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Research Article

(σ, f)-Asymptotically Lacunary Equivalent Sequences

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We introduce the strong (σ, f)-asymptotically equivalent and strong (σ, f)-asymptotically lacunary equivalent sequences which are some combinations of the definitions for asymptotically equivalent, statistical limit, modulus function, σ-convergence, and lacunary sequences. Then we use these definitions to prove strong (σ, f)-asymptotically equivalent and strong (σ, f)-asymptotically lacunary equivalent analogues of Connor's results in Connor, 1988, Fridy and Orhan's results in Fridy and Orhan, 1993, and Das and Patel's results in Das and Patel, 1989.

1. Introduction

Let s, ℓ_∞, and c denote the spaces of all real sequences, bounded and convergent sequences, respectively. Any subspace of s is called a sequence space.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k;

(ii) $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$;

(iii) $\phi(x) = \phi(\sigma x)$ for all $x \in \ell_\infty$, where $\sigma x = (x_{\sigma(n)})$. The mapping σ is one to one with $\sigma^k(n) \neq n$ for all positive integers n and k, where σ^k denotes the kth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$. If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$.

Several authors including Bilgin [1], Mursaleen [2], Savas [3], Schaefer [4], and others have studied invariant convergent sequences.

The notion of modulus function was introduced by Nakano [5]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$, (iii) f is increasing, and (iv) f is continuous from the right at 0. Hence f must be continuous everywhere on $[0, \infty)$. Kolk [6], Maddox [7], Öztürk and Bilgin [8], Pehlivan and Fisher [9], Ruckle [10], and others used a modulus function to construct sequence spaces.

Following Freedman et al. [11], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r/k_{r-1}$. These notations will be used throughout the paper. The sequence space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [11] as follows:

$$N_\theta = \left\{ x = (x_i) \in s : h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \right\}. \tag{1}$$

Lacunary convergent sequences have been studied by Bilgin [12], Das and Mishra [13], Das and Patel [14], Savas and Patterson [15], and others. Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [16]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in [17].

Recently, the concept of asymptotically equivalent was generalized by Bilgin [12], Kumar and Sharma [18], Patterson and Savas [19], Savas and Basarr [20], and Patterson and

Savas [15]. In this paper we introduce the strong (σ, f)-asymptotically equivalent and strong (σ, f)-asymptotically lacunary equivalent sequences which are some combinations of the definitions for asymptotically equivalent, statistical limit, modulus function, σ-convergence, and lacunary sequences.

In addition to these definitions, natural inclusion theorems will also be presented.

2. Definitions and Notations

Now we recall some definitions of sequence spaces (see [7, 12, 15–17, 21–25]).

Definition 1. A sequence $[x]$ is statistically convergent to L if

$$\lim_n \frac{1}{n}\{\text{the number of } k \le n : |x_k - L| \ge \varepsilon\} = 0 \quad \text{for every } \varepsilon > 0, \quad (\text{denoted by } st - \lim x = L). \tag{2}$$

Definition 2. A sequence $[x]$ is strongly (Cesàro) summable to L if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} |x_k - L| = 0, \quad (\text{denoted by } w - \lim x = L). \tag{3}$$

Definition 3. Let f be any modulus; the sequence $[x]$ is strongly (Cesàro) summable to L with respect to a modulus if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f(|x_k - L|) = 0, \quad (\text{denoted by } w_f - \lim x = L). \tag{4}$$

Definition 4. Two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically equivalent if $\lim_k (x_k/y_k) = 1$, (denoted by $x \backsim y$).

Definition 5. Two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left\{\text{the number of } k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} = 0, \quad \left(\text{denoted by } x \overset{S}{\backsim} y\right) \tag{5}$$

and simply asymptotically statistical equivalent, if $L = 1$.

Definition 6. Two nonnegative sequences $[x]$ and $[y]$ are said to be strong asymptotically equivalent of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} \left|\frac{x_k}{y_k} - L\right| = 0 \quad \left(\text{denoted by } x \overset{w}{\backsim} y\right) \tag{6}$$

and simply strong asymptotically equivalent, if $L = 1$.

Definition 7. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left\{\text{the number of } k \in I_r : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} = 0, \quad \left(\text{denoted by } x \overset{S_\theta}{\backsim} y\right) \tag{7}$$

and simply asymptotically lacunary statistical equivalent, if $L = 1$.

Definition 8. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} \left|\frac{x_k}{y_k} - L\right| = 0 \quad \left(\text{denoted by } x \overset{N_\theta}{\backsim} y\right) \tag{8}$$

and simply strong asymptotically lacunary equivalent, if $L = 1$.

Definition 9. Let f be any modulus; the two nonnegative sequences $[x]$ and $[y]$ are said to be f-asymptotically equivalent of multiple L provided that

$$\lim_k f\left(\left|\frac{x_k}{y_k} - L\right|\right) = 0 \quad \left(\text{denoted by } x \overset{f}{\backsim} y\right) \tag{9}$$

and simply strong f-asymptotically equivalent, if $L = 1$.

Definition 10. Let f be any modulus; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong f-asymptotically equivalent of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right) = 0 \quad \left(\text{denoted by } x \overset{w_f}{\backsim} y\right) \tag{10}$$

and simply strong f-asymptotically equivalent, if $L = 1$.

Definition 11. Two nonnegative sequences $[x]$ and $[y]$ are said to be S_σ-asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left\{\text{the number of } k \le n : \left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| \ge \varepsilon\right\} = 0, \quad \text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{S_\sigma}{\backsim} y\right) \tag{11}$$

and simply σ-asymptotically statistical equivalent, if $L = 1$.

Definition 12. Let θ be a lacunary sequence; two nonnegative sequences $[x]$ and $[y]$ are said to be $S_{\sigma,\theta}$-asymptotically

lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left\{\text{the number of } k \in I_r : \left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| \geq \varepsilon\right\} = 0,$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{S_{\sigma,\theta}}{\sim} y\right) \tag{12}$$

and simply $S_{\sigma,\theta}$-asymptotically lacunary statistical equivalent, if $L = 1$.

Definition 13. Let θ be a lacunary sequence; two nonnegative sequences $[x]$ and $[y]$ are said to be strong σ-asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| = 0,$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{N_{\sigma,\theta}}{\sim} y\right) \tag{13}$$

and simply σ-asymptotically lacunary statistical equivalent, if $L = 1$.

Definition 14. Let f be any modulus and let θ be a lacunary sequence; two nonnegative sequences $[x]$ and $[y]$ are said to be strong f-asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k} - L\right|\right) = 0, \quad \left(\text{denoted by } x \overset{N_{\theta,f}}{\sim} y\right) \tag{14}$$

and simply f-asymptotically lacunary equivalent, if $L = 1$.

Definition 15. Let f be any modulus and let θ be a lacunary sequence; two nonnegative sequences $[x]$ and $[y]$ are said to be strong (σ, f)-asymptotically equivalent of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) = 0,$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{w_{\sigma,f}}{\sim} y\right) \tag{15}$$

and simply (σ, f)-asymptotically equivalent, if $L = 1$.

For $(i) = i + 1$ we write $[w_f]$ for $w_{\sigma,f}$. Hence the two nonnegative sequences $[x]$ and $[y]$ are said to be strong almost f-asymptotically equivalent of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{k+i}}{y_{k+i}} - L\right|\right) = 0,$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{[w_f]}{\sim} y\right) \tag{16}$$

and simply f-asymptotically equivalent, if $L = 1$.

For $f(x) = x$ for all $x \in [0, \infty)$ we write $[w]$ for $[w_f]$. Hence the two nonnegative sequences $[x]$ and $[y]$ are said to be strong almost asymptotically equivalent of multiple L provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_{k+i}}{y_{k+i}} - L\right| = 0$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{[w_f]}{\sim} y\right) \tag{17}$$

and simply strong almost asymptotically equivalent, if $L = 1$.

Definition 16. Let f be any modulus and let θ be a lacunary sequence; two nonnegative sequences $[x]$ and $[y]$ are said to be strong (σ, f)-asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) = 0,$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{N_{\sigma,\theta,f}}{\sim} y\right) \tag{18}$$

and simply (σ, f)-asymptotically lacunary equivalent, if $L = 1$.

For $(i) = i + 1$ we write $M_{\theta,f}$ for $N_{\sigma,\theta,f}$. Hence the two nonnegative sequences $[x]$ and $[y]$ are said to be strong almost f-asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{k+i}}{y_{k+i}} - L\right|\right) = 0,$$
$$\text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{M_{\theta,f}}{\sim} y\right) \tag{19}$$

and simply strong almost f-asymptotically lacunary equivalent, if $L = 1$.

For $f(x) = x$ for all $x \in [0, \infty)$ we write $[w_\theta]$ for $w_{\theta,f}$. Hence the two nonnegative sequences $[x]$ and $[y]$ are

said to be strong almost asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{k+i}}{y_{k+i}}-L\right|=0,$$
$$\text{uniformly in } i=1,2,3,\ldots,\quad \left(\text{denoted by } x \overset{M_\theta}{\sim} y\right) \tag{20}$$

and simply strong almost asymptotically lacunary equivalent, if $L=1$.

3. Main Theorems

We now give lemma to be used later.

Lemma 17. *Let f be any modulus. Suppose for given $\varepsilon>0$, there exist n_o and i_o such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)<\varepsilon \quad \forall n\geq n_o,\ i\geq i_o. \tag{21}$$

Then $x \overset{w_{\sigma,f}}{\sim} y$.

Proof. Let $\varepsilon>0$ be given. Chose n_1 and i_o such that

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)<\frac{\varepsilon}{2} \quad \forall n\geq n_1,\ i\geq i_o. \tag{22}$$

It is sufficient to prove that there exists n_2 such that for $n\geq n_2$, $i_o\geq i\geq 0$

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)<\varepsilon \tag{23}$$

since, taking $n_o=\max(n_1,n_2)$, (23) holds for $n>n_o$ and for all i, which gives the result. Once i_o has been chosen, i_o is fixed, so

$$\sum_{k=0}^{i_o} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)=R\,(\text{say}). \tag{24}$$

Now, taking $i_o\geq i\geq 0$ and $n>i_o$, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)&=\frac{1}{n}\left(\sum_{k=0}^{i_o-1}\sum_{k=i_o}^{n-1}\right) f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)\\&\leq\frac{R}{n}+\frac{\varepsilon}{2}.\end{aligned} \tag{25}$$

Taking n sufficiently large, we can make $R/n+\varepsilon/2<\varepsilon$ which gives (23) and hence the result. □

The next theorems show the relationship between the strong (σ,f)-asymptotically equivalence and the strong (σ,f)-asymptotically lacunary equivalence.

Theorem 18. *Let f be any modulus. Then $x \overset{N_{\sigma,\theta,f}}{\sim} y \Leftrightarrow x \overset{w_{\sigma,f}}{\sim} y$ for every lacunary sequence θ.*

Proof. Let $x \overset{N_{\sigma,\theta,f}}{\sim} y$. Then, given $\varepsilon>0$, there exist r_o and L such that

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)<\varepsilon \tag{26}$$

for $r\geq r_o$ and $i=k_{r-1}+1+v$, $v\geq 0$. Let $n\geq h_r$; write $n=mh_r+p$, where $0\leq p\leq h_r$; m is an integer. Since $n\geq h_r$, $m\geq 0$. We have

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)&\leq\frac{1}{n}\sum_{k=0}^{(m+1)h_{r-1}} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)\\&=\frac{1}{n}\sum_{j=0}^{m}\sum_{k=jh_r}^{(j+1)h_{r-1}} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)\\&\leq\frac{(m+1)\,h_r\varepsilon}{n}\leq\frac{2mh_r\varepsilon}{n}\quad\text{for } m\geq 0.\end{aligned} \tag{27}$$

For $h_r/n\leq 1$, since $mh_r/n\leq 1$, therefore,

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}}-L\right|\right)\leq 2\varepsilon. \tag{28}$$

Then by lemma $x \overset{N_{\sigma,\theta,f}}{\sim} y$ implies $x \overset{w_{\sigma,f}}{\sim} y$. It is easy to see that $x \overset{w_{\sigma,f}}{\sim} y$ implies $x \overset{N_{\sigma,\theta,f}}{\sim} y$ for every θ. □

Proposition 19. *Let f be any modulus. Then $x \overset{M_{\theta,f}}{\sim} y \Leftrightarrow x \overset{[w_f]}{\sim} y$ for every lacunary sequence θ.*

Proof. It follows from Theorem 18 for $\sigma(i)=i+1$ for all $i=1,2,3,\ldots$. □

Proposition 20. *$x \overset{M_\theta}{\sim} y \Leftrightarrow x \overset{[w]}{\sim} y$ for every lacunary sequence θ.*

Proof. It follows from Proposition 19 for $f(x)=x$ for all $x\in[0,\infty)$. □

Theorem 21. *Let f be any modulus. Then*

(1) *if $\liminf_r q_r>1$, then $x \overset{w_{\sigma,f}}{\sim} y$ implies $x \overset{N_{\sigma,\theta,f}}{\sim} y$;*

(2) *if $\limsup_r q_r<\infty$, then $x \overset{N_{\sigma,\theta,f}}{\sim} y$ implies $x \overset{w_{\sigma,f}}{\sim} y$;*

(3) *if $1<\liminf_r q_r\leq\limsup_r q_r<\infty$, then $x \overset{w_{\sigma,f}}{\sim} y \Leftrightarrow x \overset{N_{\sigma,\theta,f}}{\sim} y$.*

Proof. Part (1): let $x \overset{w_{\sigma,f}}{\sim} y$ and $\liminf_r q_r > 1$. There exists $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for sufficiently large r. We have, for sufficiently large r, that $(h_r/k_r) \geq \delta/(1+\delta)$. Then

$$\begin{aligned}\frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) &\geq \frac{1}{k_{r-1}}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\\ &= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\\ &\geq \left[\frac{\delta}{1+\delta}\right]\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\end{aligned} \tag{29}$$

which yields that $x \overset{N_{\sigma,\theta,f}}{\sim} y$.

Part (2): if $\limsup_r q_r < \infty$, then there exists $K > 0$ such that $q_r < K$ for every r.

Now suppose that $x \overset{N_{\sigma,\theta,f}}{\sim} y$ and $\varepsilon > 0$. There exists m_0 such that for every $m \geq m_0$,

$$H_m = \frac{1}{h_m}\sum_{k\in I_m} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) < \varepsilon, \quad \forall i. \tag{30}$$

We can also find $R > 0$ such that $H_m \leq R$ for all m. Let n be any integer with $k_r \geq n > k_{r-1}$ where $r > m_0$. Now write

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\\ &\quad\leq \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\\ &\quad= \frac{1}{k_{r-1}}\left[\sum_{k\in I_1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) + \sum_{k\in I_2} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\right.\\ &\qquad\left. + \cdots + \sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\right]\\ &\quad= \frac{1}{k_{r-1}}\left[\frac{k_1}{k_1}\sum_{k\in I_1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) + \frac{k_2-k_1}{k_2-k_1}\right.\\ &\qquad\times\sum_{k\in I_2} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\\ &\qquad+ \cdots + \frac{k_{m_0}-k_{m_0-1}}{k_{m_0}-k_{m_0-1}}\sum_{k\in I_{m_0}} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\\ &\qquad\left.+ \cdots + \frac{k_r-k_{r-1}}{k_r-k_{r-1}}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\right]\\ &\quad= \frac{k_1}{k_{r-1}}H_1 + \frac{k_2-k_1}{k_{r-1}}H_2 + \cdots + \frac{k_{m_0}-k_{m_0-1}}{k_{r-1}}H_{m_0}\\ &\qquad+ \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}H_r\\ &\quad\leq \frac{k_{m_0}}{k_{r-1}}\sup_{1<k<k_{m_0}} H_k + \sup_{k>k_{m_0}} H_k\frac{(k_r-k_{m_0})}{k_{r-1}}\\ &\quad< R\frac{k_{m_0}}{k_{r-1}} + \varepsilon K\end{aligned} \tag{31}$$

from which we deduce that $x \overset{w_{\sigma,f}}{\sim} y$.

Part (3): this immediately follows from (1) and (2). □

Proposition 22. *Let f be any modulus. Then*

(1) *if $\liminf_r q_r > 1$, then $x \overset{[w_f]}{\sim} y$ implies $x \overset{M_{\theta,f}}{\sim} y$;*

(2) *if $\limsup_r q_r < \infty$, then $x \overset{M_{\theta,f}}{\sim} y$ implies $x \overset{[w_f]}{\sim} y$;*

(3) *if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $x \overset{[w_f]}{\sim} y \Leftrightarrow x \overset{M_{\theta,f}}{\sim} y$.*

Proof. It follows from Theorem 21 for $\sigma(i) = i + 1$ for all $i = 1, 2, 3, \ldots$. □

Proposition 23. *Consider the following:*

(1) *if $\liminf_r q_r > 1$, then $x \overset{[w]}{\sim} y$ implies $x \overset{M_\theta}{\sim} y$;*

(2) *if $\limsup_r q_r < \infty$, then $x \overset{M_\theta}{\sim} y$ implies $x \overset{[w]}{\sim} y$;*

(3) *if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $x \overset{[w]}{\sim} y \Leftrightarrow x \overset{M_\theta}{\sim} y$.*

Proof. It follows from Proposition 22 for $f(x) = x$ for all $x \in [0, \infty)$. □

In the following theorem we study the relationship between the strong (σ, f)-asymptotically lacunary equivalence and the strong σ-asymptotically lacunary equivalence.

Theorem 24. *Let f be any modulus. Then*

(1) *if $x \overset{N_{\sigma,\theta}}{\sim} y$, then $x \overset{N_{\sigma,\theta,f}}{\sim} y$;*

(2) *if $\lim_{t\to\infty}(f(t)/t) = \beta > 0$, then $x \overset{N_{\sigma,\theta}}{\sim} y \Leftrightarrow x \overset{N_{\sigma,\theta,f}}{\sim} y$.*

Proof. Part (1): let $x \overset{N_{\sigma,\theta}}{\sim} y$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \le u \le \delta$. We can write

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) = \frac{1}{h_r}\sum_{1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) + \frac{1}{h_r}\sum_{2} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right), \tag{32}$$

where the first summation is over $|x_{\sigma^k(i)}/y_{\sigma^k(i)} - L| \le \delta$ and the second summation over $|x_{\sigma^k(i)}/y_{\sigma^k(i)} - L| > \delta$ with $k \in I_r$. By definition of f, we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \le \varepsilon + 2f(1)\delta^{-1}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right). \tag{33}$$

Therefore, $x \overset{N_{\sigma,\theta,f}}{\sim} y$.

Part (2): if $\lim_{t\to\infty}(f(t)/t) = \beta > 0$, then $f(t) \ge \beta t$ for all $t > 0$. Let $x \overset{N_{\sigma,\theta,f}}{\sim} y$; clearly

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \ge \frac{1}{h_r}\sum_{k\in I_r} \beta\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| = \beta\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|. \tag{34}$$

Therefore, $x \overset{N_{\sigma,\theta}}{\sim} y$. By using (23) the proof is complete. □

Proposition 25. *Let f be any modulus. Then*

(1) *if $x \overset{M_\theta}{\sim} y$, then $x \overset{M_{\theta,f}}{\sim} y$;*

(2) *if $\lim_{t\to\infty}(f(t)/t) = \beta > 0$, then $x \overset{M_\theta}{\sim} y \Leftrightarrow x \overset{M_{\theta,f}}{\sim} y$.*

Proof. It follows from Theorem 24 for $\sigma(i) = i + 1$ for all $i = 1, 2, 3, \ldots$. □

Finally we give the relation between $S_{\sigma,\theta}$-asymptotically lacunary statistical equivalence and strong (σ, f)-asymptotically lacunary equivalence. Also we give relation between $S_{\sigma,\theta}$-asymptotically lacunary statistical equivalence and strong (σ, f)-asymptotically equivalence.

Theorem 26. *Let f be any modulus. Then*

(1) *if $x \overset{N_{\sigma,\theta,f}}{\sim} y$, then $x \overset{S_{\sigma,\theta}}{\sim} y$;*

(2) *if f is bounded and $x \overset{S_{\sigma,\theta}}{\sim} y$, then $x \overset{N_{\sigma,\theta,f}}{\sim} y$;*

(3) *if f is bounded, then $x \overset{N_{\sigma,\theta,f}}{\sim} y \Leftrightarrow x \overset{S_{\sigma,\theta}}{\sim} y$.*

Proof. Part (1): take $\varepsilon > 0$ and $x \overset{N_{\sigma,\theta,f}}{\sim} y$. Let $\sum_1$ denote the sum over $k \in I_r$ and $|x_{\sigma^k(i)}/y_{\sigma^k(i)} - L| \ge \varepsilon$.

Then

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \ge \frac{1}{h_r}\sum_{1} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \ge f(\varepsilon)\left\{\text{the number of } k \in I_r : \left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| \ge \varepsilon\right\}. \tag{35}$$

Therefore, $x \overset{S_{\sigma,\theta}}{\sim} y$.

Part (2): suppose that f is bounded and $x \overset{S_{\sigma,\theta}}{\sim} y$. Since f is bounded, there exists an integer T such that $|f(x)| \le T$ for all $x \ge 0$. Given $\varepsilon > 0$,

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \le T\frac{1}{h_r}\left\{\text{the number of } k \in I_r : \left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| \ge \varepsilon\right\} + f(\varepsilon). \tag{36}$$

Therefore, $x \overset{N_{\sigma,\theta,f}}{\sim} y$.

Part (3): follows from (1) and (2).

For $(i) = i + 1$ we write $[S_\theta]$ for $S_{\sigma,\theta}$. Hence the two nonnegative sequences $[x]$ and $[y]$ are said to be $[S_\theta]$-asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left\{\text{the number of } k \in I_r : \left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| \ge \varepsilon\right\} = 0, \quad \text{uniformly in } i = 1, 2, 3, \ldots, \quad \left(\text{denoted by } x \overset{[S_\theta]}{\sim} y\right) \tag{37}$$

and $[S_\theta]$-asymptotically lacunary statistical equivalent, if $L = 1$. Hence we have the following. □

Proposition 27. *Let f be any modulus. Then*

(1) *if $x \overset{M_{\theta,f}}{\sim} y$, then $x \overset{[S_\theta]}{\sim} y$;*

(2) *if f is bounded and $x \overset{[S_\theta]}{\sim} y$, then $x \overset{M_{\theta,f}}{\sim} y$;*

(3) *if f is bounded, then $x \overset{M_{\theta,f}}{\sim} y \Leftrightarrow x \overset{[S_\theta]}{\sim} y$.*

Proof. It follows from Theorem 26 for $\sigma(i) = i + 1$ for all $i = 1, 2, 3, \ldots$. □

Proposition 28. *$x \overset{M_\theta}{\sim} y$ implies $x \overset{[S_\theta]}{\sim} y$.*

Proof. It follows from Proposition 27(1) for $f(x) = x$ for all $x \in [0, \infty)$. □

Theorem 29. *Let f be bounded; then $x \overset{S_{\sigma,\theta}}{\smile} y$ implies $x \overset{w_{\sigma,f}}{\smile} y$ for every lacunary sequence θ.*

Proof. Let n be any integer with $n \in I_r$; then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) = \frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) + \frac{1}{n}\sum_{k=1+k_{r-1}}^{n} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right). \tag{38}$$

Consider the first term on the right in (38):

$$\begin{aligned}\frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) &\le \frac{1}{k_{r-1}}\sum_{p=1}^{r-1}\sum_{k\in I_p} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \\ &= \frac{1}{k_{r-1}}\sum_{p=1}^{r-1} h_p\left(\frac{1}{h_p}\sum_{k\in I_p} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\right).\end{aligned} \tag{39}$$

Since f is bounded and $x \overset{S_{\sigma,\theta}}{\smile} y$, it follows from Theorem 26(2) that

$$\frac{1}{h_p}\sum_{k\in I_p} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \longrightarrow 0. \tag{40}$$

Hence

$$\frac{1}{k_{r-1}}\sum_{p=1}^{r-1} h_p\left(\frac{1}{h_p}\sum_{k\in I_p} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right)\right) \longrightarrow 0. \tag{41}$$

Consider the second term on the right in (38); since f is bounded, there exists an integer T such that $|f(x)| \le T$ for all $x \ge 0$. We split the sum for $k_{r-1} < k \le n$ into sums over $|x_{\sigma^k(i)}/y_{\sigma^k(i)} - L| \ge \varepsilon$ and $|x_{\sigma^k(i)}/y_{\sigma^k(i)} - L| < \varepsilon$. Therefore, we have for every $\varepsilon > 0$ that

$$\frac{1}{n}\sum_{k=1+k_{r-1}}^{n} f\left(\left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right|\right) \le T\frac{1}{h_r}\left\{\text{the number of } k \in I_r : \left|\frac{x_{\sigma^k(i)}}{y_{\sigma^k(i)}} - L\right| \ge \varepsilon\right\} + f(\varepsilon). \tag{42}$$

Since $x \overset{S_{\sigma,\theta}}{\smile} y$, f is continuous from the right at 0 and ε is arbitrary; the expression on left side of (42) tends to zero as $r \to \infty$, uniformly in i. Hence (38), (41), and (42) imply that $x \overset{w_{\sigma,f}}{\smile} y$. □

Proposition 30. *Let f be bounded; then $x \overset{[S_\theta]}{\smile} y$ implies $x \overset{[w_f]}{\smile} y$ for every lacunary sequence θ.*

Proof. It follows from Theorem 29 for $\sigma(i) = i + 1$ for all $i = 1, 2, 3, \ldots$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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