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Research Article

Existence of Mild Solutions for Impulsive Fractional Stochastic Differential Inclusions with State-Dependent Delay

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We study the existence of mild solutions for a class of impulsive fractional stochastic differential inclusions with state-dependent delay. Sufficient conditions for the existence of solutions are derived by using the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan. An example is given to illustrate the theory.

1. Introduction

During the past two decades, fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, and engineering [1–3]. There has been a great deal of interest in the solutions of fractional differential equations in analytical and numerical senses. One can see the monographs of Kilbas et al. [2], Miller and Ross [4], Podlubny [5], and Lakshmikantham et al. [6] and the survey of Agarwal et al. [7, 8].

To study the theory of abstract differential equations with fractional derivatives in infinite dimensional spaces, the first step is how to introduce new concepts of mild solutions. A pioneering work has been reported by El-Borai [9, 10]. Very recently, Hernández et al. [11] showed that some recent papers of fractional differential equations in Banach spaces were incorrect and used another approach to treat abstract equations with fractional derivatives based on the well-developed theory of resolvent operators for integral equations. Moreover, Wang and Zhou [12], Zhou and Jiao [13] also introduced a suitable definition of mild solutions based on Laplace transform and probability density functions.

On the other hand, the theory of impulsive differential equations or inclusions has become an active area of investigation due to its applications in fields such as mechanics, electrical engineering, medicine, biology, and ecology. One can refer to [14, 15] and the references therein. Recently, the problems of existence of solutions and controllability of impulsive differential equations and differential inclusions have been extensively studied [16, 17]. Benedetti in [18] proved an existence result for impulsive functional differential inclusions in Banach spaces. Obukhovskii and Yao [19] considered local and global existence results for semilinear functional differential inclusions with infinite delay and impulse characteristics in a Banach space. Some existence results were obtained for certain classes of functional differential equations and inclusions in Banach spaces under assumption that the linear part generates an compact semigroup (see, e.g., [20–22]). The existence results of impulsive differential equations and inclusions have been generalized to stochastic differential equations with impulsive conditions [23, 24] and for stochastic impulsive differential inclusions [25–27].

We would like to mention that the impulsive effects also widely exist in fractional stochastic differential systems [28–30], and it is important and necessary to discuss the qualitative properties for stochastic fractional equations with impulsive perturbations with state-dependent delay. However, to the authors' knowledge, no result has been reported on the existence problem of impulsive fractional stochastic differential inclusions with state-dependent delay and the aim of this paper is to fill this gap.

Motivated by this consideration, in this paper we will discuss the existence of mild solutions for a class of impulsive fractional stochastic differential inclusions with state-dependent delay in Hilbert spaces. Specifically, sufficient

conditions for the existence are given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan.

2. Preliminaries and Basic Properties

In this section, we provide definitions, lemmas, and notations necessary to establish our main results. Throughout this paper, we use the following notations. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathscr{F}_t$, $t \in J = [0, b]$ satisfying the usual conditions (i.e., right continuous and $\mathscr{F}_0$ containing all $\mathbb{P}$-null sets). We consider two real separable Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ with inner product $(\cdot,\cdot)_{\mathscr{H}}$, $(\cdot,\cdot)_{\mathscr{K}}$ and norm $\|\cdot\|_{\mathscr{H}}$, $\|\cdot\|_{\mathscr{K}}$. Let $w = (w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathscr{F}_b, \mathbb{P})$ with the linear bounded covariance operator Q such that $\mathrm{Tr}(Q) < \infty$. Assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathscr{K}$, a bounded sequence of nonnegative real numbers $\{\lambda_k\}$ such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$, and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$(w(t), e)_{\mathscr{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathscr{K}} \beta_k(t), \quad e \in \mathscr{K},\ t \in [0, b] \tag{1}$$

and $\mathscr{F}_t = \mathscr{F}_t^w$, where $\mathscr{F}_t^w$ is the sigma algebra generated by $\{w(s),\ 0 \leq s \leq t\}$. Let $L(\mathscr{K}, \mathscr{H})$ denote the space of all bounded linear operators from $\mathscr{K}$ to $\mathscr{H}$ equipped with the usual operator norm $\|\cdot\|_{L(\mathscr{K},\mathscr{H})}$. For $\psi \in L(\mathscr{K}, \mathscr{H})$ we define

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{k=1}^{\infty} \left\| \sqrt{\lambda_k} \psi e_k \right\|^2. \tag{2}$$

If $\|\psi\|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(\mathscr{K}, \mathscr{H})$ denote the space of all Q-Hilbert-Schmidt operators ψ. The completion $L_Q(\mathscr{K}, \mathscr{H})$ of $L(\mathscr{K}, \mathscr{H})$ with respect to the topology is induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology. Let $L_2(\Omega, \mathscr{H})$ be a Banach space of all strongly measurable, square integrable, $\mathscr{H}$-valued random variables equipped with the norm $\|x(\cdot)\|_{L_2} = (\mathbb{E}\|x(\cdot, \omega)\|^2)$, where $\mathbb{E}(\cdot)$ denote the expectation with respect to the measure $\mathbb{P}$. Let $\mathscr{C}(J, L_2(\Omega, \mathscr{H}))$ be the Banach space of all continuous maps from J to $L_2(\Omega, \mathscr{H})$ satisfying the condition $\sup_{0\leq t\leq b}\mathbb{E}\|x(t)\|^2 < \infty$. Let $L_2^0(\Omega, \mathscr{H})$ denote the family of all $\mathscr{F}_0$-measurable, $\mathscr{H}$-valued random variables $x(0)$.

The purpose of this paper is to investigate the existence of mild solutions for a class of impulsive fractional stochastic differential inclusions with state-dependent delay of the form

$$\begin{gathered} D_t^\alpha x(t) - Ax(t) \in f\left(t, x_{\rho(t,x_t)}\right) + \Sigma\left(t, x_{\rho(t,x_t)}\right) \frac{dw(t)}{dt}, \\ t \in J := [0, b],\ t \neq t_k,\ k = 1, \ldots, m, \\ \Delta x(t_k) = I_k\left(x_{t_k}\right), \quad k = 1, \ldots, m, \\ x_0 = \phi \in \mathscr{B}, \end{gathered} \tag{3}$$

where D_t^α is the Caputo fractional derivative of order α, $0 < \alpha < 1$; $x(\cdot)$ takes the value in the separable Hilbert space $\mathscr{H}$; and $A : \mathscr{D}(A) \subset \mathscr{H} \to \mathscr{H}$ is the generator of an α-resolvent operator family $S_\alpha(t)$, $t \geq 0$ The history $x_t : (-\infty, 0] \to \mathscr{H}$, $x_t(\theta) = x(t + \theta)$, $\theta \leq 0$, belongs to an abstract phase space $\mathscr{B}$ defined axiomatically; f, Σ, ρ, and I_k, $k = 1, \ldots, m$, are given functions to be specified later. Here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathscr{F}_0$-measurable, $\mathscr{B}$-valued random variable independent of w with finite second moments.

Recall the following known definitions. For more details see [2].

Definition 1. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,\ \alpha > 0 \tag{4}$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2. Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$\begin{gathered} {}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \\ t > 0,\ n-1 < \alpha < n. \end{gathered} \tag{5}$$

Definition 3. The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$\begin{gathered} {}^c D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \\ t > 0,\ n-1 < \alpha < n. \end{gathered} \tag{6}$$

If $f(t) \in C^n[0, \infty)$, then

$$\begin{gathered} {}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)\, ds = I^{n-\alpha} f^n(s), \\ t > 0,\ n-1 < \alpha < n. \end{gathered} \tag{7}$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the caputo derivative of order $\alpha > 0$ is given as

$$\begin{gathered} L\{{}^c D^\alpha f(t); s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \\ n-1 \leq \alpha < n. \end{gathered} \tag{8}$$

Definition 4 (see [31]). A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\pi/2, \pi]$, $M > 0$, such that the following two conditions are satisfied:

(i) $\rho(A) \subset \Psi_{\theta,\omega} = \{\lambda \in \mathbf{C} : \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta\}$,

(ii) $\|R(\lambda, A)\| = \|(\lambda - A)^{-1}\| \leq M/|\lambda - \omega|$, $\lambda \in \Psi_{\theta,\omega}$.

Definition 5 (see [30]). Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$, where $L(X)$ is a Banach space of all bounded linear operators from X to X and the corresponding norm is denoted by $\|\cdot\|$, such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\alpha I - A\right)^{-1} x = \int_0^\infty e^{\lambda t} S_\alpha(t)\, x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X, \tag{9}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 6. Let S_α be an α-resolvent operator family on Banach space X with generator A. Then, the following assertions hold:

(i) $S_\alpha(t)\mathscr{D}(A) \subset \mathscr{D}(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in \mathscr{D}(A)$ and $t \geq 0$,

(ii) for all $x \in X$, $I_t^\alpha S_\alpha(t)x \in \mathscr{D}(A)$ and $S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x$, $t \geq 0$,

(iii) $x \in \mathscr{D}(A)$ and $Ax = x$ if and only if $S_\alpha(t)x = x + I_t^\alpha AS_\alpha(t)x$, $t \geq 0$,

(iv) A is closed, densely defined.

Definition 7 (see [30]). Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}\left(\lambda^\alpha I - A\right)^{-1} x = \int_0^\infty e^{\lambda t} S_\alpha(t)\, x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X, \tag{10}$$

where $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [2].

Definition 8. We say that a function $x : [a, b] \to \mathscr{H}$ is a normalized piecewise continuous function on $[a, b]$ if x is piecewise continuous and left continuous on $(a, b]$.

We denote by $\mathscr{PC}([a, b], \mathscr{H})$ the space formed by normalized piecewise continuous, $\mathscr{F}_t$-adapted measurable processes from $[a, b]$ into $\mathscr{H}$. In particular, we introduce the space $\mathscr{PC}$ formed by $\mathscr{F}_t$-adapted measurable, $\mathscr{H}$-valued stochastic processes $\{x(t) : t \in [0, b]\}$ such that x is continuous at $t \neq t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1, \ldots, m$.

In this paper, we assume that $\mathscr{PC}$ is endowed with the norm $\|x\|_{\mathscr{PC}} = \left(\sup_{0\leq t\leq b}\mathbb{E}\|x(t)\|^2\right)^{1/2}$. Then $(\mathscr{PC}, \|\cdot\|_{\mathscr{PC}})$ is a Banach space [32].

We denote by $\widehat{x}_k \in \mathscr{C}([t_k, t_{k+1}]; L_2(\Omega, \mathscr{H}))$, $k = 0, \ldots, m$, the function given by

$$\widehat{x}_k(t) := \begin{cases} x(t) & \text{for } t \in (t_k, t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases} \tag{11}$$

Moreover, for $B \subseteq \mathscr{PC}$, we denote by $\widehat{B}_k$, $k = 0, \ldots, m$, the set $\widehat{B}_k = \{\widehat{x}_k : x \in B\}$. It is proved in [32] that $B \subseteq \mathscr{PC}$ is relatively compact in $\mathscr{PC}$ if, and only if, the set $\widehat{B}_k$ is relatively compact in $\mathscr{C}([t_k, t_{k+1}]; L_2(\Omega, \mathscr{H}))$, for every $k = 0, \ldots, m$. The notation $B_r(x, \mathscr{H})$ stands for the closed ball with center at x and radius $r > 0$ in $\mathscr{H}$.

Throughout this paper, we assume that the phase space $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a seminormed linear space of $\mathscr{F}_0$-measurable functions mapping $(-\infty, 0]$ into $\mathscr{H}$ and satisfying the following fundamental axioms [33].

(i) If $x : (-\infty, \tau + b] \to \mathscr{H}$, $b > 0$, $\tau \in \mathbb{R}$, is continuous on $[\tau, \tau + b)$ and $x_\tau \in \mathscr{B}$, then for every $t \in [\tau, \tau + b)$ the following conditions hold:

(a) x_t is in $\mathscr{B}$;

(b) $\|x(t)\| \leq H\|x_t\|_{\mathscr{B}}$;

(c) $\|x_t\|_{\mathscr{B}} \leq K(t - \tau)\sup\{\|x(s)\| : \tau \leq s \leq t\} + N(t-\tau)\|x_\tau\|_{\mathscr{B}}$, where $H > 0$ is a constant; $K, N : [0, \infty) \to [1, \infty)$, K is continuous, N is locally bounded, and H, K, N are independent of $x(\cdot)$.

(ii) For the function $x(\cdot)$ in (i), the function $t \to x_t$ is continuous from $[\tau, \tau + b]$ into $\mathscr{B}$.

(iii) The space $\mathscr{B}$ is complete.

The next result is a consequence of the phase space axioms. The reader can refer to [34] for the proof.

Lemma 9. *Let $x : (-\infty, b] \to \mathscr{H}$ be an $\mathscr{F}_t$-adapted measurable process such that the $\mathscr{F}_0$-adapted process $x_0 = \phi(t) \in L_2^0(\Omega, \mathscr{B})$ and $x|_J \in \mathscr{PC}(J, \mathscr{H})$. Then*

$$\|x_s\|_{\mathscr{B}} \leq N_b\mathbb{E}\|\phi\|_B + K_b \sup_{0\leq s\leq b}\mathbb{E}\|x(s)\|, \tag{12}$$

where $K_b = \sup\{K(t) : 0 \leq t \leq b\}$ and $N_b = \{N(t) : 0 \leq t \leq b\}$.

In what follows, we use the notations $\mathscr{P}(\mathscr{H})$ for the family of all nonempty subsets of $\mathscr{H}$ and denote

$$\mathscr{P}_{cl}(\mathscr{H}) = \{Y \in \mathscr{P}(\mathscr{H}) : Y \text{ is closed}\},$$

$$\mathscr{P}_{bd}(\mathscr{H}) = \{Y \in \mathscr{P}(\mathscr{H}) : Y \text{ is bounded}\},$$

$$\mathscr{P}_{cv}(\mathscr{H}) = \{Y \in \mathscr{P}(\mathscr{H}) : Y \text{ is convex}\},$$

$$\mathscr{P}_{cp}(\mathscr{H}) = \{Y \in \mathscr{P}(\mathscr{H}) : Y \text{ is compact}\},$$
$$\mathscr{P}_{cd}(\mathscr{H}) = \{Y \in \mathscr{P}(\mathscr{H}) : Y \text{ is compact-acyclic}\}. \tag{13}$$

Now, we briefly introduce some facts on multivalued analysis. For details, one can see [35].

A multivalued map $G : \mathscr{H} \to \mathscr{P}(\mathscr{H})$ is convex (closed) valued, if $G(x)$ is convex (closed) for all $x \in \mathscr{H}$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in $\mathscr{H}$, for any bounded set B of $\mathscr{H}$, that is; $\sup_{x\in B}\sup\{\|y\| \in G(x)\} < \infty$.

For $x \in \mathscr{H}$ and $Y, Z \in \mathscr{P}_{bd,cl}(\mathscr{H})$, we denote by $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$ and $\kappa(Y, Z) = \sup_{a\in Y} d(a, Z)$, and the Hausdorff metric $H_d : \mathscr{P}_{bd,cl}(\mathscr{H}) \times \mathscr{P}_{bd,cl}(\mathscr{H}) \to \mathbb{R}_+$ by $H_d(A, B) = \max\{\kappa(A, B), \kappa(B, A)\}$.

A multivalued map G is called upper semicontinuous (u.s.c. for short) on $\mathscr{H}$ if, for each $x_0 \in \mathscr{H}$, the set $G(x_0)$ is a nonempty, closed subset of $\mathscr{H}$ and if, for each open set B of $\mathscr{H}$ containing $G(x_0)$, there exists an open neighborhood $\widetilde{N}$ of x_0 such that $G(\widetilde{N}) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact, for every bounded subset $B \subseteq \mathscr{H}$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph; that is, $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

A multivalued map $G : J \to \mathscr{P}_{bd,cl,cv}(\mathscr{H})$ is said to be measurable if for each $x \in \mathscr{H}$, the function $t \to d(x, G(t))$ is measurable function on J.

Definition 10 (see [35]). Let $G : \mathscr{H} \to \mathscr{P}_{bd,cl}(\mathscr{H})$ be a multivalued map. Then G is called a multivalued contraction if there exists a constant $\vartheta \in (0, 1)$ such that for each $x, y \in \mathscr{H}$

$$H_d(G(x) - G(y)) \le \vartheta \|x - y\|. \tag{14}$$

The constant ϑ is called a contraction constant of G.

Next, we mention the statement of a nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan.

Lemma 11 (see [36]). *Let $\mathscr{H}$ be a Hilbert space with V an open convex subset of $\mathscr{H}$ and $y \in \mathscr{H}$. Suppose that*

(a) *$\Phi : \overline{V} \to \mathscr{P}_{cd}(\mathscr{H})$ has closed graph;*

(b) *$\Phi : \overline{V} \to \mathscr{P}_{cd}(\mathscr{H})$ is a condensing map with $\Phi(\overline{V})$ a subset of a bounded set in $\mathscr{H}$ hold. Then either*

(i) *Φ has a fixed point in $\overline{V}$ or*

(ii) *there exist $y \in \partial V$ and $\lambda \in (0, 1)$ with $y \in \lambda\Phi(y) + (1 - \lambda)\{y_0\}$.*

3. The Mild Solution and Existence

Before stating and proving the main result, we present the definition of the mild solution to the system (3)–(3) based on the paper [30, 31].

Let $\mathscr{S}_{\Sigma,\psi} = \{\sigma \in L^2(L(\mathscr{K}, \mathscr{H})) : \sigma(t) \in \Sigma(t, \psi)$ for a.e. $t \in J\}$ be the set of selections of Σ for each $\psi \in \mathscr{B}$, and $x(t_k^+) = x(t_k^-) + I_k(x_{t_k})$, $k = 1, 2, \ldots, m$.

Definition 12. An $\mathscr{F}_t$-adapted stochastic process $x : (-\infty, b] \to \mathscr{H}$ is called a mild solution of the system (3) if $x_0 = \phi$, $x_{\rho(t,x_t)} \in \mathscr{B}$ for every $t \in J$, $\Delta x(t_k) = I_k(x_{t_k})$, $k = 1, \ldots, m$, the restriction of $x(\cdot)$ to the interval $(t_k, t_{k+1}]$ $(k = 0, 1, \ldots, m)$ is continuous, and there exists $\sigma \in \mathscr{S}_{\Sigma,x_\rho}$ such that $x(t)$ satisfies the following integral equation:

$$x(t) = \begin{cases} T_\alpha(t)\phi(0) \\ \quad + \int_0^t S_\alpha(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in [0, t_1], \\ T_\alpha(t-t_1)\left[x(t_1^-) + I_1\left(x_{t_1}\right)\right] \\ \quad + \int_{t_1}^t S_\alpha(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds \\ \quad + \int_{t_1}^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t-t_m)\left[x(t_m^-) + I_m\left(x_{t_m}\right)\right] \\ \quad + \int_{t_m}^t S_\alpha(t-s) f\left(s, x_{\rho(s,x_s)}\right) ds \\ \quad + \int_{t_m}^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in (t_m, b], \end{cases} \tag{15}$$

where $T_\alpha(t) = (1/2\pi i)\int_{\widehat{B}_r} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda$, $S_\alpha(t) = (1/2\pi i)\int_{\widehat{B}_r} e^{\lambda t}(\lambda^\alpha - A)^{-1} d\lambda$, and $\widehat{B}_r$ denotes the Bromwich path. $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by A.

The following result on the operator $S_\alpha(t)$ appeared and is proved in [31].

Theorem 13. *If $\alpha \in (0, 1)$ and $A \in \mathscr{A}^\alpha(\theta_0, \omega_0)$ is a sectorial operator, then for any $x \in \mathscr{H}$ and $t > 0$, one has*

$$\|S_\alpha(t)\| \le Ce^{\omega t}\left(1 + t^{\alpha-1}\right), \quad t > 0,\ \omega > \omega_0, \tag{16}$$

where C is a constant depending only on θ and ω.

In order to establish the results, we first assume that the function ρ is continuous from $J \times \mathscr{B}$ into $(-\infty, b]$ and we impose the following additional hypotheses.

(H1) If $\alpha \in (0, 1)$ and $A \in \mathscr{A}^\alpha(\theta_0, \omega_0)$ is a sectorial operator, then for $x \in \mathscr{H}$ and $t > 0$,

$$\|T_\alpha(t)\| \le Me^{\omega t}, \quad \|S_\alpha(t)\| \le Ce^{\omega t}\left(1 + t^{\alpha-1}\right), \quad \omega > \omega_0. \tag{17}$$

If $\widetilde{M}_T = \sup_{0\le t\le b}\|T_\alpha(t)\|$, and $\widetilde{M}_S = \sup_{0\le t\le b} Ce^{\omega t}(1+t^{1-\alpha})$, we have

$$\|T_\alpha(t)\| \le \widetilde{M}_T, \qquad \|S_\alpha(t)\| \le t^{\alpha-1}\widetilde{M}_S \qquad \text{(for more details, see [31])}. \tag{18}$$

(H2) The function $t \to \phi_t$ is continuous from $\mathcal{Z}(\rho^-) = \{\rho(s,\psi) \le 0,\ (s,\psi) \in J\times\mathcal{B}\}$ to $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{W}^\phi : \mathcal{Z}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \le \mathcal{W}^\phi(t)\mathbb{E}\|\phi\|_{\mathcal{B}}$ for each $t \in \mathcal{Z}(\rho^-)$.

(H3) The multivalued map $F : J\times\mathcal{B} \to \mathcal{P}_{bd,cl,cv}(L(\mathcal{K},\mathcal{H}))$ is Carathédory; that is,

(i) $t \mapsto F(t,\psi)$ is measurable for each $\psi \in \mathcal{B}$;

(ii) $\psi \mapsto F(t,\psi)$ is upper semicontinuous (u.s.c.) for almost all $t \in J$,

and for each fixed $\psi \in \mathcal{B}$, the set $\mathcal{S}_{\Sigma,\psi}$ of selections of Σ is nonempty.

(H4) There exists a positive integrable function $\nu \in L^1([0,b],\mathbb{R}^+)$ such that

$$\limsup_{\|\psi\|^2_{\mathcal{B}}\to\infty} \frac{\|\Sigma(t,\psi)\|^2}{\nu(t)\|\psi\|^2_{\mathcal{B}}} = \Lambda \tag{19}$$

uniformly in $t \in J$ for a nonnegative constant Λ, where

$$\|\Sigma(t,\psi)\|^2 = \sup\left\{\mathbb{E}\|\sigma\|^2 : \sigma \in \Sigma(t,\psi)\right\}. \tag{20}$$

(H5) The function $f : J\times\mathcal{B} \to \mathcal{H}$ is continuous and there exists $M_f > 0$ such that

$$\begin{aligned} \mathbb{E}\|f(t,\psi_1) - f(t,\psi_2)\|^2 &\le M_f\|\psi_1-\psi_2\|^2_{\mathcal{B}}, \\ &\qquad t\in J,\ \psi_1,\psi_2 \in \mathcal{B}, \\ \mathbb{E}\|f(t,\psi)\|^2 &\le M_f\left(1+\|\psi\|^2_{\mathcal{B}}\right), \quad t\in J,\ \psi\in\mathcal{B}. \end{aligned} \tag{21}$$

(H6) The functions $I_k : \mathcal{B} \to \mathcal{H}$ are completely continuous and there exist constants ε_k such that

$$\limsup_{\|\psi\|^2_{\mathcal{B}}\to\infty} \frac{\|I_k(\psi)\|^2}{\|\psi\|^2_{\mathcal{B}}} = \varepsilon_k \tag{22}$$

for every $\psi \in \mathcal{B}$, $k = 1,2,\ldots,m$.

Remark 14. The condition (H2) is frequently verified by continuous and bounded functions. For more details, see, for instance, [34] (Proposition 7.1.1).

The following lemma is required for the main result. The reader can refer to [37, 38] for the lemma and to [32] for more details about the proof.

Lemma 15. *Let $x : (-\infty,b] \to \mathcal{H}$ such that $x_0 = \phi$ and $x|_J \in \mathcal{PC}(J,\mathcal{H})$. If (H2) holds, then*

$$\begin{aligned} \|x_s\|_{\mathcal{B}} \le{}& \left(N_b + \mathcal{W}^\phi_0\right)\|\phi\|_{\mathcal{B}} \\ &+ K_b \sup\{\|x(\theta)\|;\theta\in[0,\max\{0,s\}]\}, \\ &\qquad s\in\mathcal{Z}(\rho^-)\cup J, \end{aligned} \tag{23}$$

where $\mathcal{W}^\phi_0 = \sup_{t\in\mathcal{Z}(\rho^-)}\mathcal{W}^\phi(t)$.

Lemma 16 (see [39]). *Let J be a compact interval and $\mathcal{H}$ a Hilbert space. Let Σ be a multivalued map satisfying (H3) and Γ a linear continuous operator from $L^2(J,\mathcal{H})$ to $\mathcal{C}(J,\mathcal{H})$. Then the operator $\Gamma\circ\mathcal{S}_\sigma : \mathcal{C}(J,\mathcal{H}) \to \mathcal{P}_{cp,cv}(\mathcal{C}(J,\mathcal{H}))$ is a closed graph in $\mathcal{C}(J,\mathcal{H})\times\mathcal{C}(J,\mathcal{H})$.*

Theorem 17. *Assume that (H1)–(H6) hold and $x_0 \in L^0_2(\Omega,\mathcal{H})$, with $\rho(t,\psi)\le t$ for every $(t,\psi)\in J\times\mathcal{B}$. Then the problem (3) has a mild solution on J provided that*

$$\max_{1\le k\le m}\left\{6\widetilde{M}^2_T\left(1+2K^2_b\varepsilon_k\right)\right\} < 1. \tag{24}$$

Proof. Consider the space $\mathcal{BPC} = \{x : (-\infty,b) \to \mathcal{H};\ x_0 = 0,\ x|_J \in \mathcal{PC}\}$ endowed with the uniform convergence topology and define the multivalued map $\Phi : \mathcal{BPC} \to \mathcal{P}(\mathcal{BPC})$ by $\Phi x = \{z \in \mathcal{BPC}\}$ such that

$$z(t) = \begin{cases} 0, & t\in(-\infty,0], \\ T_\alpha(t)\phi(0) \\ \quad + \int_0^t S_\alpha(t-s) f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s)\,dw(s), & t\in[0,t_1], \\ T_\alpha(t-t_1)\left[\overline{z}(t_1^-) + I_1\left(\overline{z}_{t_1}\right)\right] \\ \quad + \int_{t_1}^t S_\alpha(t-s) f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds \\ \quad + \int_{t_1}^t S_\alpha(t-s)\sigma(s)\,dw(s), & t\in(t_1,t_2], \\ \vdots \\ T_\alpha(t-t_m)\left[\overline{z}(t_m^-) + I_m\left(\overline{z}_{t_m}\right)\right] \\ \quad + \int_{t_m}^t S_\alpha(t-s) f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds \\ \quad + \int_{t_m}^t S_\alpha(t-s)\sigma(s)\,dw(s), & t\in(t_m,b], \end{cases} \tag{25}$$

where $\sigma \in \mathcal{S}_{\Sigma,\overline{z}_\rho} = \{\sigma \in L^2(L(\mathcal{K},\mathcal{H})) : \sigma(t) \in \Sigma(t,\overline{z}_{\rho(t,\overline{z}_t)})$ for a.e. $t\in J\}$ and $\overline{z} : (-\infty,0] \to \mathcal{H}$ such that $\overline{z}_0 = \phi$ and $\overline{z} = x$ on J.

We shall show that Φ has a fixed point, which is then a mild solution for the problem (3). To this end we show that

Φ satisfies all the conditions of Lemma 11. For the sake of convenience, we divide the proof into several steps.

Step 1. We show that there exists an open set $V \subseteq \mathscr{BPC}$ with $x \in \lambda\Phi x$ for $\lambda \in (0,1)$ and $x \notin \partial V$. Let $\lambda \in (0,1)$ and $x \in \lambda\Phi x$, then there exists an $\sigma \in \mathscr{S}_{\Sigma,\overline{z}_\rho}$ such that

$$x(t) = \begin{cases} \lambda T_\alpha(t)\phi(0) \\ \quad +\lambda\int_0^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds \\ \quad +\lambda\int_0^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in [0, t_1], \\ \lambda T_\alpha(t-t_1)\left[\overline{z}(t_1^-) + I_1\left(\overline{z}_{t_1}\right)\right] \\ \quad +\lambda\int_{t_1}^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds \\ \quad +\lambda\int_{t_1}^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in (t_1, t_2], \\ \vdots \\ \lambda T_\alpha(t-t_m)\left[\overline{z}(t_m^-) + I_m\left(\overline{z}_{t_m}\right)\right] \\ \quad +\lambda\int_{t_m}^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds \\ \quad +\lambda\int_{t_m}^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in (t_m, b]. \end{cases} \tag{26}$$

From assumption (H4), it follows that there exist two nonnegative real numbers c_1 and c_2 such that for any $\psi \in \mathscr{B}$ and $t \in J$,

$$\|\Sigma(t,\psi)\|^2 \le c_1\nu(t) + c_2\nu(t)\|\psi\|_{\mathscr{B}}^2. \tag{27}$$

From assumption (H6), we conclude that there exist positive constants a_k $(k = 1, \ldots, m)$, c_3 such that, for $\|\psi\|_{\mathscr{B}}^2 > c_3$,

$$\begin{gathered} \mathbb{E}\|I_k(\psi)\|^2 \le (\varepsilon_k + a_k)\|\psi\|_{\mathscr{B}}^2, \\ \max_{1\le k\le m}\left\{6\widetilde{M}_T^2\left(1 + 2K_b^2(\varepsilon_k + a_k)\right)\right\} < 1. \end{gathered} \tag{28}$$

Let

$$\begin{gathered} C_1 = \left\{\psi : \|\psi\|_{\mathscr{B}}^2 \le c_3\right\}, \qquad C_2 = \left\{\psi : \|\psi\|_{\mathscr{B}}^2 > c_3\right\}, \\ C_3 = \max\left\{\mathbb{E}\|I_k(\psi)\|^2, \psi \in C_1\right\}. \end{gathered} \tag{29}$$

We have

$$\mathbb{E}\|I_k(\psi)\|^2 \le C_3 + (\varepsilon_k + a_k)\|\psi\|_{\mathscr{B}}^2. \tag{30}$$

By assumption (H5), (27) and (30), we have for $t \in [0, t_1]$

$$\begin{aligned} &\mathbb{E}\|x(t)\|^2 \\ &\le 3\mathbb{E}\|T_\alpha(t)\phi(0)\|^2 + 3\mathbb{E}\left\|\int_0^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds\right\|^2 \\ &\quad + 3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s)\, dw(s)\right\|^2 \\ &\le 3\widetilde{M}_T^2\mathbb{E}\|\phi(0)\|^2 + 3\widetilde{M}_S^2\int_0^t (t-s)^{\alpha-1} ds \\ &\quad \times \int_0^t (t-s)^{\alpha-1} M_f\left(1 + \|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2\right) ds \\ &\quad + 3\widetilde{M}_S^2\operatorname{Tr}(Q)\int_0^t (t-s)^{2(\alpha-1)}\left[c_1\nu(s) + c_2\nu(s)\right. \\ &\qquad \left.\times \|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2\right] ds \\ &\le 3\widetilde{M}_T^2 H^2\mathbb{E}\|\phi\|_{\mathscr{B}}^2 + 3\widetilde{M}_S^2 M_f\frac{b^\alpha}{\alpha} \\ &\quad \times \int_0^t (t-s)^{\alpha-1}\left(1 + \|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2\right) ds \\ &\quad + 3\widetilde{M}_S^2\operatorname{Tr}(Q)c_1\int_0^t (t-s)^{2(\alpha-1)}\nu(s)\, ds \\ &\quad + 3\widetilde{M}_S^2\operatorname{Tr}(Q)c_2\int_0^t (t-s)^{2(\alpha-1)}\nu(s)\|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2 ds. \end{aligned} \tag{31}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we have

$$\begin{aligned} &\mathbb{E}\|x(t)\|^2 \\ &\le 3\mathbb{E}\left\|T_\alpha(t-t_k)\left[\overline{z}(t_k^-) + I_k\left(\overline{z}_{t_k}\right)\right]\right\|^2 \\ &\quad + 3\mathbb{E}\left\|\int_{t_k}^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds\right\|^2 \\ &\quad + 3\mathbb{E}\left\|\int_{t_k}^t S_\alpha(t-s)\sigma(s)\, dw(s)\right\|^2 \\ &\le 6\widetilde{M}_T^2\left[\mathbb{E}\|\overline{z}(t_k^-)\|^2 + C_3 + (\varepsilon_k + a_k)\|\overline{z}_{t_k}\|_{\mathscr{B}}^2\right] \\ &\quad + 3\widetilde{M}_S^2 M_f\frac{b^\alpha}{\alpha}\int_{t_k}^t (t-s)^{\alpha-1}\left(1 + \|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2\right) ds \\ &\quad + 3\widetilde{M}_S^2\operatorname{Tr}(Q)c_1\int_{t_k}^t (t-s)^{2(\alpha-1)}\nu(s)\, ds \\ &\quad + 3\widetilde{M}_S^2\operatorname{Tr}(Q)c_2\int_{t_k}^t (t-s)^{2(\alpha-1)}\nu(s)\|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2 ds. \end{aligned} \tag{32}$$

Then, for all $t \in [0, b]$, we have

$$\begin{aligned} &\mathbb{E}\|x(t)\|^2 \\ &\le M^* + 6\widetilde{M}_T^2\left[\mathbb{E}\|\overline{z}(t_k^-)\|^2 + (\varepsilon_k + a_k)\|\overline{z}_{t_k}\|_{\mathscr{B}}^2\right] \\ &\quad + 3\widetilde{M}_S^2 M_f\frac{b^{2\alpha}}{2\alpha}\int_0^t (t-s)^{\alpha-1}\|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2 ds \\ &\quad + 3\widetilde{M}_S^2\operatorname{Tr}(Q)c_2\int_0^t (t-s)^{2(\alpha-1)}\nu(s)\|\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}^2 ds, \end{aligned} \tag{33}$$

where

$$M^* = \max\left\{3\widetilde{M}_T^2 H^2 \mathbb{E}\|\phi\|_{\mathscr{B}}^2 + 3\widetilde{M}_S^2 \operatorname{Tr}(Q) c_1 \times \int_0^t (b-s)^{2(\alpha-1)} \nu(s)\, ds, 6\widetilde{M}_T^2 C_3 + 3\widetilde{M}_S^2 \operatorname{Tr}(Q) c_1 \int_{t_k}^t (t-s)^{2(\alpha-1)} \nu(s)\, ds\right\}. \tag{34}$$

By Lemmas 9 and 15, it follows that $\rho(s, \overline{z}_s) \le s$, $s \in [0,t]$, $t \in [0,b]$, and

$$\left\|\overline{z}_{\rho(s,\overline{z}_s)}\right\|_{\mathscr{B}}^2 \le 2\left[\left(N_b + \mathscr{W}_0^{\phi}\right)\mathbb{E}\|\phi\|_{\mathscr{B}}\right]^2 + 2K_b^2 \sup_{0\le s\le b} \mathbb{E}\|x(s)\|^2. \tag{35}$$

For each $t \in [0,b]$, we have

$$\begin{aligned}\mathbb{E}\|x(t)\|^2 &\le M_* + \left[6\widetilde{M}_T^2\left(1 + 2K_b^2(\varepsilon_k + a_k)\right)\right] \times \sup_{0\le t\le b}\mathbb{E}\|x(t)\|^2 + 6\widetilde{M}_S^2 M_f K_b^2 \frac{b^{2\alpha}}{2\alpha} \\ &\times \int_0^t (t-s)^{\alpha-1} \sup_{0\le\tau\le s}\mathbb{E}\|x(\tau)\|^2 ds \\ &+ 6\widetilde{M}_S^2 K_b^2 \operatorname{Tr}(Q) c_1 \int_0^t (t-s)^{2(\alpha-1)} \nu(s) \times \sup_{0\le\tau\le s}\mathbb{E}\|x(\tau)\|^2 ds,\end{aligned} \tag{36}$$

where

$$\begin{aligned}M_* &= M^* + 6\widetilde{M}_T^2\left[C_3 + (\varepsilon_k + a_k) C_4\right] + 3\widetilde{M}_S^2 M_f C_4 \frac{b^{3\alpha}}{3\alpha} + 3\widetilde{M}_S^2 C_4 \operatorname{Tr}(Q) c_1 \times \int_0^b (b-s)^{2(\alpha-1)} \nu(s)\, ds, \\ C_4 &= 2\left[\left(N_b + \mathscr{W}_0^{\phi}\right)\mathbb{E}\|\phi\|_{\mathscr{B}}\right]^2.\end{aligned} \tag{37}$$

Since $l = \max_{1\le k\le m}\{6\widetilde{M}_T^2(1 + 2K_b^2(\varepsilon_k + a_k))\} < 1$, we have

$$\begin{aligned}\sup_{0\le t\le b}\mathbb{E}\|x(t)\|^2 &\le \frac{M_*}{1-l} + \frac{6\widetilde{M}_S^2 M_f K_b^2 b^{2\alpha}}{(1-l)2\alpha} \times \int_0^b (b-s)^{\alpha-1} \sup_{0\le\tau\le s}\mathbb{E}\|x(\tau)\|^2 ds \\ &+ \frac{6\widetilde{M}_S^2 K_b^2 \operatorname{Tr}(Q) c_1}{1-l} \times \int_0^b (b-s)^{2(\alpha-1)} \nu(s) \sup_{0\le\tau\le s}\mathbb{E}\|x(\tau)\|^2 ds.\end{aligned} \tag{38}$$

Applying Gronwall's inequality, we get

$$\sup_{0\le t\le b}\mathbb{E}\|x(t)\|^2 \le \frac{M_*}{1-l}\exp\left\{\frac{6\widetilde{M}_S^2 K_b^2}{(1-l)3\alpha}\left(M_f b^{3\alpha} + 3\alpha \operatorname{Tr}(Q) c_1 \times \int_0^b (b-s)^{2(\alpha-1)} \nu(s)\, ds\right)\right\}. \tag{39}$$

Therefore,

$$\|x\|_{\mathscr{PC}}^2 \le \frac{M_*}{1-l}\exp\left\{\frac{6\widetilde{M}_S^2 K_b^2}{(1-l)3\alpha}\left(M_f b^{3\alpha} + 3\alpha \operatorname{Tr}(Q) c_1 \times \int_0^b (b-s)^{2(\alpha-1)} \nu(s)\, ds\right)\right\} < \infty. \tag{40}$$

Then, there exists r^* such that $\|x\|_{\mathscr{PC}}^2 \ne r^*$. Set $V = \{x \in \mathscr{BPC} : \|x\|_{\mathscr{PC}}^2 < r^*\}$. Thus, from the choice of V, there is no $x \in \partial V$ such that $x \in \lambda\phi x$ for $\lambda \in (0,1)$.

Step 2. Φ has a closed graph.

Let $x^{(n)} \to x^*$, $z_n \in \Phi x^{(n)}$, $x^{(n)} \in \overline{V} = B_{r^*}(0, \mathscr{BPC})$, and $z_n \to z_*$. It is easy to see that $(\overline{z^{(n)}})_s \to \overline{z^*}_s$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. We need to show that $z_* \in \Phi\overline{x^*}$. Now $z_n \in \Phi\overline{x^{(n)}}$ means that there exists $\sigma_n \in \mathscr{S}_{\Sigma,\overline{z^{(n)}}_\rho}$ such that, for each $t \in [0,t_1]$,

$$z_n(t) = T_\alpha\phi(0) + \int_0^t S_\alpha(t-s) f\left(s, \left(\overline{z^{(n)}}\right)_{\rho(s,\overline{z^{(n)}}_s)}\right) ds + \int_0^t S_\alpha(t-s)\sigma_n(s)\, dw(s). \tag{41}$$

We must show that there exists $\sigma_* \in \mathscr{S}_{\Sigma,\overline{x^*}_\rho}$ such that, for each $t \in [0,t_1]$,

$$z_*(t) = T_\alpha\phi(0) + \int_0^t S_\alpha(t-s) f\left(s, \left(\overline{z^*}\right)_{\rho(s,\overline{z^*}_s)}\right) ds + \int_0^t S_\alpha(t-s)\sigma_*(s)\, dw(s). \tag{42}$$

Set $\Theta_n(t) = z_n(t) - T_\alpha\phi(0) - \int_0^t S_\alpha(t-s) f(s, (\overline{z^{(n)}})_{\rho(s,\overline{z^{(n)}}_s)})ds - \int_0^t S_\alpha(t-s)\sigma_n(s)dw(s)$, and $\Theta_*(t) = z_*(t) - T_\alpha\phi(0) - \int_0^t S_\alpha(t-s) f(s, (\overline{z^*})_{\rho(s,\overline{z^*}_s)})ds - \int_0^t S_\alpha(t-s)\sigma_*(s)dw(s)$.

We have, for every $t \in [0,t_1]$,

$$\left\|\Theta_n(t) - \Theta_*(t)\right\|_{\mathscr{PC}}^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{43}$$

Consider the linear continuous operator $\Gamma : L^2([0,t_1], \mathscr{H}) \to \mathscr{C}([0,t_1], \mathscr{H})$ defined by

$$\Gamma(\sigma)(t) = \int_0^t S_\alpha(t-s)\sigma(s)\, dw(s). \tag{44}$$

From Lemma 16 and the definition of Γ, it follows that $\Gamma \circ \mathcal{S}_\Sigma$ is a closed graph operator, and, for every $t \in [0, t_1]$, $\Theta_n(t) \in \Gamma(\mathcal{S}_{\Sigma,\overline{z^{(n)}}})$.

Since $(\overline{z^{(n)}})_s \to \overline{z^*}_s$ and $\Gamma \circ \mathcal{S}_\Sigma$ is a closed graph operator, then there exists $\sigma_* \in \mathcal{S}_{\Sigma,\overline{z^*}_\rho}$ such that, for every $t \in [0, t_1]$,

$$\begin{aligned} z_*(t) - T_\alpha \phi(0) - \int_0^t S_\alpha(t-s) f\left(s, \left(\overline{z^*}\right)_{\rho(s,\overline{z^*}_s)}\right) ds \\ = \int_0^t S_\alpha(t-s)\sigma_*(s)\, dw(s). \end{aligned} \tag{45}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we have

$$\begin{aligned} z_n(t) = T_\alpha(t-t_k)\left[\overline{z^{(n)}}(t_k^-) + I_k\left(\overline{z^{(n)}}_{t_k}\right)\right] \\ + \int_{t_k}^t S_\alpha(t-s) f\left(s, \left(\overline{z^{(n)}}\right)_{\rho(s,\overline{z^{(n)}}_s)}\right) ds \\ + \int_{t_k}^t S_\alpha(t-s)\sigma_n(s)\, dw(s). \end{aligned} \tag{46}$$

We must show that there exists $\sigma_* \in \mathcal{S}_{\Sigma,\overline{z^*}_\rho}$ such that, for every $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} z_*(t) = T_\alpha(t-t_k)\left[\overline{z^*}(t_k^-) + I_k\left(\overline{z^*}_{t_k}\right)\right] \\ + \int_{t_k}^t S_\alpha(t-s) f\left(s, \left(\overline{z^*}\right)_{\rho(s,\overline{z^*}_s)}\right) ds \\ + \int_{t_k}^t S_\alpha(t-s)\sigma_*(s)\, dw(s). \end{aligned} \tag{47}$$

For every $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we have

$$\left\|\Theta_n(t) - \Theta_*(t)\right\|^2_{\mathcal{PC}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{48}$$

where

$$\begin{aligned} \Theta_n(t) &= z_n(t) - T_\alpha(t-t_k)\left[\overline{z^{(n)}}(t_k^-) + I_k\left(\overline{z^{(n)}}_{t_k}\right)\right] \\ &\quad - \int_{t_k}^t S_\alpha(t-s) f\left(s, \left(\overline{z^{(n)}}\right)_{\rho(s,\overline{z^{(n)}}_s)}\right) ds \\ &\quad - \int_{t_k}^t S_\alpha(t-s)\sigma_n(s)\, dw(s), \\ \Theta_*(t) &= z_*(t) - T_\alpha(t-t_k)\left[\overline{z^*}(t_k^-) + I_k\left(\overline{z^*}_{t_k}\right)\right] \\ &\quad - \int_{t_k}^t S_\alpha(t-s) f\left(s, \left(\overline{z^*}\right)_{\rho(s,\overline{z^*}_s)}\right) ds \\ &\quad - \int_{t_k}^t S_\alpha(t-s)\sigma_*(s)\, dw(s). \end{aligned} \tag{49}$$

Now, for every $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we consider the linear continuous operator $\Gamma : L^2((t_k, t_{k+1}], \mathcal{H}) \to \mathcal{C}((t_k, t_{k+1}], \mathcal{H})$,

$$\Gamma(\sigma)(t) = \int_{t_k}^t S_\alpha(t-s)\sigma(s)\, dw(s). \tag{50}$$

From Lemma 16, it follows that $\Gamma \circ \mathcal{S}_\Sigma$ is a closed graph operator, and for every $t \in (t_k, t_{k+1}]$, $\Theta_n(t) \in \Gamma(\mathcal{S}_{\Sigma,\overline{z^{(n)}}})$.

Since $(\overline{z^{(n)}})_s \to \overline{z^*}_s$ and $\Gamma \circ \mathcal{S}_\Sigma$ is a closed graph operator, then there exists $\sigma_* \in \mathcal{S}_{\Sigma,\overline{z^*}_\rho}$ such that, for every $t \in (t_k, t_{k+1}]$

$$\begin{aligned} z_*(t) - T_\alpha(t-t_k)\left[\overline{z^*}(t_k^-) + I_k\left(\overline{z^*}_{t_k}\right)\right] \\ - \int_{t_k}^t S_\alpha(t-s) f\left(s, \left(\overline{z^*}\right)_{\rho(s,\overline{z^*}_s)}\right) ds \\ = \int_{t_k}^t S_\alpha(t-s)\sigma_*(s)\, dw(s). \end{aligned} \tag{51}$$

Hence, Φ has a closed graph.

Step 3. We show that the operator Φ is condensing. Let $\Phi_1 : \overline{V} \to \mathcal{P}(\mathcal{BPC})$ and $\Phi_2 : \overline{V} \to \mathcal{P}(\mathcal{BPC})$ be defined by $\Phi_1 x = \{z_1 \in \mathcal{BPC}\}$ and $\Phi_2 x = \{z_2 \in \mathcal{BPC}\}$ such that

$$z_1(t) = \begin{cases} 0, & t \in [0, t_1], \\ T_\alpha(t-t_1)\left[\overline{z}(t_1^-) + I_1\left(\overline{z}_{t_1}\right)\right], & t \in (t_1, t_2], \\ \vdots & \\ T_\alpha(t-t_m)\left[\overline{z}(t_m^-) + I_m\left(\overline{z}_{t_m}\right)\right], & t \in (t_m, b], \end{cases}$$

$$z_2(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ T_\alpha(t)\phi(0) & \\ \quad + \int_0^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds & \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in [0, t_1], \\ \int_{t_1}^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds & \\ \quad + \int_{t_1}^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in (t_1, t_2], \\ \vdots & \\ \int_{t_m}^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds & \\ \quad + \int_{t_m}^t S_\alpha(t-s)\sigma(s)\, dw(s), & t \in (t_m, b]. \end{cases} \tag{52}$$

We first show that Φ_1 is a contraction while Φ_2 is a completely continuous operator.

Claim 1. Φ_1 is a contraction on $\mathcal{BPC}$. Let $u, v \in \mathcal{BPC}$. From (H6), Lemmas 9 and 15, we have for every $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$,

$$\begin{aligned} &\mathbb{E}\left\|(\Phi_1 u)(t) - (\Phi_1 v)(t)\right\|^2 \\ &\le 2\left\|T_\alpha(t-t_k)\right\|^2 \left[\mathbb{E}\left\|\overline{u}(t_k^-) - \overline{v}(t_k^-)\right\|^2 \right. \\ &\qquad \left. + \mathbb{E}\left\|I_k\left(\overline{u}_{t_k}\right) - I_k\left(\overline{v}_{t_k}\right)\right\|^2\right] \\ &\le 2\widetilde{M}_T^2 \left[\sup_{0\le t\le b} \mathbb{E}\left\|\overline{u}(t) - \overline{v}(t)\right\|^2 \right. \\ &\qquad \left. + (\varepsilon_k + a_k)\left\|\overline{u}_{t_k} - \overline{v}_{t_k}\right\|^2_{\mathcal{B}}\right] \end{aligned}$$

$$\begin{aligned}
&\leq 2\widetilde{M}_T^2\left[\sup_{0\leq t\leq b}\mathbb{E}\|\overline{u}(t)-\overline{v}(t)\|^2+2\left(\varepsilon_k+a_k\right)K_b^2\right.\\
&\qquad\left.\times\sup\left\{\mathbb{E}\|\overline{u}(\tau)-\overline{v}(\tau)\|^2,0\leq\tau\leq t\right\}\right]\\
&\leq 2\widetilde{M}_T^2\left[\sup_{0\leq t\leq b}\mathbb{E}\|\overline{u}(t)-\overline{v}(t)\|^2+2\left(\varepsilon_k+a_k\right)K_b^2\right.\\
&\qquad\left.\times\sup_{0\leq s\leq b}\mathbb{E}\|\overline{u}(s)-\overline{v}(s)\|^2\right]\\
&=2\widetilde{M}_T^2\left[\sup_{0\leq t\leq b}\mathbb{E}\|u(t)-v(t)\|^2+2\left(\varepsilon_k+a_k\right)K_b^2\right.\\
&\qquad\left.\times\sup_{0\leq s\leq b}\mathbb{E}\|u(s)-v(s)\|^2\right]\quad\text{since }\overline{u}=u,\ \overline{v}=v\ \text{ on }\ J\\
&=2\widetilde{M}_T^2\left[1+2\left(\varepsilon_k+a_k\right)K_b^2\right]\|u-v\|_{\mathscr{PC}}^2.
\end{aligned}\tag{53}$$

Thus, for all $t\in[0,b]$, we have

$$\left\|(\Phi_1u)(t)-(\Phi_1v)(t)\right\|_{\mathscr{PC}}^2\leq l_0\|u-v\|_{\mathscr{PC}}^2,\tag{54}$$

where $l_0=\max_{1\leq k\leq m}\{2\widetilde{M}_T^2(1+2K_b^2(\varepsilon_k+a_k))\}<1$. Hence Φ_1 is a contraction on $\mathscr{BPC}$.

Claim 2. Φ_2 is convex for each $x\in\overline{V}$. Indeed, if z_2^1,z_2^2 belong to Φ_2x, then there exist $\sigma_1,\sigma_2\in\mathcal{S}_{\Sigma,\overline{z}_\rho}$ such that

$$\begin{aligned}
z_2^i(t)&=T_\alpha(t)\phi(0)+\int_0^tS_\alpha(t-s)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\\
&\quad+\int_0^tS_\alpha(t-s)\sigma_i(s)dw(s),\quad t\in[0,t_1],\ i=1,2.
\end{aligned}\tag{55}$$

Let $0\leq\lambda\leq1$. For each $t\in[0,t_1]$, we have

$$\begin{aligned}
&\left(\lambda z_2^1+(1-\lambda)z_2^2\right)(t)\\
&\quad=T_\alpha(t)\phi(0)+\int_0^tS_\alpha(t-s)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\\
&\qquad+\int_0^tS_\alpha(t-s)\left[\lambda\sigma_1(s)+(1-\lambda)\sigma_2(s)\right]dw(s).
\end{aligned}\tag{56}$$

Similarly, for $0\leq\lambda\leq1$ and any $t\in(t_k,t_{k+1}]$, $k=1,\ldots,m$, we have

$$\begin{aligned}
&\left(\lambda z_2^1+(1-\lambda)z_2^2\right)(t)\\
&\quad=\int_0^tS_\alpha(t-s)\left[\lambda\sigma_1(s)+(1-\lambda)\sigma_2(s)\right]dw(s),
\end{aligned}\tag{57}$$

where

$$\begin{aligned}
z_2^i(t)&=\int_0^tS_\alpha(t-s)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\\
&\quad+\int_0^tS_\alpha(t-s)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\\
&\quad+\int_0^tS_\alpha(t-s)\sigma_i(s)dw(s),\quad i=1,2.
\end{aligned}\tag{58}$$

Since $\mathcal{S}_{\Sigma,\overline{z}_\rho}$ is convex (because Σ has convex values), we have $(\lambda z_2^1+(1-\lambda)z_2^2)\in\Phi_2x$.

Claim 3. $\Phi_2(\overline{V})$ is completely continuous. First, We need to show that $\Phi_2(\overline{V})$ is equicontinuous. Let $x\in\overline{V}$. Then, from Lemmas 9 and 15, it follows that

$$\begin{aligned}
\left\|\overline{z}_{\rho(s,\overline{z}_s)}\right\|_{\mathscr{B}}^2&\leq2\left[\left(N_b+\mathscr{W}_0^\phi\right)\mathbb{E}\|\phi\|_{\mathscr{B}}\right]^2\\
&\quad+2K_b^2r^*:=r'.
\end{aligned}\tag{59}$$

Let $0<\tau_1<\tau_2\leq t_1$. For each $x\in\overline{V}$, $h_2\in\Phi_2x$, there exists $\sigma\in\mathcal{S}_{\Sigma,\overline{z}_\rho}$, such that

$$\begin{aligned}
z_2(t)&=T_\alpha(t)\phi(0)+\int_0^tS_\alpha(t-s)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\\
&\quad+\int_0^tS_\alpha(t-s)\sigma(s)dw(s).
\end{aligned}\tag{60}$$

Then

$$\begin{aligned}
&\mathbb{E}\|z_2(\tau_2)-z_2(\tau_1)\|^2\\
&\leq7\mathbb{E}\|(T_\alpha(\tau_2)-T_\alpha(\tau_1))\phi(0)\|^2\\
&\quad+7\mathbb{E}\left\|\int_0^{\tau_1-\epsilon}\left(S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\right)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\right\|^2\\
&\quad+7\mathbb{E}\left\|\int_{\tau_1-\epsilon}^{\tau_1}\left(S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\right)f\left(s,\overline{z}_{\rho(s,\overline{z}_s)}\right)ds\right\|^2\\
&\quad+7\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}S_\alpha(\tau_2-s)f(s,\overline{z}_{\rho(s,\overline{z}_s)})ds\right\|^2\\
&\quad+7\mathbb{E}\left\|\int_0^{\tau_1-\epsilon}\left(S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\right)\sigma(s)dw(s)\right\|^2\\
&\quad+7\mathbb{E}\left\|\int_{\tau_1-\epsilon}^{\tau_1}\left(S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)\right)\sigma(s)dw(s)\right\|^2\\
&\quad+7\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}S_\alpha(\tau_2-s)\sigma(s)dw(s)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq 7\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) b^{1-\alpha} \\
&\quad \times \int_0^{\tau_1-\epsilon} \left\|S_\alpha\left(\tau_2-s\right)-S_\alpha\left(\tau_1-s\right)\right\|^2 \left(\tau_1-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 7\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) \int_{\tau_1-\epsilon}^{\tau_1} \left(\tau_2-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 7\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) \int_{\tau_1-\epsilon}^{\tau_1} \left(\tau_1-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 7\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) \int_{\tau_1}^{\tau_2} \left(\tau_2-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 7\widetilde{M}_S^2 \left(c_1+c_2 r'\right) b^{2(1-\alpha)} \operatorname{Tr}(Q) \\
&\quad \times \int_0^{\tau_1-\epsilon} \left\|S_{\alpha-1}\left(\tau_2-s\right)-S_\alpha\left(\tau_1-s\right)\right\|^2 \\
&\qquad \times \left(\tau_1-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds \\
&\quad + 7\widetilde{M}_S^2 \left(c_1+c_2 r'\right) \operatorname{Tr}(Q) \int_{\tau_1-\epsilon}^{\tau_1} \left(\tau_2-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds \\
&\quad + 7\widetilde{M}_S^2 \left(c_1+c_2 r'\right) \operatorname{Tr}(Q) \int_{\tau_1-\epsilon}^{\tau_1} \left(\tau_1-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds \\
&\quad + 7\widetilde{M}_S^2 \left(c_1+c_2 r'\right) \operatorname{Tr}(Q) \int_{\tau_1}^{\tau_2} \left(\tau_2-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds.
\end{aligned} \tag{61}
$$

Similarly, for any $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, \ldots, m$, we have

$$
\begin{aligned}
z_2(t) = &\int_0^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds \\
&+ \int_0^t S_\alpha(t-s)\, \sigma(s)\, dw(s).
\end{aligned} \tag{62}
$$

Then

$$
\begin{aligned}
&\mathbb{E}\left\|z_2(\tau_2)-z_2(\tau_1)\right\|^2 \\
&\leq 6\mathbb{E}\left\|\int_{t_k}^{\tau_1-\epsilon} \left(S_\alpha\left(\tau_2-s\right)-S_\alpha\left(\tau_1-s\right)\right) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_{\tau_1-\epsilon}^{\tau_1} (S_\alpha(\tau_2-s)-S_\alpha(\tau_1-s)) f(s, \overline{z}_{\rho(s,\overline{z}_s)}) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_{\tau_1}^{\tau_2} S_\alpha(\tau_2-s) f(s, \overline{z}_{\rho(s,\overline{z}_s)}) ds\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_0^{\tau_1-\epsilon} \left(S_\alpha\left(\tau_2-s\right)-S_\alpha\left(\tau_1-s\right)\right) \sigma(s)\, dw(s)\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_{\tau_1-\epsilon}^{\tau_1} \left(S_\alpha\left(\tau_2-s\right)-S_\alpha\left(\tau_1-s\right)\right) \sigma(s)\, dw(s)\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_{\tau_1}^{\tau_2} S_\alpha\left(\tau_2-s\right) \sigma(s)\, dw(s)\right\|^2 \\
&\leq 6\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) b^{1-\alpha} \\
&\quad \times \int_{t_k}^{\tau_1-\epsilon} \left\|S_\alpha(\tau_2-s)-S_\alpha\left(\tau_1-s\right)\right\|^2 \left(\tau_1-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 6\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) \int_{\tau_1-\epsilon}^{\tau_1} \left(\tau_2-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 6\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) \int_{\tau_1-\epsilon}^{\tau_1} \left(\tau_1-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 6\widetilde{M}_S^2 M_f \frac{b^\alpha}{\alpha}\left(1+r'\right) \int_{\tau_1}^{\tau_2} \left(\tau_2-\epsilon-s\right)^{\alpha-1} ds \\
&\quad + 6\widetilde{M}_S^2 \left(c_1+c_2 r'\right) b^{2(1-\alpha)} \operatorname{Tr}(Q) \\
&\quad \times \int_{t_k}^{\tau_1-\epsilon} \left\|S_{\alpha-1}\left(\tau_2-s\right)-S_\alpha\left(\tau_1-s\right)\right\|^2 \\
&\qquad \times \left(\tau_1-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds \\
&\quad + 6\widetilde{M}_S^2 \left(c_1+c_2 r'\right) \operatorname{Tr}(Q) \int_{\tau_1-\varepsilon}^{\tau_1} \left(\tau_2-\epsilon-s\right)^{\alpha-1} \nu(s)\, ds \\
&\quad + 6\widetilde{M}_S^2 \left(c_1+c_2 r'\right) \operatorname{Tr}(Q) \int_{\tau_1-\varepsilon}^{\tau_1} \left(\tau_1-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds \\
&\quad + 6\widetilde{M}_S^2 \left(c_1+c_2 r'\right) \operatorname{Tr}(Q) \int_{\tau_1}^{\tau_2} \left(\tau_2-\epsilon-s\right)^{2(\alpha-1)} \nu(s)\, ds.
\end{aligned} \tag{63}
$$

Therefore, from the above inequalities, for ϵ sufficiently small, the right-hand side of $\mathbb{E}\|z_2(\tau_2)-z_2(\tau_1)\|^2$ tends to zero as $\tau_2-\tau_1 \to 0$, since I_k, $k = 1, \ldots, m$, are completely continuous in $\mathscr{H}$ and the compactness of $S_\alpha(t)$ for $t > 0$ (S_α is generated by the dense operator A) implies the continuity in the uniform operator topology. Thus the set $\{\Phi_2 x : x \in \overline{V}\}$ is equicontinuous.

Second, we show that $\phi_2(\overline{V})$ is relatively compact for every $t \in [0, b]$.

Let $0 < t \leq s \leq t_1$ be fixed, and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in \overline{V}$, we define

$$
\begin{aligned}
z_{2,\epsilon}(t) = &T_\alpha(t)\phi(0) + \int_0^{t-\epsilon} S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds \\
&+ \int_0^{t-\epsilon} S_\alpha(t-s)\, \sigma(s)\, dw(s),
\end{aligned} \tag{64}
$$

where $\sigma \in \mathscr{S}_{\Sigma,\overline{z}_\rho}$. Using the compactness of $T_\alpha(t)$ and $S_\alpha(t)$ for $t > 0$, we deduce that the set $U_\epsilon(t) = \{z_{2,\epsilon}(t) : x \in \overline{V}\}$ is relatively compact in $\mathscr{H}$ for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in \overline{V}$, we have

$$
\begin{aligned}
&\mathbb{E}\left\|z_2(t)-z_{2,\epsilon}(t)\right\|^2 \\
&\quad \leq 2\mathbb{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds\right\|^2 \\
&\qquad + 2\mathbb{E}\left\|\int_{t-\epsilon}^t S_\alpha(t-s)\sigma(s) dw(s)\right\|^2
\end{aligned}
$$

$$\leq 2\widetilde{M}_S^2 \frac{\epsilon^{2\alpha}}{\alpha^2} M_f \left(1 + r'\right) + 2\widetilde{M}_S^2 \left(c_1 + c_2 r'\right) \operatorname{Tr}(Q) \times \int_{t-\epsilon}^{t} (t-s)^{2(\alpha-1)} \nu(s)\, ds. \tag{65}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, let $t_k < t \leq s \leq t_{k+1}$ be fixed, and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in \overline{V}$, we define

$$z_{2,\epsilon}(t) = T_\alpha(t)\phi(0) + \int_{t_k}^{t-\epsilon} S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds + \int_{t_k}^{t-\epsilon} S_\alpha(t-s)\sigma(s)\, dw(s), \tag{66}$$

where $\sigma \in \mathcal{S}_{\Sigma,\overline{z}_\rho}$. From the compactness of $T_\alpha(t)$ and $S_\alpha(t)$ for $t > 0$, we deduce that the set $U_\epsilon(t) = \{z_{2,\epsilon}(t) : x \in \overline{V}\}$ is relatively compact in $\mathcal{H}$ for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in \overline{V}$, we have

$$\begin{aligned}
&\mathbb{E}\left\|z_2(t) - z_{2,\epsilon}(t)\right\|^2 \\
&\quad \leq 2\mathbb{E}\left\|\int_{t-\epsilon}^{t} S_\alpha(t-s) f\left(s, \overline{z}_{\rho(s,\overline{z}_s)}\right) ds\right\|^2 \\
&\qquad + 2\mathbb{E}\left\|\int_{t-\epsilon}^{t} S_\alpha(t-s)\sigma(s)\, dw(s)\right\|^2 \\
&\quad \leq 2\widetilde{M}_S^2 \frac{\epsilon^{2\alpha}}{\alpha^2} M_f \left(1 + r'\right) + 2\widetilde{M}_S^2 \left(c_1 + c_2 r'\right) \operatorname{Tr}(Q) \\
&\qquad \times \int_{t-\epsilon}^{t} (t-s)^{2(\alpha-1)} \nu(s)\, ds.
\end{aligned} \tag{67}$$

The right-hand side of the above inequality tends to zero as $\epsilon \to 0$. This implies that there are relatively compact sets arbitrarily close to the set $U(t) = \{z_1(t) : x \in \overline{V}\}$. Hence $U(t)$ is relatively compact in $\mathcal{H}$. By Arzelá-Ascoli theorem, we conclude that the operator $\Phi_2(\overline{V})$ is completely continuous.

As a consequence of the above Claims 1–3, we conclude that Φ is a condensing map. All of the conditions of Lemma 11 are satisfied; we deduce that Φ has a fixed point x in $\mathcal{BPC}$ which is a mild solution of the problem (3). □

4. An Example

To apply our abstract results, we consider the following impulsive fractional stochastic partial differential inclusions with state-dependent delay of the form

$$\begin{aligned}
&D_t^\alpha u(t,x) - \frac{\partial^2}{\partial x^2} u(x,t) \\
&\quad \in \int_{-\infty}^{t} \mu_1(t,x,s-t)\, u\left(s - \rho_1(t)\rho_2(\|u(t)\|), x\right) ds \\
&\qquad + \left[\int_{-\infty}^{t} \mu_2(t,x,s-t) \times u\left(s - \rho_1(t)\rho_2(\|u(t)\|), x\right) ds\right] \frac{d\beta(t)}{dt}, \\
&\qquad\qquad 0 \leq t \leq b,\ 0 \leq x \leq \pi, \\
&u(t,0) = u(t,\pi) = 0, \quad 0 \leq t \leq b, \\
&u(\tau,x) = \phi(\tau,x), \quad \tau \leq 0,\ 0 \leq x \leq \pi, \\
&\Delta u(t_k,x) = \int_{-\infty}^{t_k} \nu_k(s-t_k)\, u(s,x)\, ds, \quad k = 1, \ldots, m,
\end{aligned} \tag{68}$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$; D_t^α is the Caputo fractional derivative of order $0 < \alpha < 1$; ϕ is continuous; and $0 < t_1 < t_2 < \cdots < t_m < b$ are prefixed numbers.

Let $\mathcal{H} = L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ by $Ay = y''$ with the domain

$$\begin{aligned}
\mathcal{D}(A) = \{&y \in \mathcal{H}; y, y' \text{ are absolutely continuous}, \\
&y'' \in \mathcal{H},\ y(0) = y(\pi) = 0\}.
\end{aligned} \tag{69}$$

Then, $Ay = \sum_{n=1}^{\infty} n^2 (y, y_n) y_n$, $y \in \mathcal{D}(A)$, where $y_n(x) = \sqrt{(2/\pi)}\sin(nx)$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathcal{H}$ is given by

$$T(t)\, y = \sum_{n=1}^{\infty} e^{-n^2 t} (y, y_n)\, y_n, \quad \forall y \in \mathcal{H},\ t > 0. \tag{70}$$

It follows from the above expressions that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all λ in the resolvent set of A.

$r \geq 0$, $p \geq 1$ and let $z : (-\infty, r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (H-5) and (H-6) in the terminology of Hino et al. [34]. Briefly, this means that z is locally integrable and there is a non-negative, locally bounded function h on $(-\infty, 0]$ such that $z(\xi + \tau) \leq h(\xi) z(\tau)$ for all $\tau \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue's measure is zero. Let $\mathcal{PC}_r \times L^2(z, \mathcal{H})$ be the set consisting of all classes of functions $\phi : (-\infty, 0] \to \mathcal{H}$ such that $\phi_{|[-r,0]} \in \mathcal{PC}([-r,0], \mathcal{H})$, $\phi(\cdot)$ is Lebesgue measurable on $(-\infty, -r)$, and $z\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm is given by

$$\|\phi\|_{\mathcal{B}} = \sup_{-r\leq\tau\leq 0} \|\phi(\tau)\| + \left(\int_{-\infty}^{-r} z(\tau) \|\phi\|^p d\tau\right)^{1/p}. \tag{71}$$

$\mathcal{B} = \mathcal{PC}_r \times L^2(z, \mathcal{H})$ satisfies the fundamental axioms given in Section 2. When $r = 0$ and $p = 2$, we can take $H = 1$, $N(t) = h(-t)^{1/2}$, and $k(t) = 1 + \left(\int_{-t}^{0} z(\tau) d\tau\right)^{1/2}$, for $t \geq 0$ (see [34]).

Here, we assume that

(i) the functions $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous;

(ii) the functions $\mu_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2$, are continuous with $l_i = (\int_{-t}^{0}((\mu_i(s))^2/z(s))ds)^{1/2} < \infty$;

(iii) the functions $\nu_k : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\ldots,m$, are continuous with $L_k = (\int_{-t}^{0}((\mu_i(s))^2/z(s))ds)^{1/2} < \infty$ for every $k = 1,2,\ldots,m$,

and $\mathscr{B}$ will be the phase space $\mathscr{PC}_0 \times L^2(z,\mathscr{H})$. Set $\phi(\theta)(x) = \phi(\theta,x) \in \mathscr{B}$. Define $f : [0,b] \times \mathscr{B} \to \mathscr{H}$, $\Sigma : [0,b] \times \mathscr{B} \to \mathscr{P}(\mathscr{H})$ by

$$\begin{gathered} f(t,\phi)(x) = \int_{-\infty}^{0} \mu_1(t,\theta,x)\,\phi(\theta)(x)\,d\theta, \\ \Sigma(t,\phi)(x) = \int_{-\infty}^{0} \mu_2(t,\theta,x)\,\phi(\theta)(x)\,d\theta, \\ \rho(t,\phi) = \rho_1(t)\,\rho_2(\|\phi(0)\|). \end{gathered} \tag{72}$$

Thus, f, Σ are bounded operators on $\mathscr{B}$ with $\|f\| \le l_1$, $\|\Sigma\| \le l_2$ and $\|I_k\| \le L_k$, $k = 1,2,\ldots,m$. Therefore, the problem (4) can be written in the abstract form of (3). All conditions of Theorem 17 are now fulfilled, so we deduce that the system (4) has a mild solution on $[0,b]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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