Research Article

Hahn Sequence Space of Modals

T. Balasubramanian 1 and S. Zion Chella Ruth 2

1 Department of Mathematics, Kamaraj College, Tuticorin, Tamilnadu 628003, India
2 Department of Mathematics, Dr. G. U. Pope College of Engineering, Sawyerparam, Tuticorin, Tamilnadu 628251, India

Correspondence should be addressed to S. Zion Chella Ruth; routhalwin@gmail.com

Received 6 June 2014; Revised 14 October 2014; Accepted 15 October 2014; Published 9 November 2014

The history of modal intervals goes back to the very first publications on the topic of interval calculus. The modal interval analysis is used in Computer graphics and Computer Aided Design (CAD), namely, the computation of narrow bounds on Bezier and B-Spline curves. Since modal intervals are used in many fields, we introduce a new sequence space \( h(gl) \) called the Hahn sequence space of modal intervals. We have given some new definitions and theorems. Some inclusion relation and some topological properties of this space are investigated. Also dual spaces of this space are computed.

1. Introduction


Let us denote the set of all real valued closed interval by \( I \), the set of positive integers by \( N \), and the set of all real numbers by \( R \). Any element of \( I \) is called interval valued number and it is denoted by \( \tilde{x} \). That is \( \tilde{x} = \{ x \in R : \underline{x} \leq x \leq \bar{x} \} \). An interval number \( \tilde{x} \) is a closed subset of real numbers. Let \( x \) and \( \overline{x} \) be, respectively, first and last points of the interval number \( \tilde{x} \). Therefore, when \( x > \bar{x} \), \( \tilde{x} \) is not an interval number. But in modal analysis \( [\tilde{x}, \tilde{x}] \) is a valid interval. A modal \( \overline{x} = [\underline{x}, \overline{x}] : \tilde{x}, \tilde{x} \in R \) is defined by a pair of real numbers \( \tilde{x}, \tilde{x} \). Let us denote the set of all modals by \( gl \). Let us suppose that \( \tilde{x}, \tilde{y} \in gl \) then the set \( gl \) is a vector space with the metric \( d \) defined by

\[
d(\tilde{x}_1, \tilde{x}_2) = \max \left\{ \left| \tilde{x}_1 - \tilde{x}_2 \right|, \left| \tilde{x}_1 - \tilde{x}_2 \right| \right\}. \tag{1}
\]

If \( \tilde{x}, \tilde{y} \in gl \) and \( \tilde{x} \leq \tilde{x}, \tilde{y} \leq \tilde{y} \), then the set \( gl \) is a reduced ordinary set of interval numbers which is complete metric space with the metric \( d \) defined in (1) [4]. If we take \( \tilde{x}_1 = [a, a] \) and \( \tilde{x}_2 = [b, b] \), we obtain the usual metric of \( R \) with \( d(\tilde{x}_1, \tilde{x}_2) = |a - b| \), where \( a, b \in R \).

2. Definitions and Preliminaries

Let \( f \) be a function from \( N \) to \( gl \) which is defined by \( k \to f(k) = \tilde{x}, \tilde{x} = (\tilde{x}_k) \). Then \( (\tilde{x}_k) \) is called sequence of modals. We will denote the set of all sequences of modals by \( w(gl) \).

For two sequences of modals \((\tilde{x}_k)\) and \((\tilde{y}_k)\), the addition, scalar product, and multiplication are defined as follows: \( (\tilde{x}_k + \tilde{y}_k) = [\tilde{x}_k + \tilde{y}_k, \tilde{x}_k + \tilde{y}_k], (\alpha \tilde{x}_k) = [\alpha \tilde{x}_k, \alpha \tilde{x}_k], \alpha \in R, (\tilde{x}_k \tilde{y}_k) = [\min(\tilde{x}_k \tilde{y}_k, \tilde{x}_k \tilde{y}_k)], \max(\tilde{x}_k \tilde{y}_k, \tilde{x}_k \tilde{y}_k), \tilde{x}_k \tilde{y}_k, \tilde{y}_k \tilde{x}_k, \tilde{y}_k \tilde{x}_k] \), respectively.

The set \( w(gl) \) is a vector space since the vector space rules are clearly provided. The zero element of \( w(gl) \) is
the sequence \( \tilde{\theta} = (\tilde{\theta}_k) = ([0, 0]) \), all terms of which are zero interval. If \((\tilde{x}_k) \in w(gl)\) then inverse of \((\tilde{x}_k)\), according to addition, is dual\((\tilde{x}_k)\).

Let \(\lambda(gl) \subset w(gl)\). If a sequence space contains a sequence \((\tilde{c}_k)\) of modals with the property that for every \(\tilde{u} \in \lambda(gl)\) there is a unique sequence of scalars \((\tilde{t}_k)\) such that \(\lim_k d(\tilde{u}, \tilde{t}_1, \tilde{t}_2, \ldots + \tilde{t}_k, \tilde{c}_k) \to 0\), then \((\tilde{c}_k)\) is called a Schauder modal basis for \(\lambda(gl)\). The series \(\sum_{k=1}^\infty \tilde{t}_k \tilde{c}_k\) which has the sum \(\tilde{u}\) is then called the expansion of \(\tilde{u}\) with respect to \((\tilde{c}_k)\), and we write \(\tilde{u} = \sum_{k=1}^\infty \tilde{t}_k \tilde{c}_k\).

Let \(\lambda(gl)\) and \(\mu(gl)\) be linear space of modals. Then a function \(\tilde{A} : \lambda(gl) \to \mu(gl)\) is called a linear transformation if and only if, for all \(\tilde{u}_1, \tilde{u}_2 \in \lambda(gl)\) and all \(\tilde{t}_1, \tilde{t}_2 \in gl\), \(\tilde{A}(\tilde{t}_1 \tilde{u}_1 + \tilde{t}_2 \tilde{u}_2) = \tilde{t}_1 \tilde{A}\tilde{u}_1 + \tilde{t}_2 \tilde{A}\tilde{u}_2\).

**Proposition 1.** If \((\tilde{x}_k), (\tilde{y}_k),\) and \((\tilde{r}_k)\) are sequences of symmetric modal, then the following equality holds:

\[
(\tilde{x}_k)\left(\left(\tilde{y}_k\right) - (\tilde{r}_k)\right) = (\tilde{x}_k)\left(\tilde{y}_k\right) - (\tilde{x}_k)\left(\tilde{r}_k\right).
\]

**Definition 2.** A sequence \(\tilde{x} = (\tilde{x}_k) \in w(gl)\) of modals is said to be convergent to the modal \(\tilde{x}_0\), if for each \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that \(d(\tilde{x}_k, \tilde{x}_0) < \varepsilon\) for all \(k \geq n_0\) and we denote it by writing \(\lim_k \tilde{x}_k = \tilde{x}_0\). Thus, \(\lim_k \tilde{x}_k = \tilde{x}_0\Leftrightarrow \lim_{k \to \infty} \tilde{x}_k = \tilde{x}_0\) and \(\lim_{k \to \infty} \tilde{x}_k = \tilde{x}_0\).

**Definition 3.** A sequence of modals, \(\tilde{x} = (\tilde{x}_k) \in w(gl)\), is said to be modal fundamental sequence if for every \(\varepsilon > 0\) there exists \(k_0 \in N\) such that \(d(\tilde{x}_k, \tilde{x}_0) < \varepsilon\) whenever \(n, k > k_0\).

**Definition 4.** A sequence of modals \(w(gl)\) is said to be solid if \(\tilde{y} = (\tilde{y}_k) \in w(gl)\) whenever \(\|\tilde{y}_k\| \leq \|\tilde{x}_k\|\) for all \(k \in N\) and \(\tilde{x} = (\tilde{x}_k) \in w(gl)\).

**Definition 5.** A sequence of generalized intervals \(w(gl)\) is said to be monotonically convergent if \(w(gl)\) contains the canonical preimage of all its step spaces.

**Definition 6** (Weierstrass M-test). Let \(\tilde{T}_k : gl \to gl\) be given. If there exists an \(M_k \geq 0\) such that \(d(\tilde{T}_k(\tilde{x}), \tilde{0}) \leq M_k\) and the series \(\sum_{k=1}^\infty M_k\) converges, then the series \(\sum_{k=1}^\infty \tilde{T}_k(\tilde{x})\) is uniformly and absolutely convergent in \(gl\).

The following spaces are needed for work:

- \(c_0(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \lim_{k \to \infty} d(\tilde{u}_k, \tilde{0}) = 0 \right\}\)
- \(c(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \lim_{k \to \infty} d(\tilde{u}_k, \tilde{u}_0) = 0 \right\}\)
- \(l_\infty(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sup_k d(\tilde{u}_k, \tilde{0}) < \infty \right\}\)
- \(cs(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \lim_{n \to \infty} d\left(\sum_{k=n}^\infty \tilde{u}_k, \tilde{0}\right) = 0 \right\}\)

\[bs(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sup_n d\left(\sum_{k=1}^n \tilde{u}_k, \tilde{0}\right) < \infty \right\}\]

- \(l_1(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sum_{k=1}^\infty d(\tilde{u}_k, \tilde{0}) < \infty \right\}\)

- \(bv(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sum_{k=1}^\infty d(\tilde{u}_k - \tilde{u}_{k+1}, \tilde{0}) < \infty \right\}\)

- \(bv_0(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sum_{k=1}^\infty d(\tilde{u}_k - \tilde{u}_{k+1}, \tilde{0}) < \infty \right\}\)

- \(\sigma_{oo}(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sup_n d\left(\sum_{k=1}^n \tilde{u}_k, \tilde{0}\right) < \infty \right\}\)

- \(E(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \left(\tilde{u}_k, \tilde{0}\right) \in E(gl) \right\}\)

\[dE(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \left(\frac{1}{k}, \tilde{0}\right) \in E(gl) \right\}\]

is the integrated space of \(E(gl)\).

is the differentiated space of \(E(gl)\).

**3. Main Results**

Define a sequence \(\tilde{y} = (\tilde{y}_k)\) which will be frequently used as the \(T\)-transform of a sequence \(\tilde{x} = (\tilde{x}_k) \in w(gl)\). That is, \(\tilde{y}_k = (\tilde{T}\tilde{x}_k) = k(\tilde{x}_k - \tilde{x}_{k+1}), k \geq 1\).

We introduce the sequence spaces \(h(gl)\) and \(h_\infty(gl)\) as the set of all sequences such that \(T\)-transforms of them are in \(l(gl)\).

That is,

\[h(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sum_{k=1}^\infty d\left(\tilde{T}\tilde{u}_k, \tilde{0}\right) < \infty \right\}\]

\[h_\infty(gl) = \left\{ \tilde{u} = (\tilde{u}_k) \in w(gl) : \sup_k d\left(\tilde{T}\tilde{u}_k, \tilde{0}\right) < \infty \right\}\]

\[h(gl)\] is a normed space with the norm \(\|\tilde{u}\| = \sum_{k=1}^\infty d(\tilde{T}\tilde{u}_k, \tilde{0})\).

**Example 7.** Consider the sequence \(\tilde{u} = (\tilde{u}_k)\) defined by

\[\tilde{u} = (\tilde{u}_k) = \begin{cases} \left[1, 1\right], & 1 \leq k \leq n \\ \left[0, 0\right], & k > n. \end{cases}\]
Note that $\sum_{k=1}^{\infty} d((T\bar{u})_k, \vec{0}) = \sum_{k=1}^{\infty} d(k(\bar{u}_k - \bar{u}_{k+1}), \vec{0}) = 0$, which is convergent.

Also $\lim_{k \to \infty} d(\bar{u}_k, \vec{0}) = 0$.

Hence $\bar{u} = (\bar{u}_k) \in h(gI)$.

**Theorem 8.** $h(gI)$ and $h_{co}(gI)$ are complete metric spaces with the metrics $\bar{d}_h$ and $\bar{d}_{hc}$ defined by

$$\bar{d}_h (\bar{u}, \bar{v}) = \sum_{k=1}^{\infty} d\left( (T\bar{u})_k, (T\bar{v})_k \right),$$

$$\bar{d}_{hc} (\bar{u}, \bar{v}) = \sup_k d\left( (T\bar{u})_k, (T\bar{v})_k \right),$$

respectively, where $\bar{u} = (\bar{u}_k)$ and $\bar{v} = (\bar{v}_k)$ are the elements of the space $h(gI)$ or $h_{co}(gI)$.

**Proof.** Let $(\bar{u}^{(i)})$ be any fundamental sequence in the space $h(gI)$, where $(\bar{u}^{(i)}) = (\bar{u}_0^{(i)}, \bar{u}_1^{(i)}, \ldots)$. Then for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$\sum_{k=1}^{\infty} d\left( (T\bar{u})_k, (T\bar{v})_k \right) < \varepsilon,$$

for $i, j \geq n_0(\varepsilon)$. (9)

We obtain for each fixed $k \in \mathbb{N}$ from (8) that

$$d\left( (T\bar{u}^{(i)})_k, (T\bar{v}^{(i)})_k \right) < \varepsilon$$

which leads to the fact that $\{T\bar{u}^{(i)}_k\}$ is a fundamental sequence in $gI$ for every fixed $k \in \mathbb{N}$.

Since $gI$ is complete, $\{T\bar{u}^{(i)}_k\}$ converges to $(T\bar{u})_k$ as $i \to \infty$.

Consider the sequence $\{T\bar{u}_i, T\bar{v}_i, \ldots\}$; we have from (9) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\varepsilon)$ that

$$\sum_{k=1}^{m} d\left( (T\bar{u}^{(i)})_k, (T\bar{v}^{(j)})_k \right) \leq \bar{d}_h \left( \bar{u}^{(i)}, \bar{v}^{(j)} \right) < \varepsilon,$$

for $i, j \geq n_0(\varepsilon)$. (10)

Take any $i \geq n_0(\varepsilon)$ and take limit as $j \to \infty$ first and then let $m \to \infty$ in (8); we obtain

$$\bar{d}_h \left( \bar{u}^{(i)}, \bar{u} \right) < \varepsilon.$$ (11)

Therefore $\bar{u}^{(i)} \to \bar{u}$.

We have to prove $\bar{u} \in h(gI)$.

Since $\{\bar{u}^{(i)}\}$ is a fundamental sequence in $h(gI)$, we have

$$\sum_{k=1}^{\infty} d\left( (T\bar{u})_k, \vec{0} \right) = 0.$$ (12)

Now,

$$d\left( (T\bar{u})_k, \vec{0} \right) \leq d\left( (T\bar{u})_k, (T\bar{u}^{(i)})_k \right) + d\left( (T\bar{u}^{(i)})_k, (T\bar{u}^{(j)})_k \right) + d\left( (T\bar{u}^{(j)})_k, \vec{0} \right).$$ (13)

Hence

$$\sum_{k=1}^{\infty} d\left( (T\bar{u})_k, \vec{0} \right) \leq \sum_{k=1}^{\infty} d\left( (T\bar{u})_k, (T\bar{u}^{(i)})_k \right) + \sum_{k=1}^{\infty} d\left( (T\bar{u}^{(i)})_k, (T\bar{u}^{(j)})_k \right) + \sum_{k=1}^{\infty} d\left( (T\bar{u}^{(j)})_k, \vec{0} \right).$$ (14)

Also from (10) and (11), $\lim_{k \to \infty} d((T\bar{u})_k, \vec{0}) = 0$. Hence $\bar{u} \in h(gI)$. Since $\{\bar{u}^{(i)}\}$ is an arbitrary fundamental sequence, the space $h(gI)$ is complete.

Similarly, we can prove $h_{co}(gI)$ is complete.

**Theorem 9.** The space $h(gI)$ is monotone.

**Proof.** Let $n < m$; it follows from

$$d\left( \bar{u}_m, \vec{0} \right) \leq d\left( \bar{u}_m - \bar{u}_{m+1}, \vec{0} \right) + d\left( \bar{u}_{m+1} - \bar{u}_n, \vec{0} \right)$$

$$+ \cdots + d\left( \bar{u}_{m-1} - \bar{u}_m, \vec{0} \right) + d\left( \bar{u}_n, \vec{0} \right)$$

that

$$\sum_{i=1}^{m-n} d\left( \bar{u}_i, \vec{0} \right) \leq \sum_{k=1}^{m-n} k d\left( \bar{u}_k - \bar{u}_{k+1}, \vec{0} \right) + n d\left( \bar{u}_n - \bar{u}_{n+1}, \vec{0} \right) + \cdots + d\left( \bar{u}_{m-1} - \bar{u}_m, \vec{0} \right) + d\left( \bar{u}_n, \vec{0} \right).$$ (15)

The sequence $\{\|\bar{u}^{(n)}\|\}$ is monotone increasing; it thus follows from $\bar{u} = \lim_{n \to \infty} \bar{u}^{(n)}$ that $\|\bar{u}\| = \lim_{n \to \infty} \|\bar{u}^{(n)}\| = \sup_n \|\bar{u}^{(n)}\|$. This completes the proof.

**Theorem 10.** The space $h(gI)$ is not rotund.

**Proof.** Consider the sequences $\bar{u} = (\bar{u}_k)$ and $\bar{v} = (\bar{v}_k)$ defined by

$$\bar{u} = (\bar{u}_k) = \{[1, 1], \vec{0}, \vec{0}, \ldots\},$$

$$\bar{v} = (\bar{v}_k) = \{[1/2, 1/2], [1/2, 1/2], \vec{0}, \vec{0}, \ldots\}.$$ (17)

Then

$$\sum_{k=1}^{\infty} d\left( (T\bar{u})_k, \vec{0} \right) < \infty, \sum_{k=1}^{\infty} d\left( (T\bar{v})_k, \vec{0} \right) < \infty.$$ (18)

Also,

$$\lim_{k \to \infty} d\left( \bar{u}_k, \vec{0} \right) = 0, \lim_{k \to \infty} d\left( \bar{v}_k, \vec{0} \right) = 0.$$ (19)

Thus $\bar{u} = (\bar{u}_k)$ and $\bar{v} = (\bar{v}_k)$ are in $h(gI)$ and $\bar{d}_h(\bar{u}, \vec{0}) = \bar{d}_h(\bar{v}, \vec{0}) = 1$.

Note that $\bar{u} \neq \bar{v}$, but $\bar{d}_h((\bar{u} + \bar{v})/2, \vec{0}) = 1$. Therefore $h(gI)$ is not rotund.
Theorem 11. The space $h(gl)$ is not solid.

Proof. Consider the sequence 
\[
\tilde{u} = (\tilde{u}_k) = \begin{cases} [1,1], & 1 \leq k \leq n \\ [0,0], & k > n, \end{cases} \quad (20)
\]
\[
\tilde{v} = (\tilde{v}_k) = \left( \left[(-1)^k, (-1)^k \right] \right).
\]
Since
\[
d\left(\tilde{v}_k, \tilde{0}\right) = d\left(\tilde{u}_k, \tilde{0}\right) = 1, \quad d\left((T\tilde{u})_k, \tilde{0}\right) = 0, \quad (21)
\]
it immediately follows that $\tilde{u} = (\tilde{u}_k) \in h(gl)$.

However, it is trivial that $d((T\tilde{v})_k, \tilde{0}) = 2, d((T\tilde{v})_k, \tilde{0}) \notin d((T\tilde{u})_k, \tilde{0})$, which implies $\tilde{v} = (\tilde{v}_k) \notin h(gl)$. This completes the proof.

4. Dual Space of $h(gl)$

Definition 12. The $\alpha$-dual, $\beta$-dual, and $\gamma$-dual of $s(gl) \subset w(gl)$ are, respectively, defined by

\[
\begin{align*}
\{s(gl)\}^\alpha &= \{(\tilde{u}_k) \in w(gl) : (\tilde{u}_k\tilde{v}_k) \in l_1(gl) \forall (\tilde{v}_k) \in s(gl)\} \\
\{s(gl)\}^\beta &= \{(\tilde{u}_k) \in w(gl) : (\tilde{u}_k\tilde{v}_k) \in cs(gl) \forall (\tilde{v}_k) \in s(gl)\} \\
\{s(gl)\}^\gamma &= \{(\tilde{u}_k) \in w(gl) : (\tilde{u}_k\tilde{v}_k) \in bs(gl) \forall (\tilde{v}_k) \in s(gl)\}.
\end{align*}
\]
(22)

It is trivial that the following inclusions hold: \{s(gl)\}^\alpha \subset \{s(gl)\}^\beta \subset \{s(gl)\}^\gamma.

Theorem 13. Let $E(gl)$ and $E_i(gl)$ be the sets of sequences of modals. Then the following statements hold.

(i) $E(gl) \subset \{E(gl)\}^\beta$.
(ii) $\{E(gl)\}^\beta = \{E(gl)\}^\beta$.
(iii) If $E_i(gl) \supset E(gl)$ then $\{E_i(gl)\}^\beta \subset \{E(gl)\}^\beta$.

The same results hold for dual also.

Proof. Let $\tilde{u} = (\tilde{u}_k), \tilde{v} = (\tilde{v}_k)$, and $\tilde{w} = (\tilde{w}_k)$ be sequences of modal intervals.

(i) Suppose $\tilde{u} = (\tilde{u}_k) \notin \{E(gl)\}^\beta$; then $(\tilde{u}_k\tilde{v}_k) \notin cs(gl)$ for at least one $\tilde{v} = (\tilde{v}_k) \in \{E(gl)\}^\beta$. But $\tilde{v} = (\tilde{v}_k) \in \{E(gl)\}^\beta$ implies that $(\tilde{u}_k\tilde{v}_k) \in cs(gl)$ for all $\tilde{w} = (\tilde{w}_k) \in E(gl)$.

This means that $\tilde{u}$ is not a member of $E(gl)$. Hence $E(gl) \subset \{E(gl)\}^\beta$. The proof for (ii) follows similarly.

(ii) Suppose $\tilde{u} = (\tilde{u}_k) \in \{E_i(gl)\}^\beta$; then $(\tilde{u}_k\tilde{v}_k) \in cs(gl)$ for all $\tilde{v} = (\tilde{v}_k) \in E_i(gl)$. Since $E_i(gl) \supset E(gl)$, $\tilde{v} = (\tilde{v}_k) \in E(gl)$. Thus $\{E_i(gl)\}^\beta \subset \{E(gl)\}^\beta$.

Define the sequence $\tilde{y} = (\tilde{y}_k)$ which will be frequently used as the $B$-transform of a sequence $\tilde{x} = (\tilde{x}_k) \in w(gl)$.

That is,
\[
\tilde{y}_k = (B\tilde{x})_k = \frac{1}{k} \sum_{i=1}^{k} \tilde{x}_i, \quad (23)
\]

The Cesaro space of $l_\infty(gl)$ is the set of all sequences such that the $B$-transforms of them are in $l_\infty(gl)$. That is, $\sigma(l_\infty(gl)) = \{\tilde{x} = (\tilde{x}_k), sup_k d((B\tilde{x})_k, \tilde{0}) < \infty\}$.

Theorem 14. $\sigma(l_\infty(gl))$ is a complete metric space with the metric $d_{\sigma}(\tilde{u}, \tilde{v}) = sup_k d((B\tilde{u})_k, (B\tilde{v})_k)$ where $\tilde{u} = (\tilde{u}_k)$ and $\tilde{v} = (\tilde{v}_k)$ are the elements of space $\sigma(l_\infty(gl))$.

Proof. Let $\tilde{u}^{(i)}$ be any fundamental sequence in the space $\sigma(l_\infty(gl))$ where $\tilde{u}^{(i)} = (\tilde{u}_0^{(i)}, \tilde{u}_1^{(i)}, \ldots)$. Then for a given $\epsilon > 0$, there exists a positive integer $n_\epsilon(e)$ such that
\[
\tilde{d}_{\sigma}(\tilde{u}^{(i)}_k, \tilde{u}^{(j)}_k) = sup_k \left| (B\tilde{u}^{(i)}_k) - (B\tilde{u}^{(j)}_k) \right| < \epsilon, \quad (24)
\]
\[
\forall i, j \geq n_\epsilon(\epsilon). \quad (25)
\]

We obtain for each fixed $k \in N$ from (24) that
\[
d\left(\left( B\tilde{u}^{(i)}_k \right), \left( B\tilde{u}^{(j)}_k \right) \right) < \epsilon \quad (26)
\]

which leads to the fact that $(B\tilde{u}^{(i)}_k)$ is a fundamental sequence for every fixed $k \in N$. Since $gl$ is complete, $(B\tilde{u}^{(i)}_k) \rightarrow (B\tilde{u})_k$ as $i \rightarrow \infty$.

Consider the sequence $\{(B\tilde{u}), (B\tilde{u})_2, \ldots\}$. We have from (25) for each $m \in N$ and $i, j \geq n_\epsilon(\epsilon)$ that
\[
sup_k \left| (B\tilde{u}^{(i)}_k) - (B\tilde{u}^{(j)}_k) \right| \leq \tilde{d}_{\sigma}(\tilde{u}^{(i)}_k, \tilde{u}^{(j)}_k) < \epsilon, \quad (27)
\]
\[
\forall k = 1, 2, \ldots, m, \forall i, j \geq n_\epsilon(\epsilon). \quad (28)
\]

Finally, we proceed to prove $\tilde{u} \in \sigma(l_\infty(gl))$. Since $\{\tilde{u}^{(i)}\}$ is a fundamental sequence in $\sigma(l_\infty(gl))$, we have $sup_k d((B\tilde{u}^{(i)}_k), \tilde{0}) < \infty$.

Now,
\[
d\left( (B\tilde{u})_k, \tilde{0} \right) \leq d\left( (B\tilde{u})_k, (B\tilde{u}^{(i)}_k) \right) + d\left( (B\tilde{u}^{(i)}_k), (B\tilde{u}^{(i)}_k) \right) + d\left( (B\tilde{u}^{(i)}_k), \tilde{0} \right),
\]
\[
sup_k \left| (B\tilde{u}^{(i)}_k), \tilde{0} \right| \leq sup_k d\left( (B\tilde{u}^{(i)}_k), (B\tilde{u}^{(i)}_k) \right) + sup_k d\left( (B\tilde{u}^{(i)}_k), \tilde{0} \right) < \infty.
\]

Hence $\tilde{u} \in \sigma(l_\infty(gl))$. 

International Scholarly Research Notices
Since \( \{\overline{u}^{(b)}\} \) is an arbitrary fundamental sequence, the space \( \sigma(l_{\infty}(g I)) \) is complete. \( \square \)

**Theorem 15.** The \( \beta \)-dual and \( \gamma \)-dual of \( h(g I) \) are \( \sigma(l_{\infty}(g I)) \).

**Proof.** Let \( \overline{u} = (\overline{u}_k) \in h(g I) \) and \( (\overline{v}_k) \in \sigma(l_{\infty}(g I)) \). Since \( (\overline{u}_k) \in h(g I) \), we have \( \lim_{k \to \infty} d[\overline{u}_k, \overline{0}] = 0 \). Therefore for given \( \varepsilon > 0 \), there exists \( n_0 \) such that \( d[\overline{u}_k, \overline{0}] < \varepsilon \).

Since \( (\overline{v}_k) \in \sigma(l_{\infty}(g I)) \), \( \sup_k d[(B\overline{v})_k, \overline{0}] < \infty \). Thus \( d[\overline{v}_k, \overline{0}] < \infty \) for all \( k \) and \( n \).

Hence there exists an \( M > 0 \) such that \( d[\overline{v}_k, \overline{0}] < M \) for all \( k \) and \( n \).

Now,
\[
\sum_{k=1}^{\infty} \overline{u}_k \overline{v}_k \text{ converges uniformly by Weierstrass } M\text{-test.}
\]

Thus
\[
\sigma(l_{\infty}(g I)) \subset [h(g I)]^{\beta}.
\] (30)

Conversely, suppose \( (\overline{v}_k) \in [h(g I)]^{\beta} \). Then the series \( \sum_{k=1}^{\infty} \overline{u}_k \overline{v}_k \) converges for all \( (\overline{u}_k) \in h(g I) \). This also holds for the sequence of modals \( (\overline{u}_k) \) defined by \( (\overline{u}_k) = ([1-1], 1) \) for all \( k \in \mathbb{N} \).

\[
\sum_{k=1}^{\infty} \overline{u}_k \overline{v}_k = \sum_{k=1}^{\infty} [1-1] \overline{v}_k = \sum_{k=1}^{\infty} \max[\overline{v}_k, |\overline{v}_k|] \text{ converges uniformly.}
\]

Thus \( \sup_k d[(B\overline{v})_k, \overline{0}] < \infty \).

Hence
\[
(\overline{v}_k) \in \sigma(l_{\infty}(g I)), \quad [h(g I)]^{\beta} \subset \sigma(l_{\infty}(g I)).
\] (31)

From (30) and (31), \( [h(g I)]^{\beta} = \sigma(l_{\infty}(g I)) \). This completes the proof. \( \square \)

**Theorem 16.** Consider
(i) \( h(g I) \subset l_1(g I) \cap \int c_0(g I) \).
(ii) \( h(g I) = l_1(g I) \cap \int bv(g I) = l_1(g I) \cap \int bv_0(g I) \).

**Proof.** (i) Let \( \overline{u} = (\overline{u}_k) \in h(g I) \). Then \( \sum_{k=1}^{\infty} d((T\overline{u})_k, \overline{0}) < \infty \) and \( \lim_{k \to \infty} d[\overline{u}_k, \overline{0}] = 0 \). Consider \( \sum_{k=1}^{\infty} d[\overline{u}_k, \overline{0}] \leq \sum_{k=1}^{\infty} d((T\overline{u})_k, \overline{0}) < \infty \). Therefore, \( \overline{u} = (\overline{u}_k) \in l_1(g I) \).

And also since \( \lim_{k \to \infty} d[\overline{u}_k, \overline{0}] = \lim_{k \to \infty} k d[\overline{u}_k, \overline{0}] = 0 \), \( \overline{u} = (\overline{u}_k) \in \int c_0(g I) \).

Therefore, \( \overline{u} = (\overline{u}_k) \in l_1(g I) \cap \int c_0(g I) \). Hence \( h(g I) \subset l_1(g I) \cap \int c_0(g I) \).

(ii) For \( k = 1, 2, \ldots \),
\[
k(\overline{u}_k - \overline{u}_{k+1}) = \overline{u}_{k+1} + k\overline{u}_k - k\overline{u}_{k+1} - \overline{u}_{k+1} = \overline{u}_{k+1} + [k\overline{u}_k - (k + 1) \overline{u}_{k+1}]. \] (32)

\( \overline{u} = (\overline{u}_k) \in h(g I) \) implies
\[
\sum_{k=1}^{\infty} d((T\overline{u})_k, \overline{0}) = \sum_{k=1}^{\infty} d(k(\overline{u}_k - \overline{u}_{k+1}), \overline{0}) = \sum_{k=1}^{\infty} d(k\overline{u}_k - (k + 1)\overline{u}_{k+1}, \overline{0}) \geq \sum_{k=1}^{\infty} d[k\overline{u}_k - (k + 1)\overline{u}_{k+1}, \overline{0}] + \sum_{k=1}^{\infty} d(\overline{u}_{k+1}, \overline{0}).
\] (33)

The last series is convergent since \( h(g I) \subset l_1(g I) \). Hence also \( \sum_{k=1}^{\infty} d[k\overline{u}_k - (k + 1)\overline{u}_{k+1}, \overline{0}] < \infty \) and therefore \( h(g I) \subset \int bv(g I) \).

Hence
\[
h(g I) \subset l_1(g I) \cap \int bv(g I).
\] (34)

Conversely, (32) implies for \( (\overline{u}_k) \in l_1(g I) \cap \int bv(g I) \)
\[
\sum_{k=1}^{\infty} d((T\overline{u})_k, \overline{0}) \leq \sum_{k=1}^{\infty} d(k(\overline{u}_k - \overline{u}_{k+1}), \overline{0}) \leq \sum_{k=1}^{\infty} d[k\overline{u}_k - (k + 1)\overline{u}_{k+1}, \overline{0}] + \sum_{k=1}^{\infty} d(\overline{u}_{k+1}, \overline{0}) < \infty.
\] (35)

Thus \( (\overline{u}_k) \in h(g I) \).

Therefore,
\[
l_1(g I) \cap \int bv(g I) \subset h(g I).
\] (36)

Hence from (34) and (36), we have shown that \( h(g I) = l_1(g I) \cap \int bv(g I) \).

Similarly, we can prove other equalities. \( \square \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


