 is SSC, there exists $\phi(c) \in L_2$ such that $0_{L_2} < \phi(c) \leq \phi(a)$ and $\phi(c) \wedge \phi(b) = 0_{L_2}$. Since $b \neq 0_{L_1}$, we have $\phi(b) \neq 0_{L_2}$. This gives that $\phi(c)$ and $\phi(b)$ are adjacent in $G_{\{0\}}(L_2)$. Since ϕ is a graph isomorphism, we have that c and b are adjacent which further imply that $c \wedge b = 0_{L_1}$. Hence, $c \wedge a = 0_{L_1}$ in L_1 . Thus a and c are adjacent in $G_{\{0\}}(L_1)$. Hence, we have $\phi(c) \wedge \phi(a) = 0_{L_2}$. But $\phi(c) \leq \phi(a)$ gives $\phi(c) = 0_{L_2}$, a contradiction. Hence $f(a) \leq f(b)$ in L_2 .

Conversely, assume that $f(a) \leq f(b)$ in L_2 . We claim that $a \leq b$. If any of $f(a)$ or $f(b)$ is 0_{L_2} or 1_{L_2} , then we are through. Let us assume that $f(a), f(b) \in L_2 \setminus \{0_{L_2}, 1_{L_2}\}$. This yields $a, b \notin \{0_{L_1}, 1_{L_1}\}$. Now, suppose on the contrary that $a \not\leq b$ for $a, b \notin \{0_{L_1}, 1_{L_1}\}$. Since L_1 is SSC, there exists $c \in L_1$ such that $0_{L_1} < c \leq a$ and $c \wedge b = 0_{L_1}$. Thus c and b are adjacent in $G_{\{0\}}(L_1)$. Then $\phi(c)$ and $\phi(b)$ are adjacent in $G_{\{0\}}(L_2)$. Then $\phi(c) \wedge \phi(b) = 0_{L_2}$ in L_2 . Replacing the role of a, b in the above proof by c, a , respectively, we have $\phi(c) \leq \phi(a)$. This implies that $\phi(c) \leq \phi(a) \leq \phi(b)$. But this gives $\phi(c) = \phi(c) \wedge \phi(b) = 0_{L_2}$, a contradiction. Hence $a \leq b$. Thus $L_1 \cong L_2$.
(3) $\Rightarrow$ (2) : Obvious. $\square$

Lemma 16. *Let L_1 be a bounded SSC meet semilattice, and L_2 be a bounded semicomplemented meet semilattice such that $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$. Then L_2 is also SSC.*

Proof. Since L_1 and L_2 are semicomplemented, $V(G_{\{0\}}(L_i)) = L_i \setminus \{0_{L_i}, 1_{L_i}\}$ for $i = 1, 2$. Let $\phi : V(G_{\{0\}}(L_1)) \rightarrow V(G_{\{0\}}(L_2))$ be a graph isomorphism. Define a map $f : L_1 \rightarrow L_2$ such that $f(a) = \phi(a)$, for all $a \in V(G_{\{0\}}(L_1))$ along with $f(0_{L_1}) = 0_{L_2}$ and $f(1_{L_1}) = 1_{L_2}$, where $0_{L_i}(1_{L_i})$ is the smallest (largest) element of L_i . Since ϕ is a bijective map, f is bijective. Let $f(a) < f(b)$ in L_2 . Since L_2 is semicomplemented, and if $f(a) = 0_{L_2}$ or $f(b) = 1_{L_2}$, then we are through. Hence assume that $f(a) \neq 0_{L_2}$ and $f(b) \neq 1_{L_2}$. Then clearly, $a \neq b$ and $b \neq 1_{L_1}$. Since L_1 is SSC and $a \wedge b < b$, there exists $0_{L_1} < c \leq b$ and $c \wedge a = 0_{L_1}$. Since c and a are adjacent in $G_{\{0\}}(L_1)$, we have $\phi(c)$ and $\phi(a)$ are adjacent in $G_{\{0\}}(L_2)$; that is, $\phi(c) \wedge \phi(a) = 0_{L_2}$. Further b and c are nonadjacent in $G_{\{0\}}(L_1)$. Then $\phi(b)$ and $\phi(c)$ are nonadjacent in $G_{\{0\}}(L_2)$. Consider $0_{L_2} \neq \phi(b) \wedge \phi(c) = f(b) \wedge f(c) = f(d)$ say. Clearly, $0_{L_2} < f(d) \leq f(b)$ and $f(d) \wedge f(a) = 0_{L_2}$ (as $\phi(c) \wedge \phi(a) = 0_{L_2}$ implies $f(c) \wedge f(a) = 0_{L_2}$). Thus L_2 is SSC. $\square$

Theorem 17. *Let L_1 and L_2 be bounded semicomplemented meet semilattices. If $G_E(L_1) \cong G_{\{0\}}(L_2)$, then L_2 is SSC.*

Proof. For the given lattice L_1 , we can construct an SSC meet semilattice L'_1 (as given in Lemma 2), such that $G_E(L_1) \cong G_{\{0\}}(L'_1)$. This gives $G_{\{0\}}(L'_1) \cong G_{\{0\}}(L_2)$. By Lemma 16, L_2 is SSC. $\square$

The above result is analogous to the following result of Anderson and LaGrange [14].

Theorem 18 (Anderson and LaGrange [14, Theorem 2.6]). *Let R and S be commutative reduced rings with $1 \neq 0$ and $Z(S) \neq \{0\}$. If $\Gamma_E(R) \cong \Gamma(S)$, then S is a Boolean ring.*

We quote the result of Joshi [6, Theorem 2.4] when the corresponding poset is a meet semilattice.

Theorem 19 (Joshi [6, Theorem 2.4]). *Let L be a meet semilattice with 0. Then the graph $G_{\{0\}}(L)$ is connected and $\text{diam}(G_{\{0\}}(L)) \leq 3$.*

Theorem 20. *Let L be a meet semilattice with 0. Then the graph $G_E(L)$ is connected and $\text{diam}(G_E(L)) \leq 3$.*

Proof. Follows from Corollary 5, Theorem 19, and the fact that adjacency in $G_E(L)$ implies adjacency in $G_{\{0\}}(L)$. Hence $\text{diam}(G_E(L)) \leq \text{diam}(G_{\{0\}}(L)) \leq 3$. $\square$

Theorem 21. *Let L be an SSC lattice. Then $G_{\{0\}}(L)$ is either K_2 or contains a cycle of length 3. Hence $\text{girth}(G_{\{0\}}(L)) = 3$.*

Proof. If $V(G_{\{0\}}(L)) = 2$, then $G_{\{0\}}(L) \cong K_2$, by Theorem 19. Now assume that $|V(G_{\{0\}}(L))| \geq 3$. Choose $a, b, c \in V(G_{\{0\}}(L))$ such that $a - b - c$ is a path with $a \neq c$ and $a \wedge c \neq 0$. Since L is SSC and without loss of generality assume that $a \wedge c < a$, there exists x such that $0 < x \leq a$ and $c \wedge x = 0$. Clearly, $x \in V(G_{\{0\}}(L))$. Since $a \wedge b = 0$, we have $x \wedge b = 0$. Thus we have a cycle $b - x - c - b$ of length 3. $\square$

An immediate consequence of the above theorem is the following corollary.

Corollary 22. *Let L be a meet semilattice with 0; then $G_E(L)$ is either K_2 or contains a cycle of length 3. Hence, if $G_E(L) \not\cong K_2$, then $\text{girth}(G_E(L)) = 3 = \text{girth}(G_{\{0\}}(L))$.*

Proof. It follows from Corollary 5 and Theorem 21. $\square$

Remark 23. In the above result, the condition of existence of a cycle in $G_E(L)$ is necessary. In Figure 1, we can observe that $G_{\{0\}}(L)$ has a cycle but $G_E(L)$ has no cycle.

3. Correlation between Diameter of $G_{\{0\}}(L)$ and $G_E(L)$

The following result gives the correlation between diameter of $G_{\{0\}}(L)$ and $G_E(L)$.

Theorem 24. *The following statements are true for a meet semilattice L with 0.*

- (a) *If $\text{diam}(G_{\{0\}}(L)) = 1$, then $\text{diam}(G_E(L)) = 1$.*
- (b) *If $\text{diam}(G_{\{0\}}(L)) = 2$, then $\text{diam}(G_E(L)) = 1$ or 2.*
- (c) *If $\text{diam}(G_{\{0\}}(L)) = 3$, if and only if $\text{diam}(G_E(L)) = 3$.*
- (d) *If $\text{diam}(G_E(L)) = 1$, then $\text{diam}(G_{\{0\}}(L)) = 1$ or 2.*
- (e) *If $\text{diam}(G_E(L)) = 2$, then $\text{diam}(G_{\{0\}}(L)) = 2$.*

Proof. (a) It is clear that $\text{diam}(G_E(L)) \neq 0$. Hence the result follows from Theorem 20.

(b) Suppose that $\text{diam}(G_{\{0\}}(L)) = 2$. So we have a path $a - b - c$ in $G_{\{0\}}(L)$. We have two possibilities.

Case 1. If $[a]$, $[b]$, and $[c]$ are distinct vertices of $G_E(L)$, we get the path $[a] - [b] - [c]$. Hence by Theorem 20, $\text{diam}(G_E(L)) = 2$.

Case 2. The only possibility in this case is $[a] = [c]$, and hence, $\text{diam}(G_E(L)) = 1$.

(c) Let $\text{diam}(G_{\{0\}}(L)) = 3$. Then there exists a path $a - c - d - b$. Since $c \in \text{ann}(a)$ gives $[a] \neq [b]$; otherwise $d(a, b) = 2$, a contradiction. Now assume that $d([a], [b]) = 2$ in $G_E(L)$. Then there exists $[x] \in V(G_E(L))$ such that $[a] - [x] - [b]$. This yields that $d(a, b) = 2$, again a contradiction. Thus by

FIGURE 5: $\text{diam}(G_E(L)) = \text{diam}(G_{\{0\}}(L)) = 1$.

FIGURE 6: $\text{diam}(G_E(L)) = \text{diam}(G_{\{0\}}(L)) = 2$.

Theorem 20, $\text{diam}(G_E(L)) = 3$. The converse follows from Theorem 20.

(d) Suppose that $\text{diam}(G_E(L)) = 1$. Then $\text{diam}(G_{\{0\}}(L)) \neq 3$, by (c). Hence the result.

(e) Suppose that $\text{diam}(G_E(L)) = 2$; then there exists a path $[a] - [c] - [b]$ in $G_E(L)$. This yields a path $a - c - b$ in $G_{\{0\}}(L)$. Thus $\text{diam}(G_{\{0\}}(L)) \neq 1$. The result follows from (c) and $\text{diam}(G_{\{0\}}(L)) \neq 3$. $\square$

Remark 25. The assertions (a) and (d) of Theorem 24 are justified in Figures 1 and 5, whereas the assertions (b) and (e) are justified in Figures 5 and 6. Figure 3 verifies the assertion (c).

Corollary 26. *Let L be a meet semilattice with 0. Then $\text{diam}(G_E(L)) \leq 2$ if and only if $\text{diam}(G_{\{0\}}(L)) \leq 2$.*

4. Cut Vertices in $G_E(L)$

In this section, we examine the properties of cut vertices of $G_E(L)$.

Definition 27. A lattice L with 0 is said to be 0-distributive, if $a \wedge b = a \wedge c = 0$ implies that $a \wedge (b \vee c) = 0$; Varlet [23]. More details about 0-distributive posets can be found in Joshi and Waphare [24]; see also Joshi and Mundlik [25].

Theorem 28. *Let L be a 0-distributive lattice. If $[a]$ is a cut vertex of $G_E(L)$, then $[a] \cup \{0\}$ forms an ideal of L .*

Proof. Let $[a]$ be a cut vertex of $G_E(L)$. Let a_1 and a_2 be nonzero elements of $[a]$. Since $a_1, a_2 \in [a]$, $\text{ann}(a_1) = \text{ann}(a) = \text{ann}(a_2)$. Now, it is enough to show that $\text{ann}(a_1 \vee a_2) = \text{ann}(a)$. Clearly, $\text{ann}(a_1 \vee a_2) \subseteq \text{ann}(a)$.

Let $t \in \text{ann}(a)$. This gives $t \wedge a_1 = t \wedge a_2 = t \wedge a = 0$. Since L is 0-distributive, we have $t \wedge (a_1 \vee a_2) = 0$; that is, $t \in \text{ann}(a_1 \vee a_2)$. Thus $\text{ann}(a) \subseteq \text{ann}(a_1 \vee a_2)$. This implies that $a_1 \vee a_2 \in [a]$.

Now suppose that $x \in [a] \cup \{0\}$ and $y \leq x$, to show that $y \in [a] \cup \{0\}$. If $x = 0$ or $y = 0$, then $y \in [a] \cup \{0\}$. Let $y \neq 0$ and $y \leq x$. Then $\text{ann}(x) \subseteq \text{ann}(y)$. Since $[a]$ is a cut vertex of $G_E(L)$,

therefore for any two arbitrary vertices $[b], [c] \in V(G_E(L))$, we have a path $[b] - [a] - [c]$. Therefore $b \in \text{ann}(a) = \text{ann}(x)$; that is, $b \wedge x = 0$, which further yields $b \wedge y = 0$. Similarly, $c \wedge y = 0$. Therefore, we get a path $[b] - [y] - [c]$. Since $[a]$ is cut vertex, $[a] = [y]$. Thus $[a] \cup \{0\}$ is an ideal. $\square$

Remark 29. It is clear from the proof of the above theorem that in a general lattice, if $[a]$ is a cut vertex of $G_E(L)$, then $[a] \cup \{0\}$ is a semi-ideal.

Lemma 30. *Let L be a meet semilattice with 0. If $[a]$ is a cut vertex of $G_E(L)$, then $\text{ann}(a)$ is maximal in $\mathcal{L} = \{\text{ann}(x) \mid 0 \neq x \in L\}$. Hence $\text{ann}(a)$ is a prime semi-ideal.*

Proof. Let $[a]$ be a cut vertex of $G_E(L)$, and let G_1 and G_2 be mutually separated subgraphs of $G_E(L)$ with $V(G_1 \cup G_2) = V(G_E(L)) \setminus [a]$. Let $[b] \in G_1$ and $[c] \in G_2$, therefore we have the path $[b] - [a] - [c]$. Suppose that $\text{ann}(a) \subseteq \text{ann}(x)$ for some $\text{ann}(x) \in \mathcal{L}$. Since $[b] - [a]$ is an edge, $b \in \text{ann}(a) \subseteq \text{ann}(x)$. Then $[b] \wedge [x] = \{0\}$. Similarly, $[c] \wedge [x] = \{0\}$. Thus we have another path $[b] - [x] - [c]$ passing through of $[x]$. Since $[a]$ is a cut vertex of $G_E(L)$, then $[x] = [a]$. Thus $\text{ann}(x) = \text{ann}(a)$. It is easy to prove that $\text{ann}(a)$ is a prime semi-ideal. $\square$

Lemma 31. *If $[a]$ is a cut vertex of $G_E(L)$, then all other associated primes of $G_E(L)$ are contained in only one component of $G_E(L) \setminus [a]$.*

Proof. Suppose that G_1 and G_2 are two mutually separated connected components of $G_E(L) \setminus [a]$, and each contains an associated prime. It is easy to observe that the associated primes are adjacent, and hence G_1, G_2 are connected, a contradiction. $\square$

Alizadeh et al. [2] raised the following problem.

Problem 32. Characterize those posets P for which $(\text{diam}(G_{\{0\}}(P)), \text{girth}(G_{\{0\}}(P))) = (2, 3)$ or $(3, 3)$.

In view of Corollary 5 and the following result, it is clear that for an SSC meet semilattice L , $(\text{diam}(G_{\{0\}}(L)), \text{girth}(G_{\{0\}}(L))) = (3, 3)$.

Theorem 33. *Let L be a meet semilattice with 0. If $|V(G_E(L))| \geq 4$ and $G_E(L)$ have at least 2 cut vertices, then $(\text{diam}(G_E(L)), \text{girth}(G_E(L))) = (3, 3)$.*

Proof. Let $[a]$ and $[b]$ be cut vertices of $G_E(L)$. Since $[a]$ is a cut vertex of $G_E(L)$, there is some $[a_1]$ such that any path connecting $[a_1]$ and $[b]$ must include $[a]$. Similarly, since $[b]$ is a cut vertex, there is some $[b_1]$ such that any path connecting $[b_1]$ and $[a]$ must include $[b]$. Clearly, $[a_1] \neq [b_1]$ and any path from $[a_1]$ to $[b_1]$ must include $[a]$ and $[b]$ and so $\text{diam}([a], [b]) \geq 3$. By Theorem 20, $\text{diam}(G_E(L)) = 3$. Thus $(\text{diam}(G_E(L)), \text{girth}(G_E(L))) = (3, 3)$, by Corollary 22. $\square$

It is easy to see that Figure 3 fulfills the conditions of Theorem 33.

5. Beck's Conjecture for $G_E(L)$

Joshi [6] (see also Halaš and Jukl [4], Nimbhorkar et al. [26], and Lu and Wu [12]) proved that Beck's Conjecture is true for the zero divisor graphs of posets. We quote this result when the poset is a meet semilattice and ideal is a zero ideal.

Theorem 34 (Joshi [6, Theorem 2.9]). *Let L be a meet semilattice with 0. If $\text{Clique}(G_{\{0\}}(L)) < \infty$, then $\text{Clique}(G_{\{0\}}(L)) = \chi(G_{\{0\}}(L)) = n$, where n is the number of minimal prime semi-ideals of L .*

From Theorem 34 and Corollary 5, it is clear that Beck's Conjecture is true for the graph $G_E(L)$. In the sequel, we calculate the chromatic number of $G_E(L)$. As a preparation, we need the following easy lemma.

Lemma 35. *Let P be a minimal prime semi-ideal of a meet semilattice L with 0. Then for any $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$.*

Proof. It follows from the fact that $L \setminus P$ is a maximal filter of L and $x \notin L \setminus P$. $\square$

Lemma 36. *Let L be a meet semilattice with 0, and let L' be a meet semilattice with 0 (as constructed in Lemma 2). If P is a minimal prime semi-ideal of L , then $P' = \{[x] \mid x \in P\}$ is a minimal prime semi-ideal of L' .*

Proof. First, we prove that P' is a semi-ideal. Let $[y] < [x] \in P'$. On the contrary, assume that $[y] \notin P'$. Hence $\text{ann}(y) \subseteq P$. Further, $[y] < [x]$ gives $\text{ann}(x) \subsetneq \text{ann}(y)$ which yields $\text{ann}(x) \subseteq P$. Since $x \in P$, by Lemma 35, there exists $y \notin P$ such that $x \wedge y = 0$; that is, $y \in \text{ann}(x)$. But then $y \in \text{ann}(x) \subseteq P$, a contradiction. Thus P' is a semi-ideal.

Let $[x] \wedge [y] \in P'$. Then $x \wedge y \in P$. By primeness of P , either $x \in P$ or $y \in P$. This gives either $[x] \in P'$ or $[y] \in P'$. Thus P' is prime. Let Q' be a prime semi-ideal of L' such that $Q' \subsetneq P'$. Hence there exists $[x] \in P'$ such that $[x] \notin Q'$. Again by Lemma 35, there exists $y \notin P$ such that $x \wedge y = 0$. This gives $[x] \wedge [y] = [0] \in Q'$. But then $[y] \in Q' \subseteq P'$, a contradiction. Thus P' is a minimal prime semi-ideal of L' . $\square$

On similar lines, we can prove the following result.

Lemma 37. *Let L be a meet semilattice with 0, and let L' be a meet semilattice with 0 (as constructed in Lemma 2). If P' is a minimal prime semi-ideal of L' , then $P = \{x \mid [x] \in P'\}$ is a minimal prime semi-ideal of L .*

Let us denote the set of all minimal prime semi-ideals of L by $\text{Min}_s(L)$.

Theorem 38. *Let L be a meet semilattice with 0, and let L' be a meet semilattice with 0 (as constructed in Lemma 2). Let $\phi : \text{Min}_s(L) \rightarrow \text{Min}_s(L')$ be a map defined by $\phi(P) = P' = \{[x] \mid x \in P\}$. Then ϕ is bijective.*

Proof. Let $\phi : \text{Min}_s(L) \rightarrow \text{Min}_s(L')$ be a map defined by $\phi(P) = P' = \{[x] \mid x \in P\}$. By Lemma 36, P' is a minimal prime semi-ideal. First, we show that ϕ is one-to-one. Let

$\phi(P) = \phi(Q)$. Then for $x \in P$, we have $[x] \in P' = \phi(P) = \phi(Q) = Q'$. This gives $x \in Q$. Hence $P \subseteq Q$. Similarly, $Q \subseteq P$. Thus ϕ is one-to-one. Let Q' be any minimal prime semi-ideal in L' . Then $Q = \{x \mid [x] \in Q'\}$ is a minimal prime semi-ideal of L , by Lemma 37. It is clear that $\phi(Q) = Q'$. Thus ϕ is onto. $\square$

With this preparation, we now prove Beck's Conjecture for $G_E(L)$.

Theorem 39. *Let L be a meet semilattice with 0. If $\text{Clique}(G_E(L)) < \infty$, then $\chi(G_E(L)) = \chi(G_{\{0\}}(L)) = \text{Clique}(G_E(L)) = \text{Clique}(G_{\{0\}}(L)) = n$, where n is the number of all minimal prime semi-ideals of L .*

Proof. Let $\text{Clique}(G_E(L)) < \infty$. Then $\text{Clique}(G_{\{0\}}(L')) < \infty$, by Corollary 5, where L' is the semilattice constructed as in Lemma 2. By Theorem 34, $\text{Clique}(G_{\{0\}}(L')) = \chi(G_{\{0\}}(L')) = n$, where n is the number of minimal prime semi-ideals of L' . Now, by Theorem 38, the number of minimal prime semi-ideals of L is also n . Thus we have $\text{Clique}(G_E(L)) = \chi(G_E(L)) = n$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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