Research Article

On Some Numbers Related to Extremal Combinatorial Sum Problems

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Let n , d , and r be three integers such that $1 \leq r, d \leq n$. Chiaselotti (2002) defined $\gamma(n, d, r)$ as the minimum number of the nonnegative partial sums with d summands of a sum $\sum_{i=1}^n a_i \geq 0$, where $a_1, \dots, a_n$ are n real numbers arbitrarily chosen in such a way that r of them are nonnegative and the remaining $n - r$ are negative. Chiaselotti (2002) and Chiaselotti et al. (2008) determine the values of $\gamma(n, d, r)$ for particular infinite ranges of the integer parameters n , d , and r . In this paper we continue their approach on this problem and we prove the following results: (i) $\gamma(n, d, r) \leq \binom{r}{d} + \binom{r}{d-1}$ for all values of n , d , and r such that $((d-1)/d)(n-1) \leq r \leq ((d-1)/d)n$; (ii) $\gamma(d+2, d, d) = d+1$.

1. Introduction

Let n , d , and r be three fixed integers such that $1 \leq r, d \leq n$. We set $[n] := \{1, \dots, n\}$ and

$$W_n := \left\{ f : [n] \longrightarrow \mathbb{R} : \sum_{x \in [n]} f(x) \geq 0 \right\}. \quad (1)$$

The elements of W_n are called n -weight functions and, if $f \in W_n$, we set $f^+ = |\{a \in [n] : f(a) \geq 0\}|$. For example, if $f(1) = f(2) = f(3) = 1$, $f(4) = f(5) = -1/2$, and $f(6) = f(7) = -1$, then $f^+ = 3$. If $f \in W_n$, we also set $\Phi(f, d) := \{A \subseteq [n] : |A| = d, \sum_{a \in A} f(a) \geq 0\}$ (we call (d^+, n) -subset of f a generic element of $\Phi(f, d)$), $\phi(f, d) := |\Phi(f, d)|$, and

$$\gamma(n, d, r) := \min \{\phi(f, d) : f \in W_n, f^+ = r\}. \quad (2)$$

These numbers $\gamma(n, d, r)$ were introduced in [1] in order to refine the study of a conjecture of Manickam-Miklós-Singhi (for further information on this conjecture and on its links with the numbers $\gamma(n, d, r)$ see [1–4]). The complete determination of the numbers $\gamma(n, d, r)$ is a very difficult task and actually they are known only for a relatively small range of the integer parameters n , d , and r . In [1–3] some of the numbers $\gamma(n, d, r)$ have been determined, and we report these values:

$$\begin{aligned} & \binom{n-1}{d-1} \text{ if } r \leq d \leq n/2, \\ & \binom{n-r}{d-r} \text{ if } r \leq d < n \text{ and } r < (n/(n-d)), \\ & \binom{r}{d} \text{ if } d < r < n \text{ and } r > ((d-1)/d)n, \\ & \binom{n-1}{d-1} \text{ if } r = 1, \\ & \binom{r}{d} + \binom{r-1}{d-1} \text{ if } r \geq d \text{ and } r = ((d-1)/d)n, \\ & \binom{r}{d} + \binom{r-1}{d-1} \text{ if } n = 2d+2, r = 2d-1 \text{ and } d \geq 2. \end{aligned}$$

In particular, in [3] the authors prove the last of these results using Hall's matching theorem.

Also, in [2, 5] the numbers $\gamma(n, d, r)$ were linked within the context of the combinatorial order theory. More in detail, in [2] the authors introduce two new classes of lattices of signed integer partitions, $S(n, r)$ and $S(n, d, r)$, and they show that the numbers $\gamma(n, d, r)$ can be interpreted as the cardinality of particular types of up-sets in the previous lattices.

On the other hand, the lattices $S(n, r)$ and $S(n, d, r)$ can also be considered as particular types of discrete dynamical systems. In this context many properties of the numbers $\gamma(n, d, r)$ can be related to the evolution rules that characterize $S(n, r)$ and $S(n, d, r)$ as discrete dynamical systems (see [6, 7]). For very recent studies concerning the discrete dynamical systems see [8–12].

In this paper we determine some new identities and new bounds for the numbers $\gamma(n, d, r)$. In particular, we show that

- (i) $\gamma(n, d, r) \leq \binom{r}{d} + \binom{r}{d-1}$ for all values of n, d , and r satisfying $((d-1)/d)(n-1) \leq r \leq ((d-1)/d)n$ (Corollary 5),
- (ii) $\gamma(n, d, r) = \binom{r}{d} + \binom{r}{d-1} = d+1$ in the case $n = d+2$ and $r = d$ (Proposition 8).

Finally we provide a combinatorial interpretation of the inequality $((d-1)/d)(n-1) < r \leq ((d-1)/d)n$.

The remaining part of this paper is structured as follows. In Section 2 we provide the necessary notations for the sequel. In Section 3 we establish our results and, finally, in Section 4 we briefly describe conclusions and possible future research approaches.

2. Notations

In the sequel, we will assume that a generic weight function $f \in W_n$, with $f^+ = r$, has the form

$$\begin{array}{ccccccccc} 1 & \cdots & r & r+1 & \cdots & n \\ \downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow \\ x_1 & \cdots & x_r & y_1 & \cdots & y_{n-r}, \end{array} \quad (3)$$

with

$$x_1 \geq x_2 \geq \cdots \geq x_r \geq 0 > y_1 \geq y_2 \geq \cdots \geq y_{n-r}. \quad (4)$$

Let us call the indexes $1, \dots, r$ the *nonnegative elements* of f and the indexes $r+1, \dots, n$ the *negative elements* of f . The real numbers $x_1, \dots, x_r$ are said to be the *nonnegative values* of f and the numbers $y_1, \dots, y_{n-r}$ are said to be the *negative values* of f .

If $i_1, \dots, i_\alpha$ are nonnegative elements of f and $j_1, \dots, j_\beta$ are negative elements of f , with $i_1 < \cdots < i_\alpha$ and $j_1 < \cdots < j_\beta$, a subset A of $\{1, \dots, n\}$ is said to be of type

$$[i_1, \dots, i_\alpha]_a^+ [j_1, \dots, j_\beta]_b^- \quad (5)$$

if A is made of a elements chosen in $\{i_1, \dots, i_\alpha\}$ and b elements chosen in $\{j_1, \dots, j_\beta\}$.

Let X be a finite set of integers. If q is an integer less than or equal to $|X|$, we call q -string on X a sequence $a_1, \dots, a_q$, where $a_1, \dots, a_q$ are distinct elements of X such that $a_1 < \cdots < a_q$. In this paper, each subset Y of X with q elements will be identified with the q -string of its elements ordered in increasing way. When $i_1, \dots, i_k$ are nonnegative elements of f and $j_1, \dots, j_l$ are negative elements of f , with $i_1 < \cdots < i_k < j_1 < \cdots < j_l$, the $(k+l)$ -string $i_1 \cdots i_k j_1 \cdots j_l$ will be written in the form

$$i_1 \cdots i_k \mid (j_1 - r) \cdots (j_l - r) \quad (6)$$

(thus $j_1 - r, \dots, j_l - r \in \{1, \dots, n-r\}$).

For example, if $n = 10$ and $r = 7$, the 4-string 1269 will be written in the form $126 \mid 9$.

Using the string-terminology instead of the set-terminology, in the sequel, we will call a (d^+, n) -subset of f a (d^+, n) -string of f .

3. The Results

The range of values of the integer parameters n, d , and r that we are going to study is the following: $d \leq r \leq n$. As reported in the first section, we already know that the next result holds for $\gamma(n, d, r)$ in the case $r > ((d-1)/d)n$. We state it referring to its proof.

Proposition 1. *If $r \geq d$ and $r > ((d-1)/d)n$, then $\gamma(n, d, r) = \binom{r}{d}$.*

Proof. See [1]. □

Therefore we concentrate our attention on the case $r \leq ((d-1)/d)n$. In order to examine it, we start by considering the partition $\mathcal{P}$ of the real interval $(0, ((d-1)/d)n]$:

$$\mathcal{P} = \left\{ 0, \frac{d-1}{d}, 2\frac{d-1}{d}, \dots, (n-1)\frac{d-1}{d}, n\frac{d-1}{d} \right\}. \quad (7)$$

The following proposition establishes when an interval determined by $\mathcal{P}$ contains an integer.

Proposition 2. *If $k = 1, \dots, n$ and if $n - k \not\equiv_d 0$, there exists a unique integer r such that*

$$\frac{d-1}{d}(n-k) < r \leq \frac{d-1}{d}(n-k+1), \quad (8)$$

and r coincides with $\lfloor ((d-1)/d)(n-k+1) \rfloor$. Furthermore if $n - k \equiv_d 0$, no integer r satisfies (8).

Proof. Let $k \in \{1, \dots, n\}$ and set $m = n - k + 1$. Since the interval $((d-1)/d)(m-1), ((d-1)/d)m]$ has length $((d-1)/d) < 1$, there is at most one integer r that satisfies (8). Let us now write m in the form

$$m = \tilde{q}d + s, \quad (9)$$

where $\tilde{q}, s$ are integers such that $\tilde{q} \geq 0$, $1 \leq s \leq d$. Let us suppose now that $n - k \not\equiv_d 0$; that is, $m \not\equiv_d 1$; then we have $2 \leq s \leq d$.

Let $r = \lfloor ((d-1)/d)m \rfloor$. We show that r satisfies (8).

Firstly, the second inequality is straightforward; secondly, for the first inequality we observe

$$\begin{aligned} \frac{d-1}{d}(m-1) &= \frac{d-1}{d}(\tilde{q}d + s - 1) \\ &= \tilde{q}(d-1) + (s-1)\frac{d-1}{d}. \end{aligned} \quad (10)$$

Furthermore

$$\begin{aligned} r &= \left\lfloor \frac{d-1}{d}(\tilde{q}d + s) \right\rfloor = \left\lfloor \tilde{q}(d-1) + s - \frac{s}{d} \right\rfloor \\ &= \tilde{q}(d-1) + (s-1). \end{aligned} \quad (11)$$

Therefore $r > ((d-1)/d)(m-1)$, since $s \geq 2$.

If $n - k \equiv_d 0$, that is, $m \equiv_d 1$, in (9), we have $s = 1$ and (8) becomes

$$\tilde{q}(d-1) < r \leq \tilde{q}(d-1) + \frac{d-1}{d}. \quad (12)$$

Note that (12) has no integer solutions. □

Lemma 3. Let r be a positive integer such that

$$d \leq r \leq \frac{d-1}{d}n. \quad (13)$$

Then there exists a unique positive integer $b(r) \in \{1, \dots, n-r-1\}$ that satisfies

$$\frac{d-1}{d}(n-b(r)) < r \leq \frac{d-1}{d}(n-b(r)+1). \quad (14)$$

Proof. By construction of partition $\mathcal{P}$, as in (7), there exists a unique $b(r) \in \{1, \dots, n\}$ such that (14) holds.

We now show that $b(r)$ cannot exceed $n-r-1$.

Firstly, we suppose that $b(r) > n-r$. Then, we write $b(r)$ in the form $b(r) = n-r+\zeta$, with ζ integer such that $1 \leq \zeta \leq r$. Since r satisfies (14), we have

$$\frac{d-1}{d}(r-\zeta) < r \leq \frac{d-1}{d}(r-\zeta+1); \quad (15)$$

that is,

$$\zeta(1-d) < r \leq (d-1)(1-\zeta). \quad (16)$$

Since $\zeta(1-d) < 0$ and $(d-1)(1-\zeta) \leq 0$, there is no positive integer r that satisfies (16).

Secondly, if $b(r) = n-r$, (14) becomes

$$\frac{d-1}{d}r < r \leq \frac{d-1}{d}(r+1); \quad (17)$$

that is,

$$0 < r \leq d-1, \quad (18)$$

contradicting the hypothesis (13). $\square$

Using the previous lemma, we find a useful and appreciable upper bound for $\gamma(n, d, r)$. This is one of our important steps to establish, under suitable hypotheses for n , d , and r , an exact value for $\gamma(n, d, r)$.

Proposition 4. Let r be a positive integer that satisfies

$$d \leq r \leq \frac{d-1}{d}n; \quad (19)$$

then

$$\gamma(n, d, r) \leq \sum_{j=0}^{\min\{b(r), d-1\}} \binom{b(r)}{j} \binom{r}{d-j}. \quad (20)$$

Proof. Since $1 \leq b(r) \leq n-r-1$, we construct a weight function $f \in W_n$, with $f^+ = r$, such that

$$\phi(f, d) = \sum_{j=0}^{\min\{b(r), d-1\}} \binom{b(r)}{j} \binom{r}{d-j}. \quad (21)$$

This is sufficient to prove the (20).

Let $h = \min\{b(r), d-1\}$. Let α be a positive real number. In order to simplify the notation, we call β the number $((r + b(r)(-\alpha))/(n-r-b(r)))$, in such a way that

$$r + b(r)(-\alpha) + (n-r-b(r))(-\beta) = 0 \quad (22)$$

holds.

At this point we define the function

$$\begin{array}{ccccccccccc} 1 & \cdots & r & r+1 & \cdots & r+b(r) & r+(b(r)+1) & \cdots & r+(n-r) & & \\ f_\alpha : & \downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow & & \downarrow & \cdots & \downarrow \\ & 1 & \cdots & 1 & -\alpha & \cdots & -\alpha & & -\beta & \cdots & -\beta. \end{array} \quad (23)$$

We now show that, for α sufficiently small, that is,

$$0 < \alpha < \min \left\{ \frac{r}{b(r)}, \frac{d}{h} - 1, \frac{d}{b(r)} \left(r - \frac{d-1}{d}(n-b(r)) \right) \right\}, \quad (24)$$

f_α is a weight function that satisfies (21).

In fact,

(a) the denominator of β , due to Lemma 3, is a positive number. Furthermore the numerator of β is a positive number if and only if $\alpha < (r/b(r))$. Therefore (24) and the definition of β assure that f_α is a weight function.

(b) Having $\alpha < d/h - 1$ is equivalent to require

$$\underbrace{1 + \cdots + 1}_{d-h \text{ times}} + \underbrace{(-\alpha) + \cdots + (-\alpha)}_{h \text{ times}} > 0. \quad (25)$$

This condition assures that the subsets of the type

$$\begin{array}{lcl}
[1, \dots, r]_d^+ & [r+1, \dots, r+b(r)]_0^-, & \text{in total } \binom{b(r)}{0} \binom{r}{d} \\
[1, \dots, r]_{d-1}^+ & [r+1, \dots, r+b(r)]_1^-, & \text{in total } \binom{b(r)}{1} \binom{r}{d-1} \\
\vdots & \vdots & \vdots \\
[1, \dots, r]_{d-h}^+ & [r+1, \dots, r+b(r)]_h^-, & \text{in total } \binom{b(r)}{h} \binom{r}{d-h}
\end{array} \quad (26)$$

are (d^+, n) -subsets of f_α .

(c) Firstly we note that the requirement

$$\alpha < \frac{d}{b(r)} \left(r - \frac{d-1}{d} (n - b(r)) \right) \quad (27)$$

is equivalent to require

$$\frac{d-1}{d} (n - b(r)) + \alpha \frac{b(r)}{d} < r. \quad (28)$$

Lemma 3 assures the existence of a such α . Note that (28) is equivalent to

$$\underbrace{1 + \dots + 1}_{d-1 \text{ times}} + (-\beta) < 0, \quad (29)$$

which assures that the (d^+, n) -strings of f_α are only of the type (26). Therefore we have constructed a weight function $f = f_\alpha$ with r being nonnegative elements which satisfies (21). $\square$

We now concentrate our attention on the subinterval $((d-1)/d)(n-1), ((d-1)/d)n]$. By Proposition 2 there is a unique value of r contained in this subinterval. For this value formula (20) becomes simpler as expressed in the following result.

Corollary 5. Let r be a positive integer such that $r \geq d$ and $((d-1)/d)(n-1) < r \leq ((d-1)/d)n$. Then

$$\gamma(n, d, r) \leq \binom{r}{d} + \binom{r}{d-1}. \quad (30)$$

Proof. The result follows directly from Proposition 4 since $b(r) = 1$. $\square$

The aim of the next proposition is to try to individuate a lower bound for $\gamma(n, d, r)$ starting from $\phi(f, d)$. Recalling that $\gamma(n, d, r) = \min\{\phi(f, d) : f \in W_n, f^+ = r\}$, we have the following result.

Proposition 6. Let r be a positive integer such that $d \leq r \leq ((d-1)/d)n$. Let $f \in W_n$ with $f^+ = r$, as in (3). If

$$x_1 + y_{n-r} \geq 0, \quad (31)$$

then

$$\phi(f, d) \geq \binom{r-1}{d-2} (n-r) + \binom{r}{d} \geq \binom{r}{d} + \binom{r}{d-1}. \quad (32)$$

Proof. We can consider the d -strings of $\{1, \dots, n\}$ of type

$$1i_1 \dots i_{d-2} \mid (n-r), \quad (33)$$

where $i_1 \dots i_{d-2}$ are chosen in $\{2, \dots, r\}$.

By virtue of (31), each string of the type (33) is a (d^+, n) -string of f .

On the other hand, since $y_1 \geq y_2 \geq \dots \geq y_{n-r}$, each string of type

$$1i_1 \dots i_{d-2} \mid k, \quad (34)$$

where $i_1 \dots i_{d-2}$ are chosen in $\{2, \dots, r\}$ and k in $\{1, \dots, n-r\}$, will be a (d^+, n) -string of f .

The distinct strings of the type (34) are exactly $\binom{r-1}{d-2} (n-r)$. There are moreover all the (d^+, n) -strings of f that are the d -strings on $\{1, \dots, r\}$. This proves the first inequality in (32). Moreover, since $r \leq ((d-1)/d)n$, we also have $n-r \geq (r/(d-1))$. Therefore

$$\binom{r-1}{d-2} (n-r) \geq \binom{r-1}{d-2} \frac{r}{d-1} = \binom{r}{d-1}. \quad (35)$$

Thus the second inequality also holds. $\square$

Proposition 6 leads us to state the following conjecture.

Remark 7. We conjecture that

$$\gamma(n, d, r) = \binom{r}{d} + \binom{r}{d-1}, \quad (36)$$

where $r \geq d$ and $((d-1)/d)(n-1) < r \leq ((d-1)/d)n$.

Note that, in order to prove (36), by Corollary 5, it is sufficient to show

$$\gamma(n, d, r) \geq \binom{r}{d} + \binom{r}{d-1}. \quad (37)$$

In the particular case $n = 2d+2$ and $r = 2d-1$ the previous conjecture has been proved in [3].

In the next result we show that our conjecture (36) is true when $n = d+2$ and $r = d$.

Proposition 8. If $n = d+2$ and $r = d$, then

$$\gamma(n, d, r) = \binom{r}{d} + \binom{r}{d-1}. \quad (38)$$

Proof. We observe that if $n = d + 2$ and $r = d$, then $((d - 1)/d)(n - 1) < r \leq ((d - 1)/d)n$. Hence, by Corollary 5,

$$\gamma(n, d, r) \leq \binom{r}{d} + \binom{r}{d-1}. \quad (39)$$

Therefore, in order to have the thesis, we must prove that

$$\gamma(n, d, r) \geq \binom{r}{d} + \binom{r}{d-1}. \quad (40)$$

Let us note that as a direct consequence of Corollary 5 and Proposition 6 it follows that if r is a positive integer with $r \geq d$ such that $((d - 1)/d)(n - 1) < r \leq ((d - 1)/d)n$, then

$$\begin{aligned} \min \{ \phi(f, d) : f \in W_n, f^+ = r, x_1 + y_{n-r} \geq 0 \} \\ = \binom{r}{d} + \binom{r}{d-1}. \end{aligned} \quad (41)$$

Therefore by (41) inequality (40) is equivalent to the following:

$$\begin{aligned} \min \{ \phi(f, d) : f \in W_n, f^+ = r, x_k + y_{n-r} < 0, \\ \text{for every } k = 1, \dots, r \} \\ \geq \binom{r}{d} + \binom{r}{d-1}. \end{aligned} \quad (42)$$

So, to prove inequality (40), it is sufficient to prove that, for each $f \in W_n$ such that $f^+ = r$ and $x_1 + y_{n-r} = x_1 + y_2 < 0$, we have $\phi(f, d) \geq \binom{r}{d} + \binom{r}{d-1}$. We set $p = \binom{r}{d-1}$ and let $A_1, \dots, A_p$ be all the $(d - 1)$ -strings in $[r]$. At this point we consider the following configurations:

$$\begin{array}{cc} A_1 \mid 1 & i_1 \mid 2 \\ \vdots & \vdots \\ A_p \mid 1 & i_p \mid 2, \end{array} \quad (43)$$

where $\{i_j\} = [r] \setminus A_j$ ($j = 1, \dots, p$). Since f is a weight function and $x_{i_j} + y_2 < 0$ for $j = 1, \dots, p$, it follows that all the d -strings $A_1 \mid 1, \dots, A_p \mid 1$ are nonnegative d -tuple of the form $(d - 1, 1)$ associated with f . Therefore $\phi(f, d) \geq \binom{r}{d} + p = \binom{r}{d} + \binom{r}{d-1}$. This proves (42), and hence also (40) is proved. $\square$

In Table 1 we list all the known values of $\gamma(n, d, r)$ in the range $((d - 1)/d)(n - 1) < r \leq ((d - 1)/d)n$. Let us note that when the integer d is fixed and n runs in $\{d + 1, d + 2, \dots\}$, by Proposition 2, we find at most an integer value r such that $((d - 1)/d)(n - 1) < r \leq ((d - 1)/d)n$.

In Table 1, $k = 1, 2, \dots$ and $t \in \{1, \dots, d\}$. Moreover, if $t = 1$, then there does not exist an integer r such that $((d - 1)/d)(n - 1) < r \leq ((d - 1)/d)n$. In the last column we have marked the cases in which the conjecture (36) is proved. In particular, we note that, for every k , the case $t = d$ corresponds to $n = (k + 1)d$, which implies $r = (k + 1)(d - 1) = ((d - 1)/d)n$. For these values of r we know that (36) holds (see [3]) and the conjecture (36) is true.

TABLE 1

n	r	$n - r$	
$d + 1$	$\nexists$	$\nexists$	Proposition 8
$d + 2$	d	2	
$\vdots$	$\vdots$	$\vdots$	
$d + (d - 1)$	$2d - 3$	2	Proposition 2.5 in [3]
$2d$	$2d - 2$	2	
$2d + 1$	$\nexists$	$\nexists$	
$2d + 2$	$2d - 1$	3	Proposition 2.4 in [3]
$\vdots$	$\vdots$	$\vdots$	
$kd + 1$	$\nexists$	$\nexists$	
$\vdots$	$\vdots$	$\vdots$	Proposition 2.5 in [3]
$kd + t$	$k(d - 1) + (t - 1)$	$k + 1$	
$\vdots$	$\vdots$	$\vdots$	
$(k + 1)d$	$(k + 1)(d - 1)$	$(k + 1)$	Proposition 2.5 in [3]
$\vdots$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	

We give now a simple combinatorial interpretation of the inequalities

$$\frac{d - 1}{d} (n - 1) < r \leq \frac{d - 1}{d} n. \quad (44)$$

For this purpose let us note that the last inequalities are equivalent to the following:

$$(n - r - 1)(d - 1) < r \leq (n - r)(d - 1). \quad (45)$$

Let now r be a positive integer that satisfies (45) and $f \in W_n$, with $f^+ = r$, as in (3). Let us consider the following representation:

$$\begin{array}{c} \sqcup + \sqcup + \dots + \sqcup + k_1 \\ \sqcup + \sqcup + \dots + \sqcup + k_2 \\ \vdots \\ \sqcup + \sqcup + \dots + \sqcup + k_{n-r-1} \\ \sqcup + \sqcup + \dots + \sqcup + k_{n-r}, \end{array} \quad (46)$$

where every $\sqcup$ can be seen as a “box” initially empty and every row contains $d - 1$ boxes. Each of such boxes can be occupied by at most one nonnegative element of f . Thus (45) is equivalent to state that $n - r - 1$ rows in (46) must be completely occupied, whereas the last row must contain at least a nonempty box and, furthermore, the number of nonnegative elements of f cannot exceed the number of empty boxes in (46). This combinatorial interpretation of (45) suggests to examine firstly the (d^+, n) -strings of f of the form $+\dots+-$, that is, a subset with $d - 1$ nonnegative elements and only one negative.

4. Conclusions and Further Developments

In this paper we continue the research approach started in [1, 3] to the problem of determining new identities and new bounds concerning the numbers $\gamma(n, d, r)$. After a brief introduction to these numbers and their combinatorial context, we establish some preliminary results necessary to delimit the range of the integers parameters n , d , and r that we study. Next we give a relevant upper bound for numbers $\gamma(n, d, r)$ (Proposition 4). Afterwards we focus our attention on the range $((d-1)/d)(n-1) < r \leq ((d-1)/d)n$ (Corollary 5). In this context, Proposition 6 and the study of $\phi(f, d)$ bring us to conjecture a strong lower bound for $\gamma(n, d, r)$ on the subinterval $((d-1)/d)(n-1) < r \leq ((d-1)/d)n$. Finally, we prove our conjecture in the case $n = d + 2$ and $r = d$, and this provides an exact value for $\gamma(r+2, r, r)$. We think that our approach of dividing the intervals of variation of the integers parameters n , d , and r in particular types of subintervals can be useful in order to determine further upper and lower bound for $\gamma(n, d, r)$. In future papers our purpose will be:

- (i) to study other subintervals, trying to extend and generalize the validity of the results of this paper;
- (ii) to prove conjecture (36) in all the range $((d-1)/d)(n-1) < r \leq ((d-1)/d)n$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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