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Research Article

A Study of I-Function of Several Complex Variables

Prathima Jayarama,[1,2] Vasudevan Nambisan Theke Madam,[3] and Shantha Kumari Kurumujji[2,4]

[1] *Department of Mathematics, Manipal Institute of Technology, Manipal, Karnataka 576104, India*
[2] *SCSVMV, Sri Jayendra Saraswathi Street, Enathur, Kanchipuram, Tamil Nadu 631561, India*
[3] *Department of Mathematics, College of Engineering, Trikaripur, Kerala 670307, India*
[4] *Department of Mathematics, P.A. College of Engineering, Mangalore, Karnataka 574153, India*

Correspondence should be addressed to Prathima Jayarama; pamrutharaj@yahoo.co.in

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The aim of this paper is to introduce a natural generalization of the well-known, interesting, and useful Fox H-function into generalized function of several variables, namely, the I-function of "r" variables. For $r = 1$, we get the I-function introduced and studied by Arjun Rathie (1997) and, for $r = 2$, we get I-function of two variables introduced very recently by ShanthaKumari et al. (2012). Convergent conditions, elementary properties, and special cases have also been given. The results presented in this paper generalize the results of H-function of "r" variables available in the literature.

1. Introduction

In 1997, Rathie introduced the generalization of the well-known Fox's H-function [1] which has very recently found interesting applications in wireless communication [2–4]. Motivated by the I-function, very recently Shantha Kumari, Nambisan, and Rathie introduced I-function of two variables [5] which is a natural generalization of the H-function of two variables introduced earlier by Mittal and Gupta [6] and discussed some of its important properties.

In the present paper, we aim to develop I-function of "r" variables which may be regarded as the natural generalization of the H-function of "r" variables introduced earlier by Srivastava and Panda [7]. We also discussed some of the important properties.

The remainder of this paper is organized as follows.

In Section 2, we have defined the I-function of "r" variables by means of multiple Mellin-Barnes type contour integrals. In Section 3, we have given the convergence conditions for this function. In Section 4, we obtained the series representation and behaviour of the function for small values of the variables. In Section 5, we have mentioned special cases of our function giving relations with other functions available in the literature. Finally, in Section 6, we have mentioned a few important properties.

2. The I-Function of Several Variables

The generalized Fox H-function, namely, I-function of "r" variables, is defined and represented in the following manner:

$$\begin{aligned} \mathrm{I}\left[z_1,\ldots,z_r\right] &= \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\left|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right.\right] \\ &= \frac{1}{(2\pi i)^r}\int_{\mathscr{L}_1}\cdots\int_{\mathscr{L}_r}\phi\left(s_1,\ldots,s_r\right)\theta_1\left(s_1\right)\cdots\theta_r\left(s_r\right)z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \end{aligned} \tag{1}$$

where $\phi(s_1,\ldots,s_r)$, $\theta_i(s_i)$, $i=1,\ldots,r$ are given by

$$\phi(s_1,\ldots,s_r)=\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)\times\left(\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)\times\prod_{j=1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)\right)^{-1},\tag{2}$$

$$\theta_i(s_i)=\left(\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\times\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)\right)\times\left(\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\times\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)\right)^{-1},\tag{3}$$

where $i=1,\ldots,r$.

Also,

(i) $z_i\neq 0$, for $i=1,\ldots,r$;

(ii) $i=\sqrt{-1}$;

(iii) an empty product is interpreted as unity;

(iv) the parameters m_j, n_j, p_j, q_j $(j=1,\ldots,r)$, n, p, and q are nonnegative integers such that $0\le n\le p$, $q\ge 0$, $0\le n_j\le p_j$, and $0\le m_j\le q_j$ $(j=1,\ldots,r)$ (not all zero simulataneously);

(v) $\alpha_j^{(i)}$ $(j=1,\ldots,p,\ i=1,\ldots,r)$, $\beta_j^{(i)}$ $(j=1,\ldots,q,\ i=1,\ldots,r)$, $\gamma_j^{(i)}$ $(j=1,\ldots,p_i,\ i=1,\ldots,r)$, and $\delta_j^{(i)}$ $(j=1,\ldots,q_i,\ i=1,\ldots,r)$ are assumed to be positive quantities for standardisation purpose. However, the definition of I-function of "r" variables will have a meaning even if some of the quantities are zero or negative numbers. For these, we may obtain corresponding transformation formulas which will be given in a later section;

(vi) a_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$, $c_j^{(i)}$ $(j=1,\ldots,p_i,\ i=1,\ldots,r)$, and $d_j^{(i)}$ $(j=1,\ldots,q_i,\ i=1,\ldots,r)$ are complex numbers;

(vii) the exponents A_j $(j=1,\ldots,p)$, B_j $(j=1,\ldots,q)$, $C_j^{(i)}$ $(j=1,\ldots,p_i,\ i=1,\ldots,r)$, and $D_j^{(i)}$ $(j=1,\ldots,q_i,\ i=1,\ldots,r)$ of various gamma functions involved in (2) and (3) may take noninteger values;

(viii) the contour $\mathscr{L}_i$ in the complex s_i-plane is of Mellin-Barnes type which runs from $c-i\infty$ to $c+i\infty$, (c real) with indentation, if necessary, in such a manner that all singularities of $\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)$, $j=1,\ldots,m_i$ lie to the right and $\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)$, $j=1,\ldots,n_i$ are to the left of $\mathscr{L}_i$.

Following the results of Braaksma [8] the I-function of "r" variables is analytic if

$$\mu_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\le 0,\quad i=1,\ldots,r.\tag{4}$$

3. Convergence Conditions

Integral (1) converges absolutely if

$$\left|\arg(z_k)\right|<\frac{1}{2}\Delta_k\pi,\quad k=1,\ldots,r,\tag{5}$$

where

$$\Delta_k=\left[-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}\right]>0,\tag{6}$$

and if $|\arg(z_k)|=(1/2)\Delta_k\pi$ and $\Delta_k\ge 0$, $k=1,\ldots,r$, then integral (1) converges absolutely under the following conditions:

(i) $\mu_k=0$, $\Omega_k<-1$, where μ_k is given by (4) and

$$\Omega_k=\sum_{j=1}^{p}\left[\frac{1}{2}-\Re\left(a_j\right)\right]A_j-\sum_{j=1}^{q}\left[\frac{1}{2}-\Re\left(b_j\right)\right]B_j+\sum_{j=1}^{p_k}\left[\frac{1}{2}-\Re\left(c_j^{(k)}\right)\right]C_j^{(k)}-\sum_{j=1}^{q_k}\left[\frac{1}{2}-\Re\left(d_j^{(k)}\right)\right]D_j^{(k)},\quad k=1,\ldots,r;\tag{7}$$

(ii) $\mu_k\neq 0$ with $s_k=\sigma_k+it_k$, (σ_k and t_k are real, $k=1,\ldots,r$), and σ_k are chosen so that for $|t_k|\to\infty$ we have $\Omega_k+\sigma_k\mu_k<-1$.

Outline of the Proof. The convergence of integral (1) depends on the asymptotic behaviour of the functions $\phi(s_1,\ldots,s_r)$, $\theta_i(s_i)$, $i=1,\ldots,r$ defined by (2) and (3), respectively. Such

asymptotic behaviour is based on the following relation for the gamma function $\Gamma(z)$, $z = x + iy$, $x, y \in \mathbb{R}$ [9]:

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}\,|y|^{x-1/2}\exp\left(-\frac{1}{2}\pi|y|\right), \quad |y| \to \infty. \tag{8}$$

Along the contour $\mathscr{L}_k$, if we put $s_k = \sigma_k + it_k$ and take the limit as $|t_k| \to \infty$ for $k = 1, \ldots, r$, we obtain by virtue of (8) that

$$\begin{aligned}
&\left|\Gamma^{A_j}\left(1 - a_j + \sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)\right| \\
&\quad \le (2\pi)^{A_j/2}\left(\alpha_j^{(k)}|t_k|\right)^{[1/2-\Re(a_j)+\alpha_j^{(k)}\sigma_k]A_j} \\
&\qquad \times \exp\left[-\frac{\pi}{2}\left(\alpha_j^{(k)}|t_k| + \left|\Im(a_j)\right|\right)A_j\right],
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\prod_{j=1}^{n}\left|\Gamma^{A_j}\left(1 - a_j + \sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)\right| \\
&\quad \le (2\pi)^{\sum_{j=1}^{n}(A_j/2)}\prod_{j=1}^{n}\left(\alpha_j^{(k)}|t_k|\right)^{[1/2-\Re(a_j)+\alpha_j^{(k)}\sigma_k]A_j} \\
&\qquad \times \exp\left[-\frac{\Pi}{2}\sum_{j=1}^{n}\left(\alpha_j^{(k)}|t_k| + \left|\Im(a_j)\right|\right)A_j\right].
\end{aligned} \tag{10}$$

Similarly, we have

$$\begin{aligned}
&\prod_{j=n+1}^{p}\left|\Gamma^{A_j}\left(a_j - \sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)\right| \\
&\quad \ge (2\pi)^{\sum_{j=n+1}^{k}(A_j/2)}\prod_{j=n+1}^{k}\left(\alpha_j^{(k)}|t_k|\right)^{[\Re(a_j)-\alpha_j^{(k)}\sigma_k-1/2]A_j} \\
&\qquad \times \exp\left[-\frac{\pi}{2}\prod_{j=n+1}^{p}\left(\alpha_j^{(k)}|t_k| + \left|\Im(a_j)\right|\right)A_j\right], \\
&\prod_{j=1}^{q}\left|\Gamma^{B_j}\left(1 - b_j + \sum_{k=1}^{r}\beta_j^{(k)}s_k\right)\right| \\
&\quad \ge (2\pi)^{\sum_{j=1}^{q}(B_j/2)}\prod_{j=1}^{q}\left(\beta_j^{(k)}|t_k|\right)^{[1/2-\Re(b_j)+\beta_j^{(k)}\sigma_k]B_j} \\
&\qquad \times \exp\left[-\frac{\Pi}{2}\prod_{j=1}^{q}\left(\beta_j^{(k)}|t_k| + \left|\Im(a_j)\right|\right)B_j\right], \\
&\prod_{j=1}^{n_k}\left|\Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)}s_k\right)\right| \\
&\quad \le (2\pi)^{\sum_{j=1}^{n_k}(C_j^{(k)}/2)}\prod_{j=1}^{n_k}\left(\gamma_j^{(k)}|t_k|\right)^{[1/2-\Re(c_j^{(k)})+\gamma_j^{(k)}\sigma_k]C_j^{(k)}} \\
&\qquad \times \exp\left[-\frac{\Pi}{2}\prod_{j=1}^{n_k}\left(\gamma_j^{(k)}|t_k| + \left|\Im(c_j^{(k)})\right|\right)C_j^{(k)}\right], \\
&\prod_{j=n_k+1}^{p_k}\left|\Gamma^{C_j^{(k)}}\left(c_j^{(k)} - \gamma_j^{(k)}s_k\right)\right| \\
&\quad \ge (2\pi)^{\sum_{j=n_k+1}^{p_k}(C_j^{(k)}/2)}\prod_{j=n_k+1}^{p_k}\left(\gamma_j^{(k)}|t_k|\right)^{[\Re(c_j^{(k)})-\gamma_j^{(k)}\sigma_k-1/2]C_j^{(k)}} \\
&\qquad \times \exp\left[-\frac{\pi}{2}\prod_{j=n_k+1}^{p_k}\left(\gamma_j^{(k)}|t_k| + \left|\Im(c_j^{(k)})\right|\right)C_j^{(k)}\right], \\
&\prod_{j=1}^{m_k}\left|\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)}s_k\right)\right| \\
&\quad \le (2\pi)^{\sum_{j=1}^{m_k}(D_j^{(k)}/2)}\prod_{j=1}^{m_k}\left(\delta_j^{(k)}|t_k|\right)^{[\Re(d_j^{(k)})+\delta_j^{(k)}\sigma_k-1/2]D_j^{(k)}} \\
&\qquad \times \exp\left[-\frac{\pi}{2}\prod_{j=1}^{m_k}\left(\delta_j^{(k)}|t_k| + \left|\Im(d_j^{(k)})\right|\right)D_j^{(k)}\right], \\
&\prod_{j=m_k+1}^{q_k}\left|\Gamma^{D_j^{(k)}}\left(1 - d_j^{(k)} + \delta_j^{(k)}s_k\right)\right| \\
&\quad \le (2\pi)^{\sum_{j=m_k+1}^{q_k}(D_j^{(k)}/2)}\prod_{j=m_k+1}^{q_k}\left(\delta_j^{(k)}|t_k|\right)^{[1/2-\Re(d_j^{(k)})+\delta_j^{(k)}\sigma_k]D_j^{(k)}} \\
&\qquad \times \exp\left[-\frac{\pi}{2}\prod_{j=m_k+1}^{q_k}\left(\delta_j^{(k)}|t_k| + \left|\Im(d_j^{(k)})\right|\right)D_j^{(k)}\right].
\end{aligned} \tag{11}$$

Also,

$$\begin{aligned}
z_k^{s_k} &= \exp\left[(\sigma_k + it_k)(\log|z_k| + i\arg(z_k))\right] \\
&= \exp\left[\sigma_k\log|z_k| - t_k\arg(z_k)\right] \\
&= |z_k|^{\sigma_k}\exp\left[-t_k\arg(z_k)\right].
\end{aligned} \tag{12}$$

Hence, substituting (10)-(11) in (1) and using (12) we have, after much simplification,

$$\begin{aligned}
&\left|\phi(s_1, \ldots, s_r)\,\theta_k(s_k)\,z_k^{s_k}\right| \\
&\quad \sim C_k\,|t_k|^{\Omega_k+\mu_k\sigma_k}\exp\left[-t_k\arg(z_k) - \frac{\pi}{2}|t_k|\,\Delta_k\right],
\end{aligned} \tag{13}$$

where C_k is independent of t_k and Δ_k, μ_k, and Ω_k are given by (6), (7), and (8), respectively, for each $k = 1, 2, \ldots, r$.

Hence, the result follows.

Remark 1. If $D_j^{(i)} = 1$ $(j = 1, \ldots, m_i,\ i = 1, \ldots, r)$ in (1), then the function will be denoted by

$$\bar{\mathrm{I}}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \mathrm{I}_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,n:m_1,n_1;\ldots;m_r,n_r} \times \left[\begin{array}{c|c} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)};1\right)_{1,m_1},\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{m_1+1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};1\right)_{1,m_r},\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{m_r+1,q_r} \end{array}\right] = \frac{1}{(2\pi i)^r}\int_{\mathscr{L}_1}\cdots\int_{\mathscr{L}_r}\phi(s_1,\ldots,s_r)\,\overline{\theta_1}(s_1)\cdots\overline{\theta_r}(s_r)\ z_1^{s_1}\cdots z_r^{s_r}\ ds_1\cdots ds_r, \tag{14}$$

where

$$\overline{\theta_i}(s_i) = \prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right) \times\left(\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)\right)^{-1}, \tag{15}$$

where $i = 1, \ldots, r$.

Remark 2. If $C_j^{(i)} = 1$ $(j = 1, \ldots, n_i)$, $D_j^{(i)} = 1$ $(j = 1, \ldots, m_i)$, where $i = 1, \ldots, r$ and if $n = 0$ in (1), then the corresponding function will be denoted by

$$\bar{\mathrm{I}}_1\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \mathrm{I}_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,0:m_1,n_1;\ldots;m_r,n_r} \times \left[\begin{array}{c|c} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)};1\right)_{1,n_1},\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{n_1+1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{1,n_r},\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{n_r+1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)};1\right)_{1,m_1},\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{m_1+1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};1\right)_{1,m_r},\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{m_r+1,q_r} \end{array}\right] = \frac{1}{(2\pi i)^r}\int_{\mathscr{L}_1}\cdots\int_{\mathscr{L}_r}\phi_1(s_1,\ldots,s_r)\ \overline{\theta_1}(s_1)\cdots\overline{\theta_r}(s_r)\ z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \tag{16}$$

where

$$\phi_1(s_1,\ldots,s_r) = \frac{1}{\prod_{j=1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)},$$

$$\overline{\theta_i}(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \quad \forall i = 1, \ldots, r. \tag{17}$$

4. Series Representation

if

(i) $z_i \neq 0$, $(i = 1, \ldots, r)$ and $\mu_i < 0$, where μ_i is given by (4);

(ii) $\delta_{h_i}^{(i)}(d_j^{(i)}+k_i) \neq \delta_j^{(i)}(d_{h_i}^{(i)}+\eta_i)$ for $j \neq h_i$, $j, h_i = 1, \ldots, m_i$, $(i = 1, \ldots, r)$, $k_i, \eta_i = 0, 1, 2, \ldots (i = 1, \ldots, r)$,

then

$$\bar{\mathrm{I}}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \sum_{h_1=1}^{m_1}\cdots\sum_{h_r=1}^{m_r}\sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty} \times\left[\phi_1\left(\frac{dh_1^{(1)}+k_1}{\delta h_1^{(1)}},\ldots,\frac{dh_r^{(r)}+k_r}{\delta h_r^{(r)}}\right) \times\prod_{i=1}^{r}\frac{(-1)^{k_i}}{\delta h_i^{(i)}k_i!}z_i^{(dh_i+k_i)/\delta h_i}\right]_{j\neq h_i}. \tag{18}$$

This result can be proved on computing the residues at the

poles as follows:

$$s_r = \frac{dh_i^{(i)} + k_i}{\delta h_i^{(i)}}, \quad \left(h_i = 1, \ldots, m_i,\ k_i = 0, 1, 2, \ldots\right) \quad \text{for} \quad i = 1, \ldots, r. \tag{19}$$

The behaviour of the function $\bar{\mathrm{I}}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ is given by

$$\bar{\mathrm{I}}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = O\left(\prod_{j=1}^{r} |z_i|^{\phi_j}\right), \qquad \max\{|z_1|, \ldots, |z_r|\} \longrightarrow 0, \tag{20}$$

where

$$\phi_j = \min_{1 \le j \le m_i}\left[\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right], \quad (i = 1, \ldots, r). \tag{21}$$

On the other hand, when $|z_i| \to \infty$ $(i = 1, \ldots, r)$, the associated function $\bar{\mathrm{I}}_1\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ given by (16) has the behaviour

$$\bar{\mathrm{I}}_1\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = O\left(\prod_{j=1}^{r} |z_j|^{\phi_j}\right), \qquad \min\{|z_1|, \ldots, |z_r|\} \longrightarrow 0, \tag{22}$$

where

$$\phi_j = \max_{1 \le j \le n_i}\left[\mathrm{Re}\left(\frac{1 - c_j^{(i)}}{\gamma_j^{(i)}}\right)\right], \quad (i = 1, \ldots, r). \tag{23}$$

5. Elementary Special Cases

In this section, we mention some interesting and useful special cases of the I-function of "r" variables.

(i) If all the exponents A_j $(j = 1, \ldots, p)$, B_j $(j = 1, \ldots, q)$, $C_j^{(i)}$ $(j = 1, \ldots, p_i,\ i = 1, \ldots, r)$, and $D_j^{(i)}$ $(j = 1, \ldots, q_i,\ i = 1, \ldots, r)$ in (1) are equal to unity, we obtain H-function of "r" variables defined by Srivastava and Panda [7].

(ii) When $p = q = n = 0$, (1) degenerates into the product of r mutually independent I- functions of one variable introduced by Rathie [1].

(iii) When $p = q = n = 0$ and $r = 1$, (1) reduces to the I-function defined by Rathie [1].

(iv) When $n = p$, $m_i = 1$, $n_i = p_i$, $i = 1, \ldots, r$, and $A_j = B_j = C_j = D_j = 1$ and $(d_j^{(i)}, \delta_j^{(i)}; D_j^{(i)})$ is replaced by $(0, 1, 1)$, $(d_j^{(i)}, \delta_j^{(i)}; D_j^{(i)})$, (1) reduces to the generalized Lauricella function [10].

$$\begin{aligned}
&\mathrm{I}_{p,q:p_1,q_1+1;\ldots;p_r,q_r+1}^{0,p:1,p_1;\ldots;1,p_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};1\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)};1\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};1\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\beta_j^{(1)},\ldots,\beta_j^{(r)};1\right)_{1,q} : (0,1;1), \left(d_j^{(1)},\delta_j^{(1)};1\right)_{1,q_1};\ldots;(0,1;1), \left(d_j^{(r)},\delta_j^{(r)};1\right)_{1,q_r} \end{array}\right] \\
&= \frac{\prod_{j=1}^{p}\Gamma\left(1-a_j\right)\prod_{j=1}^{p_1}\Gamma\left(1-c_j^{(1)}\right)\cdots\prod_{j=1}^{p_r}\Gamma\left(1-c_j^{(r)}\right)}{\prod_{j=1}^{q}\Gamma\left(1-b_j\right)\prod_{j=1}^{q_1}\Gamma\left(1-d_j^{(1)}\right)\cdots\prod_{j=1}^{q_r}\Gamma\left(1-d_j^{(r)}\right)} \\
&\times F_{q:q_1;\ldots;q_r}^{p:p_1;\ldots;p_r}\left[\begin{array}{c|l} -z_1 & \left(1-a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p} : \left(1-c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(1-c_j^{((r))},\gamma_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ -z_r & \left(1-b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} : \left(1-d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(1-d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{array}\right].
\end{aligned} \tag{24}$$

$$\begin{aligned}
\text{(v)}\quad &\mathrm{I}_{0,0:p_1,q_1+1;\ldots;p_r,q_r+1}^{0,0:1,p_1;\ldots1,p_r}\left[\begin{array}{c|l} -z_1 & - : \left(1-c_j^{(1)},\gamma_j^{(1)};1\right)_{1,p_1};\ldots;\left(1-c_j^{(r)},\gamma_j^{(r)};1\right)_{1,p_r} \\ \vdots & \\ -z_r & - : (0,1;1), \left(1-d_j^{(1)},\delta_j^{(1)};1\right)_{1,q_1};\ldots;(0,1;1), \left(1-d_j^{(r)},\delta_j^{(r)};1\right)_{1,q_r} \end{array}\right] \\
&= {}_{p_1}\Psi_{q_1}\left[\begin{array}{c} \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1} \\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1} \end{array};z_1\right] \times \cdots \times {}_{p_r}\Psi_{q_r}\left[\begin{array}{c} \left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{array};z_r\right],
\end{aligned} \tag{25}$$

where the functions ${}_{p_i}\Psi_{q_i}$, $i = 1, \ldots, r$ are Wright's generalized hypergeometric functions [11].

$$\text{(vi)}\quad \mathrm{I}^{0,0:1,0;\ldots;1,0}_{0,0:0,2;\ldots;0,2}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} -:-;\ldots;- \\ -:(0,1;1),(-\mu_1,\alpha_1;1);\ldots;(0,1;1),(-\mu_r,\alpha_r;1) \end{array}\right] = \prod_{i=1}^{r} J_{\mu_i}^{\alpha_i}(z_i), \tag{26}$$

where the functions $J_{\mu_i}^{\alpha_i}(z_i)$ are Wright's generalized Bessel functions [12].

$$\text{(vii)}\quad \mathrm{I}^{0,0:1,2;\ldots;1,2}_{0,0:2,2;\ldots;2,2}\left[\begin{array}{c} -z_1 \\ \vdots \\ -z_r \end{array}\middle|\begin{array}{l} -:(1,1;1),(1\text{-}\alpha_1,1;\mu_1);\ldots;(1,1;1),(1\text{-}\alpha_r,1;\mu_r) \\ -:(0,1;1),(\text{-}\alpha_1,1;\mu_1);(0,1;1),(\text{-}\alpha_r,1;\mu_r) \end{array}\right] = \prod_{i=1}^{r}\Phi(z_i,\mu_i,\alpha_i), \tag{27}$$

where $\Phi(z_i, \mu_i, \alpha_i)$, $i = 1, \ldots, r$ are the generalized Riemann zeta functions [13, page 27, 1.11, (1)], which are the generalizations of Hurwitz zeta functions and Riemann zeta functions [13, page 24, 1.10, (1) and 1.12, (1)].

$$\text{(viii)}\quad \mathrm{I}^{0,0:1,2;\ldots;1,2}_{0,0:2,2;\ldots;2,2}\left[\begin{array}{c} -z_1 \\ \vdots \\ -z_r \end{array}\middle|\begin{array}{l} -:(1,1;1),(1,1;\mu_1);\ldots;(1,1;1),(1,1;\mu_r) \\ -:(0,1;1),(0,1;\mu_1);\ldots;(0,1;1),(0,1;\mu_r) \end{array}\right] = \prod_{i=1}^{r}F(z_i,\mu_i), \tag{28}$$

where $F(z_i, \mu_i)$ are the polylogarithms of order μ_i. For $\mu_i = 2$, $i = 1, \ldots, r$, the R.H.S. of (28) reduces to the product of Euler's dilogarithm [13, page 31, 1.11.1, equation (2)].

6. Elementary Properties and Transformation Formulas

The properties given below are immediate consequence of the definition (1) and hence they are given here without proof:

$$\begin{aligned}\text{(i)}\quad & \mathrm{I}^{0,0:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right] \\ & = \mathrm{I}^{0,0:n_1,m_1;\ldots;n_r,m_r}_{q,p:q_1,p_1;\ldots;q_r,p_r}\left[\begin{array}{c} z_1^{-1} \\ \vdots \\ z_r^{-1} \end{array}\middle|\begin{array}{l} \left(1-b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(1-d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(1-d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \\ \left(1-a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(1-c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\left(1-c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \end{array}\right],\end{aligned} \tag{29}$$

$$\begin{aligned}\text{(ii)}\quad & z_1^{k_1}\cdots z_r^{k_r}\mathrm{I}\left[z_1\cdots z_r\right] \\ & = \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j+\sum_{i=1}^{r}k_i\alpha_j^{(i)};\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)}+k_1\gamma_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)}+k_r\gamma_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j+\sum_{i=1}^{r}k_i\beta_j^{(i)};\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)}+k_1\delta_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)}+k_r\delta_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right],\end{aligned} \tag{30}$$

for $k_i > 0$, $i = 1, \ldots r$,

(iii) $$\frac{1}{k_1}\cdots\frac{1}{k_r}\mathrm{I}\left[z_1,\ldots z_r\right] = \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1^{k_1} \\ \vdots \\ z_r^{k_r}\end{array}\middle|\begin{array}{l}\left(a_j;k_1\alpha_j^{(1)},\ldots,k_r\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},k_1\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},k_r\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;k_1\beta_j^{(1)},\ldots,k_r\beta_j^{(r)};B_j\right)_{1,p}:\left(d_j^{(1)},k_1\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},k_r\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right], \tag{31}$$

where $k_i > 0, i = 1,\ldots,r$,

(iv) $$\mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\middle|\begin{array}{l}(a;\alpha,0,\ldots,0;A),\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{2,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1},\ldots,\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right]$$
$$= \mathrm{I}^{0,n-1:m_1,n_1+1;\ldots;m_r,n_r}_{p,q:p_1+1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\middle|\begin{array}{l}\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};\right)_{2,p}:(a,\alpha;A),\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right], \tag{32}$$

where $p \geq n \geq 1$,

(v) $$\mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\middle|\begin{array}{l}\left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p-1},(a;\alpha,0,\ldots,0;A):\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right]$$
$$= \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p-1,q:p_1+1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\middle|\begin{array}{l}\left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p-1}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1},(a,\alpha;A);\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right], \tag{33}$$

where $p-1 \geq n \geq 0$,

(vi) $$\mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\middle|\begin{array}{l}\left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q-1},(b;\beta,0,\ldots,0;B):\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right]$$
$$= \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q-1:p_1,q_1+1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\middle|\begin{array}{l}\left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q-1}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1},(b,\beta;B);\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}\end{array}\right], \tag{34}$$

where $q-1 \geq 0$,

$$\text{(vii)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} (a;0,\ldots,0;A),\left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{2,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$= \Gamma^A(1-a)\ \times\ \mathrm{I}^{0,n-1:m_1,n_1;\ldots;m_r,n_r}_{p-1,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{2,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{35}$$

where $p \geq n \geq 1$, $\Re(1-a) > 0$,

$$\text{(viii)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p-1},(a;0,\ldots,0;A):\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$= \frac{1}{\Gamma^A(a)}\ \times\ \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p-1,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p-1}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{36}$$

where $p-1 \geq n \geq 0$, $\Re(a) > 0$,

$$\text{(ix)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q-1},(b;0,\ldots,0;B):\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$= \frac{1}{\Gamma^B(1-b)} \times \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q-1:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q-1}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{37}$$

where $q-1 \geq 0$, $\Re(1-b) > 0$,

$$\text{(x)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:(c;0;C),\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{2,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$= \Gamma^C(1-c) \times \mathrm{I}^{0,n:m_1,n_1-1;\ldots;m_r,n_r}_{p,q:p_1-1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{2,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{38}$$

where $p_1 \geq n_1 \geq 1$, $\Re(1-c) > 0$,

$$\text{(xi)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1-1},(c;0;C);\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$=\frac{1}{\Gamma^C(c)}\times \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1-1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1-1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{39}$$

where $p_1-1\geq n_1\geq 0$, $\Re(c)>0$,

$$\text{(xii)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:(d;0;D),\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{2,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$=\Gamma^D(d)\times \mathrm{I}^{0,n:m_1-1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1-1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{2,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{40}$$

where $q_1\geq m_1\geq 1$, $\Re(d)>0$,

$$\text{(xiii)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1-1},(d;0;D);\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$
$$=\frac{1}{\Gamma^D(1-d)}\times \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1-1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1-1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right], \tag{41}$$

provided that $q_1-1\geq m_1\geq 0$, $\Re(1-d)\geq 0$,

$$\text{(xiv)}\quad \mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}$$
$$\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q-1},\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)};A_1\right):\left(d_j^{1},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1-1},\left(c_1^{(1)},\gamma_1^{(1)};C_1^{(1)}\right);\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r-1},\left(c_1^{(r)},\gamma_1^{(r)};C_1^{(r)}\right) \end{array}\right]$$
$$=\mathrm{I}^{0,n-1:m_1,n_1-1;\ldots;m_r,n_r-1}_{p-1,q-1:p_1-1,q_1-1;\ldots;p_r-1,q_r-1}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{2,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{2,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{2,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\beta_j^{(r)};B_j\right)_{1,q-1}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1-1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r-1} \end{array}\right], \tag{42}$$

provided that $p\geq n\geq 1$, $p_i\geq n_i\geq 1$, $i=1,\ldots,r$, and $q\geq 1$, $q_i\geq m_i+1$, $i=1,\ldots,r$,

(xv) $\mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}$

$$\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1-1},\left(d_1^{(1)},\delta_1^{(1)};D_1^{(1)}\right);\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r-1},\left(d_1^{(r)},\delta_1^{(r)};D_1^{(r)}\right) \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$

$$=\mathrm{I}^{0,n:m_1-1,n_1;\ldots;m_r-1,n_r}_{p,q:p_1-1,q_1-1;\ldots;p_r-1,q_r-1}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1-1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r-1} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{2,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{2,q_r} \end{array}\right], \tag{43}$$

provided that $p \geq n+1$, $q \geq 1$, $p_i \geq n_i + 1$, and $q_i \geq m_i \geq 1$, $i = 1,\ldots,r$,

(xvi) $\mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}$

$$\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(c_{p_1}^{(1)},\gamma_{p_1}^{(1)};C_{p_1}^{(1)}\right),\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{2,q_1};\ldots;\left(c_{p_r}^{(r)},\gamma_{p_r}^{(r)};C_{p_r}^{(r)}\right),\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{2,q_r} \end{array}\right] \tag{44}$$

$$=\mathrm{I}^{0,n:\ m_1-1,n_1;\ldots;\ m_r-1,n_r}_{p,q:\ p_1-1,q_1-1;\ldots;\ p_r-1,q_r-1}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1-1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r-1} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{2,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{2,q_r} \end{array}\right],$$

provided that $p_i \geq n_i + 1$, $q_i \geq m_i$, $i = 1,\ldots,r$,

(xvii) $$\mathrm{I}^{0,n:m_1,n_1;\ldots;m_r,n_r}_{pq:p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(d_{q_1}^{(1)},\delta_{q_1}^{(1)},D_{q_1}^{(1)}\right),\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{2,p_1};\ldots;\left(d_{q_r}^{(r)},\delta_{q_r}^{(r)},D_{q_r}^{(r)}\right),\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{2,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r} \end{array}\right]$$

$$=\mathrm{I}^{0,n:m_1,n_1-1;\ldots;m_r,n_r-1}_{p,q:p_1-1,q_1-1;\ldots;p_r-1,q_r-1}\left[\begin{array}{c|l} z_1 & \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{2,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{2,p_r} \\ \vdots & \\ z_r & \left(b_j;\beta_j^{(1)},\beta_j^{(r)};B_j\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1-1};\ldots;\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r-1} \end{array}\right], \tag{45}$$

where $p \geq n$, $q \geq 1$, $p_i \geq n_i \geq 1$, $q_i - 1 \geq m_i$, $i = 1,\ldots,r$.

7. Special Cases

When $r = 2$ and all the exponents A_j $(j = 1,\ldots,p)$, B_j $(j = 1,\ldots,q)$, $C_j^{(i)}$ $(j = 1,\ldots,p_i,\ i = 1\ldots,r)$, and $D_j^{(i)}$ $(1,\ldots,q_i,\ i = 1\ldots,r)$ the I-function of "r" variables reduces to H-function of two variables and therefore we obtain the corresponding results in H-function of two variables [14].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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