

Research Article

A Finite Difference Solution of a Simply Supported Beam of Orthotropic Composite Materials Using Displacement Potential Formulation

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Here an efficient displacement potential formulation based finite difference technique is used to solve the elastic field of a simply supported beam of orthotropic composite materials. A simply supported beam made of orthotropic composite material under uniformly distributed loading is considered and its elastic behaviors under such loading conditions are analyzed considering plane stress condition. The solutions of the problem satisfy the force equilibrium conditions as well as boundary conditions. For understanding the elastic behavior of a simply supported beam, the displacement and stress components of some important sections of the beam are shown graphically. Effects of different orthotropic composite materials on the solutions are also analyzed. Besides, at a particular section of the beam, the comparative analysis of the elastic field is carried out by using the FDM and FEM methods.

1. Introduction

Strength-to-weight ratio of a fiber reinforced composite material is usually higher than that of the corresponding isotropic material. The use of composite materials gradually increases with time, especially to satisfy the demand of dynamic vehicles such as aircrafts, rockets and so forth, to reduce energy consumptions and to absorb vibrational energy during moving. The use of composite simply supported beams in the construction of engineering structures is quite extensive. It is known that the mechanical properties like strength and toughness of a fiber reinforced composite differ significantly from those of the isotropic materials, which eventually play an important role in defining the state of stress and displacements of the corresponding structure under loading. It is noted that stress analysis of composite structures is usually carried out by the numerical method like the finite element method (FEM) but computations are extremely higher rather than the finite difference based displacement potential numerical approach [1–6]. It is worth mentioning that if the computations of numerical solutions

are decreased, the accuracy of the solution will increase and thus the present finite difference solution using the present displacement potential formulation can be used as a standard guideline for checking the reliability and accuracy of approximate solutions.

In the solution of structures, the physical conditions of bounding surfaces are mathematically modeled in terms of a mixed mode of boundary conditions, where one of the components of stress as well as displacement on the boundary is assumed to be known. However, the earlier mathematical models of elasticity were inadequate in handling the practical stress problems, as most of them are of mixed-boundary-value type. Since the numerical solution of mixed-boundary-value problems considering low computations, especially with nonisotropic materials, is beyond the scope of the existing mathematical models of elasticity, the use of a new mathematical formulation is investigated to analyze the elastic field of a simply supported beam under uniformly distributed loading.

Although the stress analysis has now become a classical subject in the field of solid mechanics, somehow these

stress analysis problems are still suffering from many shortcomings and thus are being constantly looked into [7–14]. Elasticity problems are usually formulated in terms of either deformation parameters or stress parameters. Among the existing mathematical models of plane boundary-value stress problems, the stress function approach [15] and the displacement potential formulation [16] are noticeable. The application of the stress function formulation in conjunction with finite difference technique has been reported for the solution of plane elastic problems where all of the boundary conditions are prescribed in terms of stresses only [8, 17]. Further, Conway and Ithaca [18] extended the stress function formulation in the form of Fourier integrals to the case where the material is orthotropic and obtained analytical solutions for a number of ideal problems. The shortcoming of stress approach is that it accepts boundary conditions only in terms of loadings. Boundary restraints specified in terms of displacement components cannot be satisfactorily imposed on the stress function. As most of the practical problems of elasticity are of mixed-boundary type, the stress function approach fails to provide any explicit understanding of the state of stresses at the critical regions of supports and stiffeners. The displacement formulation, on the other hand, involves finding two displacement functions simultaneously from the two second-order elliptic partial differential equations of equilibrium, which is extremely difficult, and this problem becomes more serious when the boundary conditions are mixed [16]. The elastic field of some important structural elements made of isotropic and anisotropic materials is studied analytically and numerically using displacement potential formulation [1–6, 19–38]. It is noted that the recent research and developments in using the displacement potential boundary modeling approach [1–6, 19–38] have generated renewed interest in the field of numerical solutions of stress problems. Recently, Deb Nath [3] proposed the displacement potential formulations for the solution of general anisotropic composite structures and solved cantilever beams with uniformly distributed and point loadings [4], columns with variable compressive loads at its top end [5], a one fixed panel with shear load at its ends [6], and both end fixed beams with distributed loading on its span [19] using finite difference technique. As simply supported composite beams are widely used in different structures, so detailed investigations of the stress field for its perfect design is necessary. Here an alternative numerical solution of a simply supporting beam with uniformly distributed loading using finite difference technique is obtained. Here this paper presents numerical solution of composite simply supported beam subjected to a uniformly distributed loading. Effect of different composites on the solution is also analyzed for simply supported beam. The comparisons of the elastic field obtained by the FDM and FEM of different particular sections of a simply supported beam are also reported.

2. Numerical Model of the Problem

Here, a rectangular simply supported beam of orthotropic composite material under uniformly distributed loading is

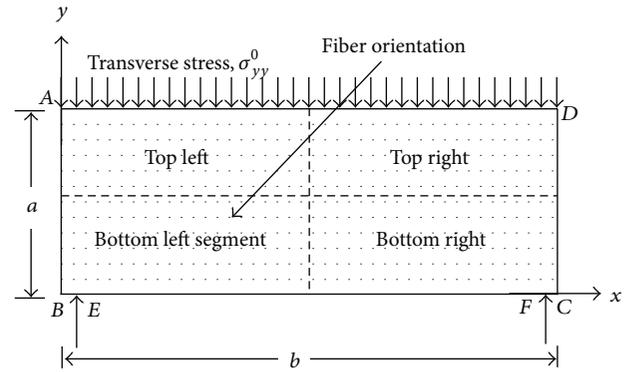


FIGURE 1: A simply supported beam of orthotropic composite material under uniformly distributed loading.

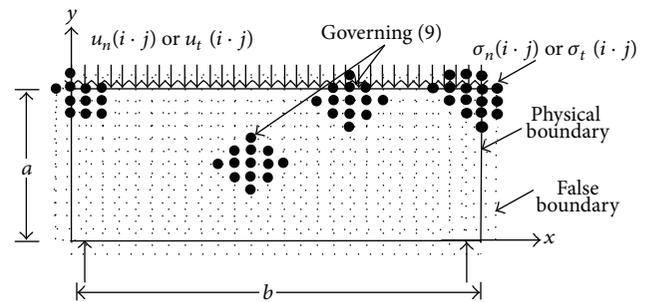


FIGURE 2: Application of the normal and tangential displacement, stress components, and the governing equation as finite difference forms.

shown in Figure 1. The detailed discretization and application of the boundary conditions and governing equation in finite difference form are shown in Figure 2. The aspect ratio of the simply supported beam is 3.0. The orientation of fibers along the length of the beam is $\theta = 0^\circ$. The boundary conditions, which are used to solve the problem, are shown in Tables 1(a) and 1(b). The used material properties are shown in Table 2. Here, it is worthy to mention that although the formulations can be applied to any composites, boron/epoxy composite is chosen merely as an example. Furthermore, all the results presented in the study correspond to the value of applied uniformly distributed loading; σ_{yy}^0 is 41.4 MPa.

3. Displacement Potential Function Formulation

With reference to a rectangular coordinate system (x, y) , the differential equations of equilibrium for the plane stress problems of orthotropic composite materials having the fiber orientation, $\theta = 0^\circ$, are as follows [3, 4, 39]:

$$\begin{aligned} & \frac{E_1^2}{E_1 - \nu_{12}^2 E_2} \frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\nu_{12} E_1 E_2}{E_1 - \nu_{12}^2 E_2} + G_{12} \right) \\ & \times \frac{\partial^2 u_y}{\partial x \partial y} + G_{12} \frac{\partial^2 u_x}{\partial y^2} = 0, \end{aligned} \quad (1)$$

TABLE 1: (a) Boundary conditions of different boundary segments of the simply supported beam as observed in Figure 1. (b) Boundary conditions of corner points of the simply supported beam problem as observed in Figure 1.

(a)				
Boundary segment	Given boundary conditions	Correspondence between mesh points and given boundary conditions		
		Mesh point on the physical boundary conditions	Mesh point on the imaginary boundary conditions	
AB	σ_n, σ_t	$\sigma_n = 0$	$\sigma_t = 0$	
BC	u_n, σ_n, σ_t	At point E and F, $u_n = 0$ Except point E and F, $\sigma_n = 0$	$\sigma_t = 0$	
CD	σ_n, σ_t	$\sigma_n = 0$	$\sigma_t = 0$	
DA	σ_n, σ_t	$\sigma_n = -\sigma_{yy}^0$	$\sigma_t = 0$	

(b)				
Corner points	Given boundary conditions	Used boundary conditions	Correspondence between mesh points and given boundary conditions	
			Mesh point on the physical boundary conditions	Mesh point on the imaginary boundary conditions
A	$[(\sigma_n, \sigma_t), (\sigma_n, \sigma_t)]$	σ_n, σ_t	$\sigma_n = -\sigma_{yy}^0$	$\sigma_t = 0$
B	$[(\sigma_n, \sigma_t), (\sigma_n, \sigma_t)]$	σ_n, σ_t	$\sigma_n = 0$	$\sigma_t = 0$
C	$[(\sigma_n, \sigma_t), (\sigma_n, \sigma_t)]$	σ_n, σ_t	$\sigma_n = 0$	$\sigma_t = 0$
D	$[(\sigma_n, \sigma_t), (\sigma_n, \sigma_t)]$	σ_n, σ_t	$\sigma_n = -\sigma_{yy}^0$	$\sigma_t = 0$

TABLE 2: Properties of composites used to obtain numerical results.

Material	Property	Boron/epoxy
Fiber	E_f (10^3 MPa)	414
	ν_f	0.20
Resin	E_r (10^3 MPa)	3.45
	ν_r	0.35
Composite	E_1 (10^3 MPa)	282.9
	E_2 (10^3 MPa)	24.2
	G_{12} (10^3 MPa)	10.4
	ν_{12}	0.27
	ν_{21}	0.023

$$\frac{E_1 E_2}{E_1 - \nu_{12}^2 E_2} \frac{\partial^2 u_y}{\partial y^2} + \left(\frac{\nu_{12} E_1 E_2}{E_1 - \nu_{12}^2 E_2} + G_{12} \right) \times \frac{\partial^2 u_x}{\partial x \partial y} + G_{12} \frac{\partial^2 u_y}{\partial x^2} = 0. \quad (2)$$

The displacement components are expressed in terms of a displacement potential function, $\psi(x, y)$, as follows:

$$\begin{aligned} u_x(x, y) &= \alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_3 \frac{\partial^2 \psi}{\partial y^2}, \\ u_y(x, y) &= \alpha_4 \frac{\partial^2 \psi}{\partial x^2} + \alpha_5 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2}, \end{aligned} \quad (3)$$

where α_i are unknown material constants.

The values of α are obtained by substituting the displacement components given by (3) into the equilibrium

equation (1). The values of the constants thus obtained for the orthotropic materials with $\theta = 0^0$ are as follows [3, 4]:

$$\begin{aligned} \alpha_1 &= \alpha_3 = \alpha_5 = 0, \\ \alpha_2 &= 1, \\ \alpha_4 &= -\frac{E_1^2}{Z_{11}}, \\ \alpha_6 &= -\frac{G_{12} (E_1 - \nu_{12}^2 E_2)}{Z_{11}}, \end{aligned} \quad (4)$$

where $Z_{11} = \nu_{12} E_1 E_2 + G_{12} (E_1 - \nu_{12}^2 E_2)$.

When the values of the constant α are substituted in the equilibrium equation (2), the equation turns into a fourth-order partial differential equation that governs the displacement potential function, $\psi(x, y)$. The governing equation in terms of the function ψ is as follows:

$$E_1 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 (E_1 - 2\nu_{12} G_{12}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_2 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0. \quad (5)$$

Similarly, substituting the value of α in (3) gives the two displacement components in terms of the displacement potential function as follows:

$$\begin{aligned} u_x(x, y) &= \frac{\partial^2 \psi}{\partial x \partial y}, \\ u_y(x, y) &= -\frac{1}{Z_{11}} \left[E_1^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} (E_1 - \nu_{12}^2 E_2) \frac{\partial^2 \psi}{\partial y^2} \right]. \end{aligned} \quad (6)$$

Using (6), the stress-strain relations, and reciprocal relationships [2] having fiber orientation $\theta = 0^0$, the stress

components in terms of the potential function ψ are obtained as follows [3, 4]:

$$\begin{aligned}\sigma_{xx}(x, y) &= \frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^2 \partial y} - \nu_{12} E_2 \frac{\partial^3 \psi}{\partial y^3} \right], \\ \sigma_{yy}(x, y) &= \frac{E_1 E_2}{Z_{11}} \left[(\nu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right], \quad (7) \\ \sigma_{xy}(x, y) &= -\frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^3} - \nu_{12} E_2 \frac{\partial^3 \psi}{\partial x \partial y^2} \right].\end{aligned}$$

From these expressions it is found that, as far as boundary conditions are concerned, either known restraints or known stresses or any combination of these can readily be converted to finite difference expressions in terms of ψ on the boundary. To analyze the state of displacements and stresses for two-dimensional regular/irregular-shaped bodies with mixed boundary condition, the fourth-order-partial differential equation (5) together with the displacement and stress components given by (6) to (7) is considered. The expressions for the displacement and stress components as given by (6) to (7) are valid for points within the body as well as on the boundary. But, for the case of practical problems, the boundary conditions are known in terms of their normal and tangential components on the boundary. Thus, the boundary equations for the known conditions on the boundary having arbitrary shape are

$$\begin{aligned}u_n(x, y) &= u_x \cdot l + u_y \cdot m \\ &= -\frac{E_1^2 m}{Z_{11}} \frac{\partial^2 \psi}{\partial x^2} - \frac{G_{12} (E_1 - \nu_{12}^2 E_2) m}{Z_{11}} \frac{\partial^2 \psi}{\partial y^2} + l \frac{\partial^2 \psi}{\partial x \partial y},\end{aligned}$$

$$\begin{aligned}u_t(x, y) &= u_y \cdot l - u_x \cdot m \\ &= -\frac{E_1^2 l}{Z_{11}} \frac{\partial^2 \psi}{\partial x^2} - \frac{l G_{12} (E_1 - \nu_{12}^2 E_2)}{Z_{11}} \frac{\partial^2 \psi}{\partial y^2} - m \frac{\partial^2 \psi}{\partial x \partial y},\end{aligned}$$

$$\begin{aligned}\sigma_n(x, y) &= l^2 \sigma_{xx}(x, y) + 2lm \sigma_{xy}(x, y) + m^2 \sigma_{yy}(x, y) \\ &= \frac{E_1}{Z_{11}} \left[-2lm E_1 G_{12} \frac{\partial^3 \psi}{\partial x^3} \right. \\ &\quad + \left\{ E_1 G_{12} l^2 + (\nu_{12} G_{12} - E_1) m^2 E_2 \right\} \frac{\partial^3 \psi}{\partial x^2 \partial y} \\ &\quad + 2lm E_2 G_{12} \nu_{12} \frac{\partial^3 \psi}{\partial x \partial y^2} \\ &\quad \left. - E_2 (G_{12} \nu_{12} l^2 + G_{12} m^2) \frac{\partial^3 \psi}{\partial y^3} \right], \\ \sigma_t(x, y) &= (l^2 - m^2) \sigma_{xy}(x, y) + lm \{ \sigma_{yy}(x, y) - \sigma_{xx}(x, y) \} \\ &= \frac{E_1}{Z_{11}} \left[-(l^2 - m^2) E_1 G_{12} \frac{\partial^3 \psi}{\partial x^3} \right.\end{aligned}$$

$$\begin{aligned}&+ \left\{ E_2 (\nu_{12} G_{12} - E_1) - E_1 G_{12} \right\} lm \frac{\partial^3 \psi}{\partial x^2 \partial y} \\ &+ (l^2 - m^2) E_2 G_{12} \nu_{12} \frac{\partial^3 \psi}{\partial x \partial y^2} \\ &\left. + lm E_2 G_{12} (\nu_{12} - 1) \frac{\partial^3 \psi}{\partial y^3} \right]. \quad (8)\end{aligned}$$

4. Finite Difference Modeling of the Problem

4.1. Finite Difference Discretization of the Governing Equation. The present computational scheme involves evaluation of a single function, ψ , at the mesh points of a uniform two-dimensional rectangular mesh network used to discretize the domain. The governing equation (5), used to evaluate the function ψ only at the internal mesh points, can be expressed in its corresponding difference form when all the derivatives are replaced by corresponding central difference expressions. The finite difference form of (5) is as follows:

$$\begin{aligned}zk1 \{ \psi(i-2, j) + \psi(i+2, j) \} \\ - zk2 \{ \psi(i-1, j) + \psi(i+1, j) \} \\ - zk3 \{ \psi(i, j+1) + \psi(i, j-1) \} + zk4 \psi(i, j) \\ + zk5 \{ \psi(i+1, j+1) + \psi(i+1, j-1) \\ + \psi(i-1, j+1) + \psi(i-1, j-1) \} \\ + Q \{ \psi(i, j-2) + \psi(i, j+2) \} = 0,\end{aligned} \quad (9)$$

where $zk1 = R^4$, $zk2 = 4R^4 + 2PR^2$, $zk3 = 2PR^2 + 4Q$, $zk4 = 6R^4 + 4PR^2 + 6Q$

$$\begin{aligned}zk5 = PR^2, \quad P = \left(\frac{E_2}{G_{12}} - \frac{2\nu_{12} E_2}{E_1} \right), \\ Q = \frac{E_2}{E_1}, \quad R = \frac{k}{h}.\end{aligned} \quad (10)$$

4.2. Application of the Governing Equation. In finite difference method, the region occupied by the structure under consideration is divided into fine meshes to get nodal points as shown in Figure 2. The solid line represents the physical boundary conditions of the structure. The discretized form of governing differential equation is applied at each of the nodal points (i, j) inside the physical boundary. This gives a set of algebraic equations for the determination of an unknown function, ψ . However, the number of unknown is greater than the number of equations, which makes the problem intractable. This is because the application of the governing equation to the interior points also involves the points on and outside the physical boundary. The line formed by connecting the points outside the physical boundary is called imaginary boundary as shown in Figure 2. The number of unknowns is equal to the number of nodal points including boundary and exterior points. To make the problem tractable,

it is necessary to generate more equations. This requirement is met up by applying the discretized boundary conditions at the nodal points on the boundary and this gives a complete set of simultaneous equations which is solved by LU decomposition.

4.3. Boundary Conditions in Finite Difference Form. The boundary conditions are discretized in such a way that their applications does not involve additional points outside the imaginary boundary. The whole body of the beam is divided into four segments. To satisfy this requirement, the same boundary conditions must be required to discretize in different forms such as *i*-forward and *j*-forward; *i*-backward and *j*-backward; *i*-backward and *j*-backward for bottom left, bottom right, top right, and top left segments, respectively.

4.3.1. Displacement and Stress Components at the Bottom Left Segment of the Boundary in Finite Difference Forms. Consider

$$\begin{aligned} u_x(x, y) = & c_1 [9\psi(i, j) - 12\{\psi(i, j+1) + \psi(i+1, j)\} \\ & + 16\psi(i+1, j+1) \\ & + 3\{\psi(i, j+2) + \psi(i+2, j)\} \\ & - 4\{\psi(i+1, j+2) + \psi(i+2, j+1)\} \\ & + \psi(i+2, j+2)], \end{aligned}$$

$$\begin{aligned} u_y(x, y) = & c_2 \{\psi(i-1, j) + \psi(i+1, j)\} \\ & + c_3 \{\psi(i, j+1) + \psi(i, j-1)\} \\ & - 2(c_2 + c_3)\psi(i, j), \end{aligned}$$

$$\begin{aligned} \sigma_{xx}(x, y) = & -3c_4 \{\psi(i-1, j) + \psi(i+1, j)\} \\ & + 4c_4 \{\psi(i-1, j+1) + \psi(i+1, j+1)\} \\ & + (6c_4 + 10c_5)\psi(i, j) - (8c_4 + 12c_5)\psi(i, j+1) \\ & + (2c_4 + 6c_5)\psi(i, j+2) - c_4\psi(i-1, j+2) \\ & - c_4\psi(i+1, j+2) \\ & - 3c_5\psi(i, j-1) - c_5\psi(i, j+3), \end{aligned}$$

$$\begin{aligned} \sigma_{yy}(x, y) = & -3c_6 \{\psi(i-1, j) + \psi(i+1, j)\} \\ & + 4c_6 \{\psi(i-1, j+1) + \psi(i+1, j+1)\} \\ & + (6c_6 + 10c_7)\psi(i, j) - (8c_6 + 12c_7)\psi(i, j+1) \\ & + (2c_6 + 6c_7)\psi(i, j+2) - c_6\psi(i-1, j+2) \\ & - c_6\psi(i+1, j+2) \\ & - 3c_7\psi(i, j-1) - c_7\psi(i, j+3), \end{aligned}$$

$$\begin{aligned} \sigma_{xy}(x, y) = & -3c_9\psi(i-1, j) + (10c_9 + 6c_8)\psi(i, j) \\ & - (12c_9 + 8c_8)\psi(i+1, j) + (2c_8 + 6c_9) \end{aligned}$$

$$\begin{aligned} & \times \psi(i+2, j) - c_9\psi(i+3, j) \\ & - 3c_8 \{\psi(i, j-1) + \psi(i, j+1)\} \\ & + 4c_8 \{\psi(i+1, j-1) + \psi(i+1, j+1)\} \\ & - c_8 \{\psi(i+2, j-1) + \psi(i+2, j+1)\}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} c_1 = \frac{1}{4kh}, \quad c_2 = -\frac{E_1^2}{Z_{11}h^2}, \\ c_3 = -\frac{G_{12}(E_1 - \nu_{12}^2E_2)}{Z_{11}k^2}, \\ c_4 = \frac{E_1^2G_{12}}{2Z_{11}h^2k}, \quad c_5 = -\frac{E_1E_2\nu_{12}G_{12}}{2Z_{11}k^3}, \\ c_6 = -\frac{E_1E_2(E_1 - \nu_{12}G_{12})}{2Z_{11}h^2k}, \\ c_7 = -\frac{E_1E_2G_{12}}{2Z_{11}k^3}, \quad c_8 = \frac{E_1E_2\nu_{12}G_{12}}{2Z_{11}hk^2}, \\ c_9 = -\frac{E_1^2G_{12}}{2Z_{11}h^3}. \end{aligned} \quad (12)$$

The discretized forms of boundary conditions derived above cannot be conveniently used for boundaries which are not parallel to the *x*- and *y*-axis. In general, the components of stress and displacements which are normal and tangential to the boundaries are used as boundary conditions. The normal and tangential form of displacement and stress components are below:

$$u_n(i, j) = u_x(i, j) \cdot l + u_y(i, j) \cdot m,$$

$$u_t(i, j) = u_y(i, j) \cdot l - u_x(i, j) \cdot m,$$

$$\sigma_n(i, j) = l^2\sigma_{xx}(i, j) + 2lm\sigma_{xy}(i, j) + m^2\sigma_{yy}(i, j),$$

$$\sigma_t(i, j) = (l^2 - m^2)\sigma_{xy}(i, j) + lm\{\sigma_{yy}(i, j) - \sigma_{xx}(i, j)\}. \quad (13)$$

5. Results and Discussion

5.1. Analysis of Elastic Behavior of the Simply Supported Beam. The geometry of the problem is shown in Figure 1. Figure 3 presents the normalized displacement components u_x as a function of *x*. At $x/b = 0.5$, the axial displacement of each section is zero. The nature of the axial displacement above and below the neutral axis is opposite and antisymmetric. Towards the center ($x/b = 0.5$) of each section, the magnitude of u_x gradually decreases to zero. On the other hand, the axial displacement component of each section with respect to *x* is the highest at the left and right ends of the beam, which gradually decreases to zero towards the center $x/b = 0.5$. Figure 4 describes the normalized lateral displacement

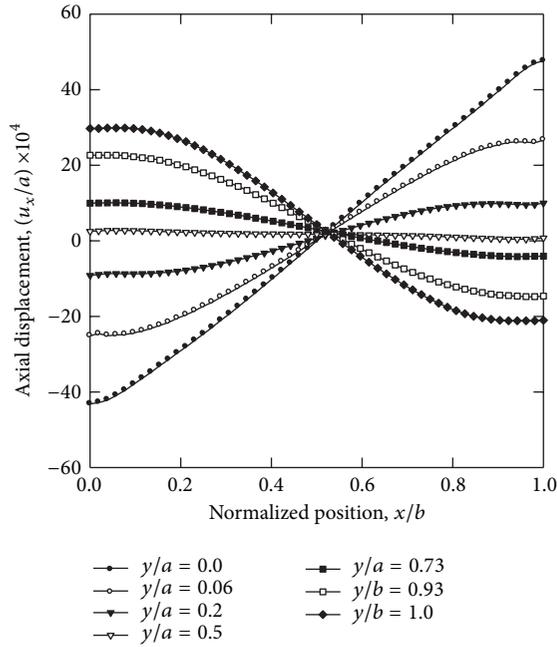


FIGURE 3: Distribution of the normalized displacement component, u_x , as a function of x/b at different sections of the beam.

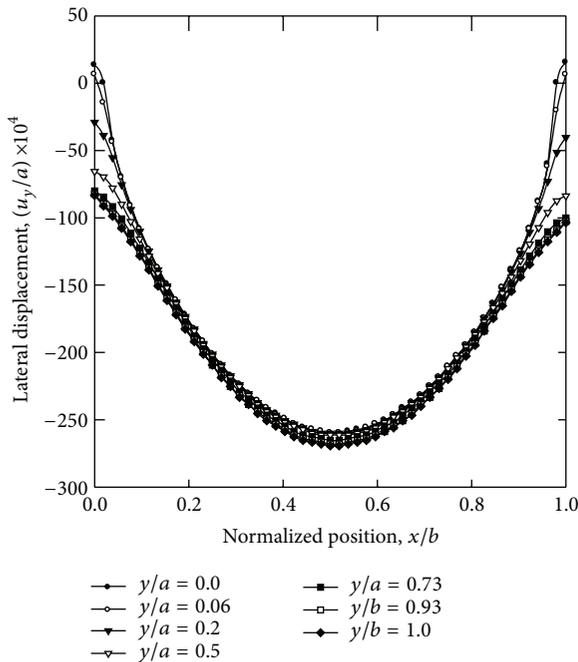


FIGURE 4: Distribution of the normalized displacement component, u_y , as a function of x/b at different sections of the beam.

component u_y , with respect to x/b at different sections of the beam. Middle region ($x/b = 0.5$) of each section shows the highest lateral displacement. Figure 5 shows the deformed shape of the beam which shows that deformation in y direction is more significant.

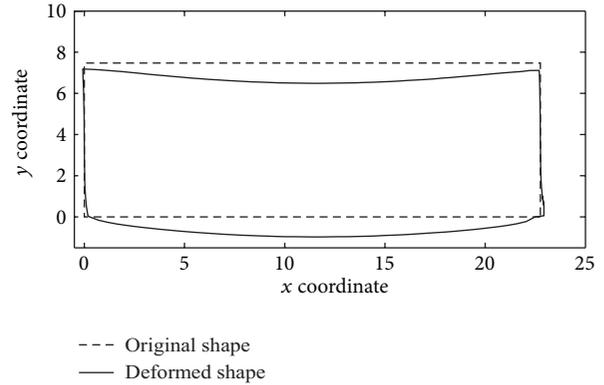


FIGURE 5: Deformed shape of the cantilever beam under uniformly distributed loading.

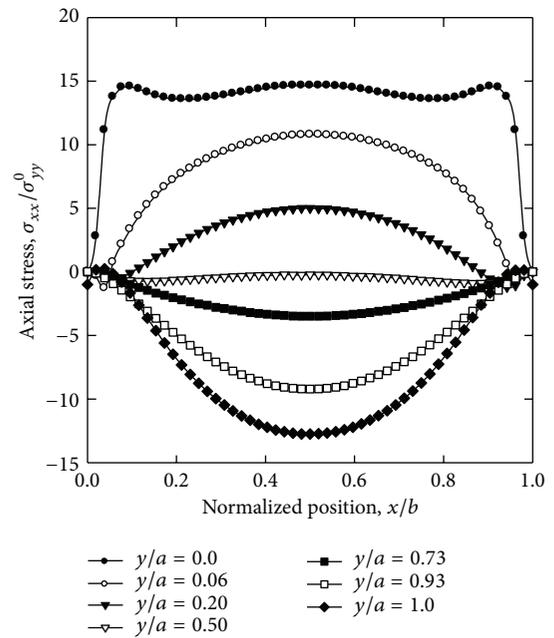


FIGURE 6: Distribution of the normalized normal stress component, $\sigma_{yy}/\sigma_{yy}^0$, as a function of x/b at different sections of the beam.

Figure 6 describes the normalized axial stress component, σ_{xx} , as a function of x/b . The stress σ_{xx} is zero at the neutral axis $y/a = 0.5$, above which σ_{xx} is negative and below which σ_{xx} is positive. The negative and positive stresses correspond that the regions are in compression and tension, respectively. This agrees with the physical characteristics of the problem. Figure 7 shows that $\sigma_{yy}/\sigma_{yy}^0$ is unity at $y/a = 1.0$ for any value of x/b . This tells that the computed stress is equal to the applied stress at the top surface which verifies the boundary condition. Further, $\sigma_{yy}/\sigma_{yy}^0$ is maximum at the two supports of the bottom surface ($y/a = 0$). This is due to the fact that the two supports equalize the total load applied at the top surface. Figure 8 exhibits the distribution of normalized shear stress σ_{xy} as a function of x/b . It is observed that the shear stress at the entire boundary is zero which conforms to the

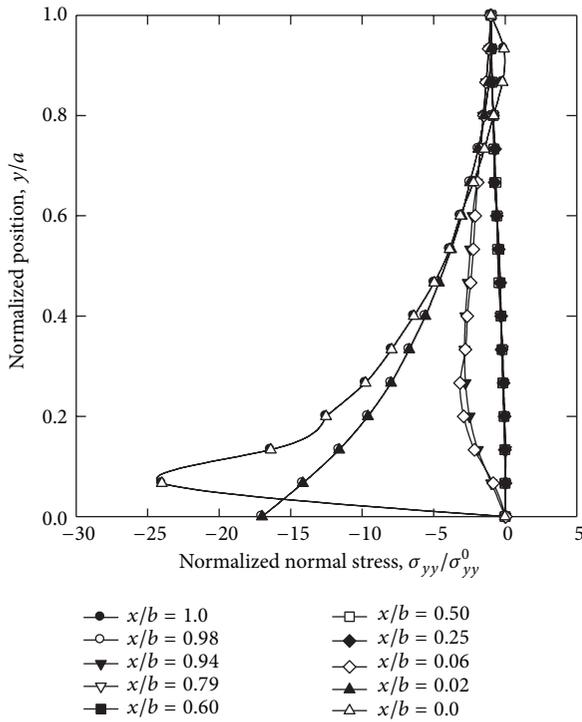


FIGURE 7: Distribution of the normalized normal stress component, $\sigma_{yy}/\sigma_{yy}^0$, as a function of y/a at different sections of the beam.

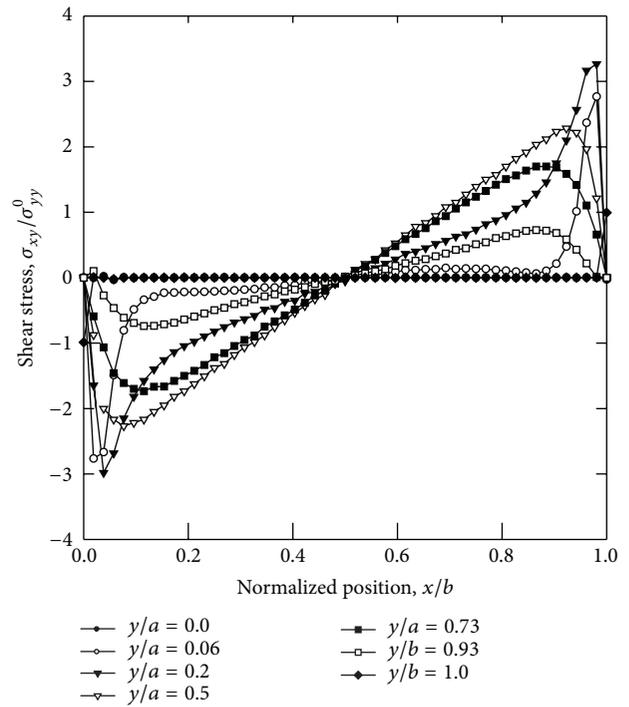


FIGURE 8: Distribution of the normalized shear stress component, $\sigma_{xy}/\sigma_{yy}^0$, as a function of x/b at different sections of the beam.

physical boundary conditions of the problem. The shear stress distribution as a function of x/a is antisymmetric as shown in Figure 8.

Figure 9 shows the distribution of normalized displacement component, u_x , at the section $x/b = 1.0$ of different composites. The axial displacement component, u_x , of glass/epoxy is the highest among those of boron/epoxy and graphite/epoxy. From Figure 10, it is observed that the displacement component, u_y , at the section $x/b = 1.0$ is the highest for glass/epoxy and the lowest for boron epoxy, and that of graphite/epoxy remains in between them. Figures 11 and 12 describe the effect of different composites on the normalized stress components, σ_{xx} and σ_{yy} , at the section $x/b = 0.5$. Effect of different composites is slightly observed on the above stress distributions.

5.2. Comparison of the Present Solution with Available Results.

Three-dimensional beam is considered two-dimensional problem considering plane stress condition. The number of elements used to mesh the geometry of the simply supported beam as observed in Figure 1 is 45×15 . Figure 13 illustrates the distribution of normalized displacement component, u_x , at the section $y/a = 0$ by the FDM and FEM methods. With increase of x/b ratio, the variation between the FDM and FEM solutions gradually increases. Figure 14 illustrates the distribution of normalized lateral displacement component, u_y , at the section $y/a = 1.0$ by the FDM and FEM methods. At the region, $x/b = 0.5$, the variation of the displacement component obtained by both of the methods is the highest and, away from the region $x/b = 0.5$, this variation decreases.

From Figure 15, it is observed that the stress component, σ_{xx} , at the section $y/a = 0.5$ obtained by the FDM is significantly higher than that of the FEM. But the stress component in y direction, at the section $y/a = 0.5$ obtained by the FD and FE methods show negligible variation as observed in Figure 16. Figure 17 illustrates the shear stress component at the section, $y/a = 0.5$ by the FD and FE methods. Here the shear stress obtained by the FD method is significantly higher than that of FEM method. The present FD method results come directly from the theoretical expressions of elasticity but the FE solutions somewhat depend on the accurateness of the assumption of the shape functions. The disadvantage of FDM is that sometimes it is difficult to handle complex boundary conditions of loaded structures.

6. Conclusions

Applying the finite difference technique on the displacement potential function formulation, a simply supported orthotropic composite beam is solved to obtain the elastic field. Simply supported composite beams are widely used in many structures. The solutions of a simply supported orthotropic composite beam using FDM technique identify the critical sections of the beam under uniformly distributed loading. Effect of different composites on the solutions is also analyzed by the present finite difference technique. The displacement and stress components at a particular section of a simply supported beam are analyzed by using FDM and FEM methods in a comparative fashion. From the comparison between the present and finite element solutions, it is

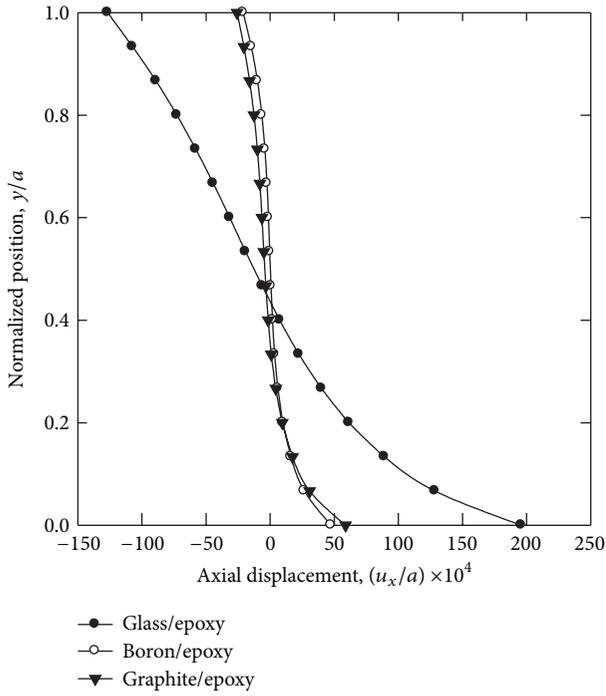


FIGURE 9: Distribution of the normalized displacement component, u_x , as a function of y/a at the section $x/b = 1.0$ of the beam under different composites.

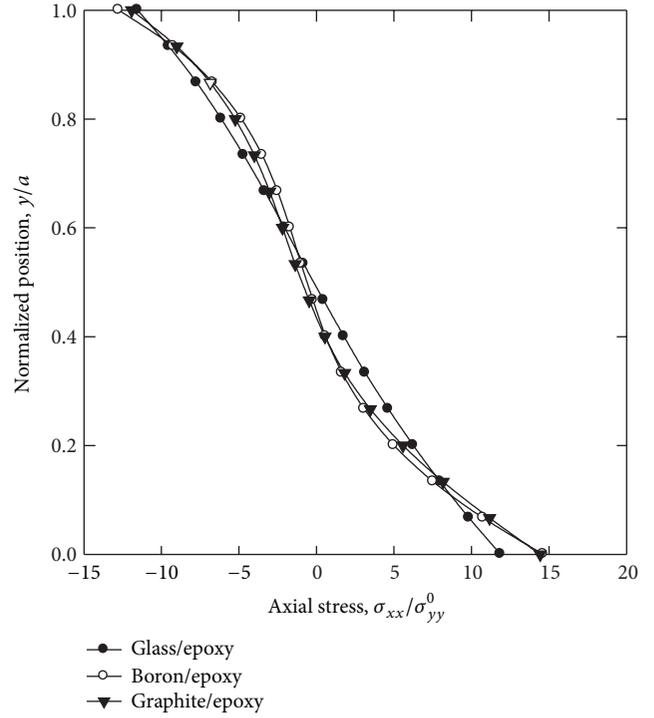


FIGURE 11: Distribution of the normalized axial stress component, $\sigma_{xx}/\sigma_{yy}^0$, as a function of y/a at the section $x/b = 0.5$ of the beam under different composites.

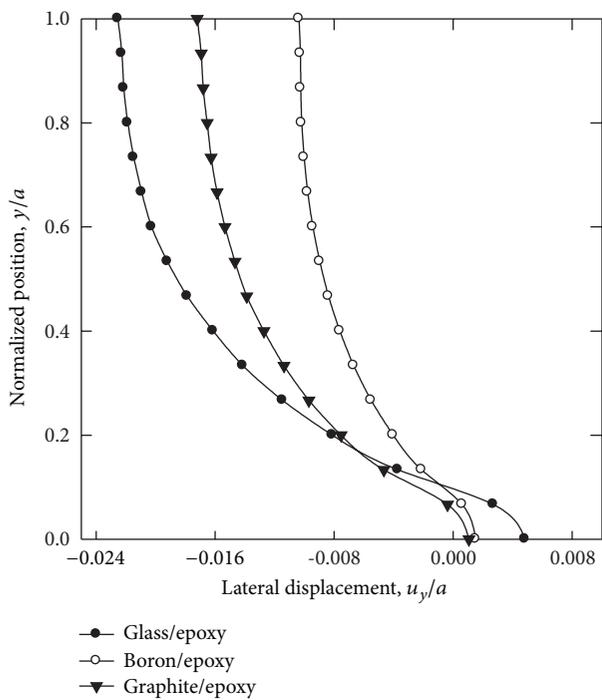


FIGURE 10: Distribution of normalized displacement component, u_y , as a function of y/a at the section $x/b = 1.0$ of the beam under different composites.

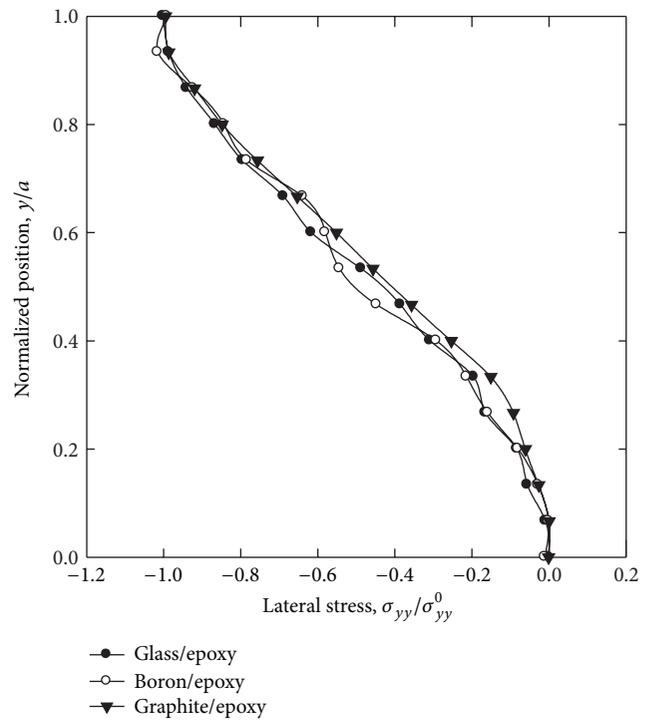


FIGURE 12: Distribution of the normalized lateral stress component, $\sigma_{yy}/\sigma_{yy}^0$, as a function of y/a at the section $x/b = 0.5$ of the beam under different composites.

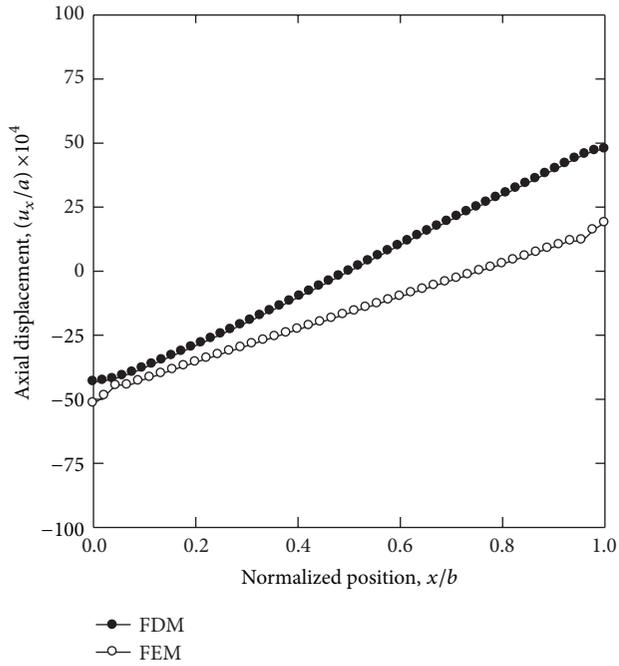


FIGURE 13: Distribution of the normalized axial displacement component, u_x , as a function of x/b at the section $y/a = 0$ of the beam under different methods.

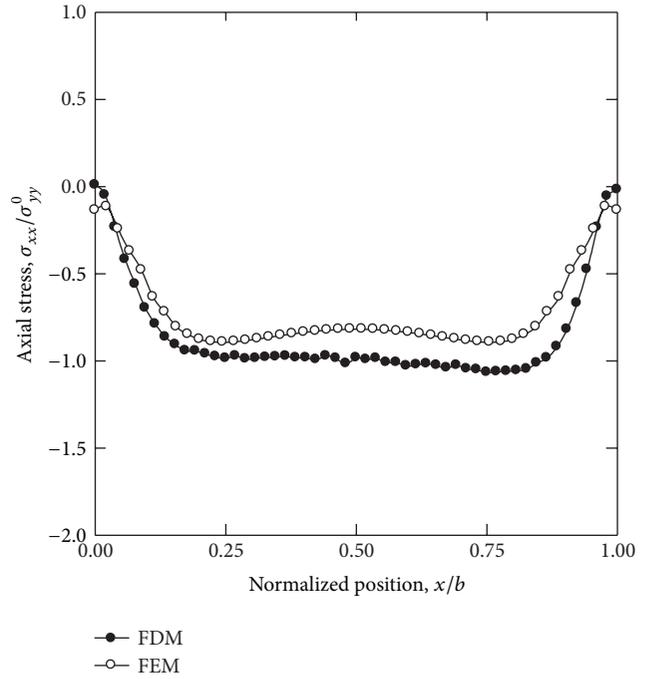


FIGURE 15: Distribution of the normalized axial stress component, σ_{xx} , as a function of x/b at the section $y/a = 0.5$ of the beam under different methods.

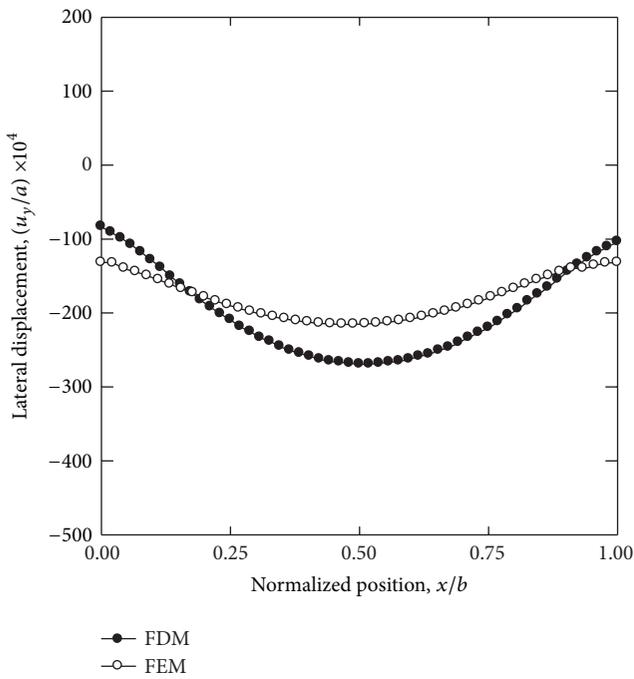


FIGURE 14: Distribution of the normalized lateral displacement component, u_y , as a function of x/b at the section $y/a = 1.0$ of the beam under different methods.

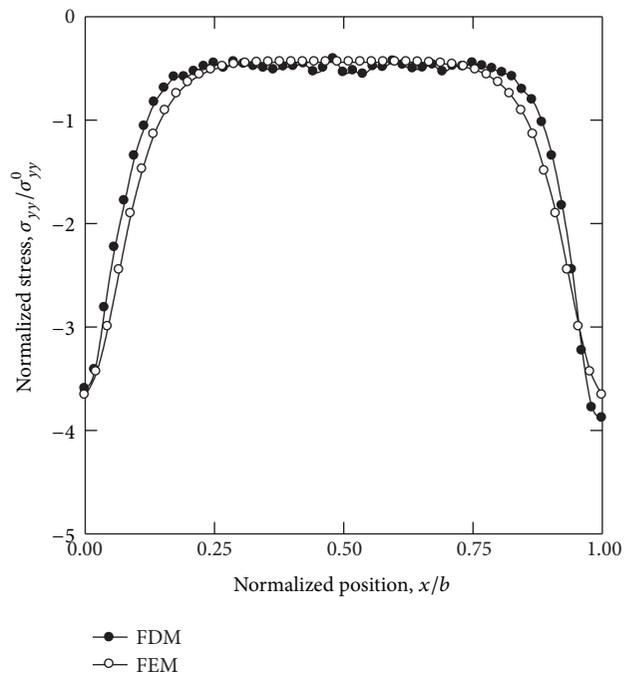


FIGURE 16: Distribution of the normalized lateral stress component, $\sigma_{yy}/\sigma_{yy}^0$, as a function of x/b at the section $y/a = 0.5$ of the beam under different methods.

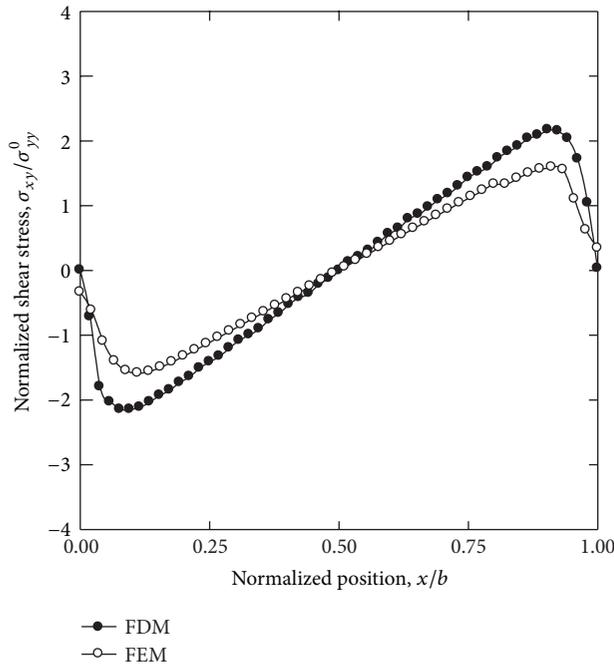


FIGURE 17: Distribution of the normalized shear stress component, $\sigma_{xy}/\sigma_{yy}^0$, as a function of x/b at the section $y/a = 0.5$ of the beam under different methods.

observed that there are some differences in some solutions of the present problem obtained by the FDM and FEM although the trend of the curves of the solutions obtained by both of the methods is similar. The present studies will be helpful for the designing of simply supported beams made of orthotropic composite materials.

Nomenclature

ψ	: Displacement potential function
E_1	: Longitudinal elastic modulus
E_2	: Transverse elastic modulus
G_{12}	: Shear modulus
ν_{12}	: Major Poisson's ratio
ν_{21}	: Minor Poisson's ratio
u_x	: Displacement component in x direction
u_y	: Displacement component in y direction
σ_{xx}	: Normal stress in x direction
σ_{yy}	: Normal stress in y direction
σ_{xy}	: Shear stress
u_n	: Normal displacement component
u_t	: Tangential displacement component
σ_n	: Normal stress component
σ_t	: Tangential displacement component
σ_{yy}^0	: Applying uniformly distributed stress
a	: Width of the beam
b	: Length of the beam

FDM: Finite difference method

FEM: Finite element method.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

- [1] S. K. Deb Nath, *A study of wear of tire treads [M.S. thesis]*, Department of Mechanical Engineering, Bangladesh University of Engineering and Technology, Dhaka, Bangladesh, 2002.
- [2] S. R. Ahmed, S. K. Deb Nath, and M. W. Uddin, "Optimum shapes of tire-treads for avoiding lateral slippage between tires and roads," *International Journal for Numerical Methods in Engineering*, vol. 64, no. 6, pp. 729–750, 2005.
- [3] S. K. Deb Nath, *Displacement potential approach to solution of elasticity problems of orthotropic composite structures [Ph.D. dissertation]*, Department of Mechanical Engineering, Bangladesh University of Engineering and Technology, Dhaka, Bangladesh, 2007.
- [4] A. M. Afsar, S. K. Deb Nath, S. R. Ahmed, and J. L. Song, "Displacement potential based finite difference solution to elastic field in a cantilever beam of orthotropic composite," *Mechanics of Advanced Materials and Structures*, vol. 15, no. 5, pp. 386–399, 2008.
- [5] S. K. Deb Nath and S. R. Ahmed, "Investigation of elastic field of a short orthotropic composite column by using finite-difference technique," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 222, no. 8, pp. 1161–1169, 2008.
- [6] S. K. Deb Nath and S. Reaz Ahmed, "A displacement potential-based numerical solution for orthotropic composite panels under end moment and shear loading," *Journal of Mechanics of Materials and Structures*, vol. 4, no. 6, pp. 987–1004, 2009.
- [7] H. D. Conway, L. Chow, and G. W. Morgan, "Analysis of deep beams," *Journal of Applied Mechanics*, vol. 18, pp. 163–172, 1951.
- [8] L. Chow, H. D. Conway, and G. Winter, "Stresses in deep beams," *Transactions of ASCE*, vol. 118, no. 2557, pp. 686–702, 1953.
- [9] C. O. Horgan and J. K. Knowles, "Recent developments concerning Saint Venant's principal," *Advances in Applied Mechanics*, vol. 23, pp. 179–269, 1983.
- [10] D. F. Parker, "The role of Saint-Venant's solutions in rod and beam theories," *Journal of Applied Mechanics*, vol. 46, no. 4, pp. 861–866, 1979.
- [11] S.-I. Suzuki, "Stress analysis of short beams," *AIAA Journal*, vol. 24, pp. 1396–1398, 1986.
- [12] S. J. Hardy and M. K. Pipelzadeh, "Static analysis of short beams," *The Journal of Strain Analysis for Engineering Design*, vol. 26, pp. 15–29, 1991.
- [13] A. V. K. Murty, "Towards a consistent beam theory," *AIAA Journal*, vol. 22, no. 6, pp. 811–816, 1984.
- [14] A. J. Durelli and B. Ranganayakamma, "Parametric solution of stresses in beams," *Journal of Engineering Mechanics*, vol. 115, no. 2, pp. 401–414, 1989.
- [15] S. Timoshenko and V. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, NY, USA, 3rd edition, 1979.
- [16] M. W. Uddin, *Finite difference solution of two-dimensional elastic problems with mixed boundary conditions [M.S. thesis]*, Carleton University, Ottawa, Canada, 1966.
- [17] R. Chapel and H. W. Smith, "Finite difference solution for plane stresses," *AIAA Journal*, vol. 6, pp. 1156–1157, 1968.

- [18] H. D. Conway and N. Y. Ithaca, "Some problems of orthotropic plane stress," *Journal of Applied Mechanics, Transactions ASME*, vol. 52, pp. 72–76, 1953.
- [19] S. K. Deb Nath and C. H. Wong, "Finite-difference solution of a both-end-fixed orthotropic composite beam under uniformly distributed loading using displacement potential function formulation," *Journal of Engineering Mechanics*, vol. 137, no. 4, pp. 258–267, 2011.
- [20] S. K. Deb Nath, M. A. S. Akanda, S. R. Ahmed, and M. W. Uddin, "Numerical investigation of bond-line stresses of tire tread section," *International Journal of Applied Mechanics and Engineering*, vol. 13, no. 1, pp. 43–61, 2008.
- [21] S. K. Deb Nath, S. R. Ahmed, S.-G. Kim, and C. H. Wong, "Effect of tire material on the prediction of optimum tire-tread sections," *International Journal of Computational Methods in Engineering Science & Mechanics*, vol. 12, no. 6, pp. 290–302, 2011.
- [22] S. Reaz Ahmed and S. K. Deb Nath, "A simplified analysis of the tire-tread contact problem using displacement potential based finite-difference technique," *Computer Modeling in Engineering & Sciences*, vol. 44, no. 1, pp. 35–64, 2009.
- [23] S. K. Deb Nath and M. A. Salam Akanda, "A finite difference solution of a steel plate with an internal hole using the displacement potential approach," *Journal of Strain Analysis for Engineering Design*, vol. 44, no. 8, pp. 713–728, 2009.
- [24] S. K. Deb Nath, S. R. Ahmed, and A. M. Afsar, "Displacement potential solution of short stiffened flat composite bars under axial loading," *International Journal of Applied Mechanics and Engineering*, vol. 11, no. 3, pp. 557–575, 2006.
- [25] S. K. Deb Nath, A. M. Afsar, and S. R. Ahmed, "Displacement potential solution of a deep stiffened cantilever beam of orthotropic composite material," *Journal of Strain Analysis for Engineering Design*, vol. 42, no. 7, pp. 529–540, 2007.
- [26] S. K. Deb Nath, A. M. Afsar, and S. R. Ahmed, "Displacement potential approach to the solution of stiffened orthotropic composite panels under uniaxial tensile load," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 221, no. 5, pp. 869–881, 2007.
- [27] S. K. Deb Nath and S. R. Ahmed, "Analytical solution of short guided orthotropic composite columns under eccentric loading using displacement potential formulation," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 222, no. 4, pp. 425–434, 2008.
- [28] S. K. Deb Nath, "Displacement Potential approach to solution of stiffened composite cantilever beams under combined loading," *International Journal of Applied Mechanics and Engineering*, vol. 13, no. 1, pp. 21–41, 2008.
- [29] S. K. D. Deb Nath and A. M. Afsar, "Analysis of the effect of fiber orientation on the elastic field in a stiffened orthotropic panel under uniform tension using displacement potential approach," *Mechanics of Advanced Materials and Structures*, vol. 16, no. 4, pp. 300–307, 2009.
- [30] S. K. Deb Nath and S. Reaz Ahmed, "Displacement potential solution of stiffened composite struts subjected to eccentric loading," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1761–1775, 2009.
- [31] S. K. Deb Nath and S. Reaz Ahmed, "Elastic analysis of short orthotropic composite columns subjected to uniform load over a part of the tip," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 223, no. 2, pp. 95–105, 2009.
- [32] S. K. Deb Nath, S. R. Ahmed, and S.-G. Kim, "Analytical solution of a stiffened orthotropic plate using alternative displacement potential approach," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 224, no. 1, pp. 89–99, 2010.
- [33] S. K. Deb Nath, S. Reaz Ahmed, and S.-G. Kim, "On the displacement potential solution of plane problems of structural mechanics with mixed boundary conditions," *Archive of Applied Mechanics*, vol. 80, no. 10, pp. 1125–1147, 2010.
- [34] M. Mohiuddin, M. W. Uddin, S. K. Deb Nath, and S. R. Ahmed, "An alternative numerical solution to a screw-thread problem using displacement potential approach," *International Journal For Computational Methods in Engineering Science and Mechanics*, vol. 13, no. 4, pp. 254–271, 2012.
- [35] S. K. Deb Nath, "Effects of fiber orientation and material isotropy on the analytical elastic solution of a stiffened orthotropic panel subjected to a combined loading," *Advances in Materials Science and Engineering*, vol. 2013, Article ID 710143, 13 pages, 2013.
- [36] S. K. Deb Nath, C. H. Wong, and S.-G. Kim, "A finite-difference solution of boron/epoxy composite plate with an internal hole subjected to uniform tension/displacements using displacement potential approach," *International Journal of Mechanical Sciences*, vol. 58, no. 1, pp. 1–12, 2012.
- [37] S. K. Deb Nath, S.-G. Kim, and C. H. Wong, "A numerical solution of a one end fixed glass/epoxy plate having a circular cutout subjected to a uniform shear using displacement potential approach," *Mechanics of Advanced Materials and Structures*, vol. 20, no. 4, pp. 297–308, 2013.
- [38] S. K. Deb Nath, "Displacement potential approach for the solution of 3-D body of orthotropic composite materials," *International Journal of Applied Mathematics and Statistics*, vol. 14, no. 09, pp. 67–77, 2009.
- [39] R. M. Jones, *Mechanics of Composite Materials*, McGraw-Hill, New York, NY, USA, 1975.



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