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Research Article

Wavelets Convergence and Unconditional Bases for $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$

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We prove that reasonable nice wavelets form unconditional bases in function space other than $L^2(\mathbb{R}^n, X)$. Moreover, characterization of convergence of wavelets series in $L^p(\mathbb{R}^n, X)$ space and Hardy space $H^1(\mathbb{R}^n, X)$ has been obtained. Here, X is a Banach space with boundedness of Riesz transform.

1. Introduction

Let X be a separable Banach space and $(e_n)_{n\in\mathbb{N}}$ a sequence of vectors in X. We say that $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis for X if for every $x \in X$ there exists a unique sequence of scalars $(c_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n\in\mathbb{N}} c_n e_n, \tag{1}$$

where the series converges in the norm and as the net of sums over finite subsets of $\mathbb{N}$. In other words we require that $\sum_{n=1}^{\infty} c_{\sigma(n)} e_{\sigma(n)}$ converges for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$. Again, a consequence of the uniform boundedness principle is that the coefficient functionals $f_n : x \mapsto c_n$ are continuous. If X is reflexive, then f_n is an unconditional basis for the dual space X^*.

Wavelet systems provide explicit unconditional bases for many function spaces. For instance, any orthonormal wavelet ψ such that ψ and ψ' have a common radial decreasing L^1-majorant also generates an unconditional basis for $L^p(R)$, $1 < p < \infty$, and $H^1(\mathbb{R})$, although it has less regularity.

It can be shown [1] that if two coefficient spaces are equal (as subsets of $\mathbb{C}^{\mathbb{N}}$), then their norms are equivalent and corresponding Banach spaces $L^p(\mathbb{R})$, $1 < p < \infty$, and Hardy space $H^1(\mathbb{R})$ are (topologically) isomorphic through an isomorphism of one basis to another.

In this paper we will consider the spaces $L^p(\mathbb{R})$, $1 < p < \infty$, and $H^1(\mathbb{R}^n)$ and try to study that the reasonable nice wavelets form unconditional bases in these function spaces. Also, the convergence of wavelet series in these spaces will be characterized. There are several equivalent definitions of real Hardy space $H^1(\mathbb{R}^n)$. In the real variable theory, one finds (at least) four major types of characterizations of H^1: integrability of maximal functions, integrability of square functions, integrability of conjugate functions (the Riesz transform or its variants), and atomic decompositions.

Let T be a classical Calderón-Zygmund singular integral operator on $\mathbb{R}^n$ with smooth kernel K and let T^* be the associated maximal singular integral

$$T^* f(x) = \sup_{\varepsilon>0} \left|T^\varepsilon f(x)\right|, \tag{2}$$

where $T^\varepsilon f(x)$ is the truncation at level ε:

$$T^\varepsilon f(x) = \int_{|y-x|>\varepsilon} f(x-y) K(y)\, dy. \tag{3}$$

The jth Riesz transform, $1 \le j \le n$, is the singular integral operator

$$\begin{aligned} \mathbb{R}_j f(x) &= \text{P.V.} \int f(x-y) \frac{y_j}{|y|^{n+1}} dy \\ &\equiv \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}} dy. \end{aligned} \tag{4}$$

The principal value integral above exists for all x if f is a compactly supported smooth function and one shows for such functions the $L^p(\mathbb{R}^n)$ estimates:

$$\left\|\mathbb{R}_j f\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \tag{5}$$

for some positive constant C independent of f. Then a bounded operator R_j can be defined on $L^p(\mathbb{R}^n)$ by the density argument. A subissue arises when one tries to show that the principal value in (4) exists for almost all x in $\mathbb{R}^n$ for each function f in $L^p(\mathbb{R}^n)$. Following a well-known principle, one looks for $L^p(\mathbb{R}^n)$ estimates for the maximal Riesz transform $\mathbb{R}_j^* f$ and one indeed proves that [2, 3]

$$\left\|\mathbb{R}_j^* f\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{6}$$

The following results [4] improve inequality (6).

For $1 < p \leq \infty$ there exists a constant $C = C_{p,n}$ such that

$$\left\|\mathbb{R}_j^* f\right\|_{L^p(\mathbb{R}^n)} \leq C\left\|\mathbb{R}_j f\right\|_{L^p(\mathbb{R}^n)} \tag{7}$$

for each function f belonging to some $L^q(\mathbb{R}^n)$, $1 \leq q \leq \infty$.

Inequality (7) still holds in the limiting case $p = 1$ and the answer is provided by next result.

Given j, $1 \leq j \leq n$, and a positive constant C, there exists a function f in $L^1(\mathbb{R}^n)$ such that

$$\left\|\mathbb{R}_j^* f\right\|_{(L^1,L^\infty)(\mathbb{R}^n)} \geq C\left\|\mathbb{R}_j f\right\|_{L^1(\mathbb{R}^n)}. \tag{8}$$

Remark 1. Inequality (8) does not hold even for the Hilbert transform, which is the case $n = 1$.

Stein and Weiss originally defined the real Hardy space $H^1(\mathbb{R})$ to be

$$\begin{gathered} H^1(\mathbb{R}) = \left\{f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R})\right\}, \\ \|f\|_{H^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}, \end{gathered} \tag{9}$$

where H denotes the Hilbert transform $(Hf)(x) = \text{P.V.} \int_{\mathbb{R}} (f(x-t)/\pi t)dt$ for $n > 1$, and we use Riesz transform.

In order to define real Hardy space $H^1(\mathbb{R}^n)$, set $b = (1, 0, \ldots, 0)$ and $a = (-1, 0, \ldots, 0)$ and let μ be the length measure on the segment joining a and b. For an appropriate constant C_n we have

$$\mu = C_n\left(\frac{1}{|x|^{n-1}} * \sum_{j=1}^{n} \mathbb{R}_j\left(\partial_j \mu\right)\right). \tag{10}$$

Taking the Fourier transforms on both sides, we obtain

$$\delta_a - \delta_b = \partial_1 \mu = \mathbb{R}_1\left(C_n \sum_{j=1}^{n} \mathbb{R}_j\left(\partial_j \mu\right)\right). \tag{11}$$

Let φ be a nonnegative continuously differentiable and compactly supported function in the unit ball such that $\widehat{\varphi}(0) = 1$, and set the approximate identity as

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right). \tag{12}$$

Convolving identity (10) by approximate identity φ_ε, we get

$$\varphi_\varepsilon(x-a) - \varphi_\varepsilon(x-b) = \mathbb{R}_1\left(C_n \sum_{j=1}^{n} \mathbb{R}_j\left(\partial_j \mu\right) * \varphi_\varepsilon\right). \tag{13}$$

Set $f_\varepsilon = C_n \sum_{j=1}^{n} \mathbb{R}_j(\partial_j \mu) * \varphi_\varepsilon = C_n \sum_{j=1}^{n} \mathbb{R}_j(\mu * \partial_j \varphi_\varepsilon)$.

Since $\mu * \partial_j \varphi_\varepsilon$ is a compactly supported function in $L^\infty(\mathbb{R}^n)$ with zero integral, thus $\mu * \partial_j \varphi_\varepsilon$ is a function in the Hardy space $H^1(\mathbb{R}^n)$ and so $f_\varepsilon \in L^1(\mathbb{R}^n)$. From (13) we see that

$$\|(\mathbb{R}_1)(f_\varepsilon)\|_1 \leq 2. \tag{14}$$

Now we define

$$\begin{gathered} H^1(\mathbb{R}^n) = \left\{f \in L^1(\mathbb{R}^n) : \mathbb{R}_j f \in L^1(\mathbb{R}^n)\right\}, \\ \|f\|_{H^1(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)} + \left\|\mathbb{R}_j f\right\|_{L^1(\mathbb{R}^n)}, \end{gathered} \tag{15}$$

where $\mathbb{R}_j$ denotes the Riesz transform.

The Hardy space $H^1(\mathbb{R}^n, X)$ is initially defined in terms of the atomic decomposition as follows.

Atoms are those measurable functions $a_i : \mathbb{R}^n \to \mathbb{C}$ for which there exist balls $\overline{B}_i$ in $\mathbb{R}^n$ such that

$$\begin{gathered} \operatorname{supp}(a_i) \subset \overline{B}_i, \qquad \int a_i(x)\,dx = 0, \\ \sum_{i=1}^{\infty} \|a_i\|_{L^p(\mathbb{R}^n)} |B_i|^{1/p} < \infty \end{gathered} \tag{16}$$

for some value of $p \in]1, \infty[$ being fixed and p' denotes the conjugate exponent, $1/p + 1/p' = 1$.

We say that $f \in H^1(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$ has an expansion of the form $f(x) = \sum_{i=1}^{\infty} a_i(x)$, and the norm is defined as

$$\|f\|_{H^1(\mathbb{R}^n)} = \inf\left\{\sum_{i=1}^{\infty} \|a_i\|_{L^p(\mathbb{R}^n)} \left|\overline{B}_i\right|^{1/p'}\right\}. \tag{17}$$

Atomic Hardy space is a Banach space

$$H^1(\mathbb{R}^n) = \left\{\sum_{i=1}^{\infty} a_i x : a_i \text{ are atoms}\right\}. \tag{18}$$

Notice that $H^1(\mathbb{R}^n) \subseteq \{f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f = 0\}$, but the converse inclusion is false.

If in the definition of atoms we consider only dyadic intervals in place of balls in $\mathbb{R}^n$, we obtain a "smaller" space called the dyadic real Hardy space $H^1_{\text{dyadic}}(\mathbb{R})$. The spaces $H^1(\mathbb{R})$ and $H^1_{\text{dyadic}}(\mathbb{R})$ are (topologically) isomorphic Banach spaces. This isomorphism leads to a nonconstructive proof of the existence of an unconditional basis for $H^1_{\mathbb{R}}$. Namely, the Haar system provides a perfect unconditional basis for $H^1_{\text{dyadic}}(\mathbb{R})$, because no regularity is needed in the dyadic setting. In this paper our approach is different and based on the fact that smoother wavelets will directly provide an unconditional basis for $H^1(\mathbb{R}^n)$; we will characterize the coefficient space by taking the wavelet basis in $H^1(\mathbb{R}^n)$.

It should be significant to mention here that it would be better to obtain constants depending only on p and regularity of wavelet, but no other information about it. For that reason, a better approach is to prove square function estimates using Calderón-Zygmund decomposition and real interpolation [5].

A Calderón-Zygmund operator is a bounded linear operator $T^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with operator norm at most k and such that there exists a $\mathbb{C}^1$-function $k(x, y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$\begin{gathered} |k(x,y)| \le k|x-y|^{-n}, \\ |\nabla_x k(x,y)| + |\nabla_y k(x,y)| \le k|x-y|^{-n-1}, \\ T^* f(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy, \\ \text{for } f \in L_c^2(\mathbb{R}^n), \quad x \notin \operatorname{supp}(f). \end{gathered} \tag{19}$$

A function with these properties is called a Calderón-Zygmund kernel or a standard singular kernel.

2. Some Definitions and Auxiliary Results

Definition 2. A Banach space X is UMD if for some (and then all, cf. [6]) $1 < p < \infty$ there is a finite constant C so that

$$\left\| \sum_{k=1}^{n} \varepsilon_k d_k \right\|_{L^p(\Omega,X)} \le C \left\| \sum_{k=1}^{n} d_k \right\|_{L^p(\Omega,X)} \tag{20}$$

for all $n \in \mathbb{Z}_+$, whenever $(\varepsilon_k)_{k=1}^n \in \{-1,+1\}^n$ and $(d_k)_{k=1}^n \in L^p(\Omega, X)^n$, a martingale difference sequence or an arbitrary probability space (Ω, A, μ) (i.e., there are sub-σ-algebras $A_0 \subset A_1 \subset \cdots \subset A_n \subset A$), such that, for all $k = 1, \ldots, n$, the function d_k is A_k-measurable and $\int_A d_k d\mu = 0$ for all $A \in A_{k-1}$.

Remark 3. The validity of Theorem 1.9 of [7] on $H^1(\mathbb{R}^n, X)$ actually characterizes the UMD-property of X. Indeed, let X be any complex Banach space and let conclusion of Theorem 1.9 of [7] be satisfied. Since $\|\mathbb{R}_j f\|_{L^p(\mathbb{R}^n,X)} \le C\|f\|_{L^p(\mathbb{R}^n,X)}$, this is equivalent to the UMD-property of X. We also have $\|\mathbb{R}_j f\| \le C\|f\|_{L^1(\mathbb{R}^n,X)}$. This implies that $\mathbb{R}_j$ is bounded on $H^1(\mathbb{R}^n, X)$.

In view of above remark we can get without modification in [8] the following.

Proposition 4. *Let $k(x, y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the standard estimates*

$$\begin{gathered} |k(x,y)| \le k|x-y|^{-n}, \\ |\nabla_x k(x,y)| + |\nabla_y k(x,y)| \le k|x,y|^{-n-1}. \end{gathered} \tag{21}$$

Assume, moreover, that T is bounded on $L^2(\mathbb{R}^n)$ with operator norm at most k. Then T is also bounded on $L^p(\mathbb{R}^n, X)$, where

$$\|\mathbb{R}_j f\|_{L^p(\mathbb{R}^n,X)} \le C_{p,n}(X)\, k \|f\|_{L^p(\mathbb{R}^n,X)} \tag{22}$$

for all $p \in]1, \infty[$, and it is bounded from $H^1(\mathbb{R}^n, X)$ to $L^1(\mathbb{R}^n, X)$ with norm $C_{1,n}(X)k$. If, in addition,

$$[T1](y) = \int_{\mathbb{R}^n} k(x,y)\, dx \equiv 0, \tag{23}$$

then T is bounded on $H^1(\mathbb{R}^n, X)$ with norm $\le C_0(X)k$.

Let us note that atomic definition of $H^1(\mathbb{R}^n, X)$ is known to agree with the one given in terms of various maximal functions, even for an arbitrary Banach space X. One can check that the proof of this fact in the scalar case, as given, for example, in Stein's book [9], goes through word by word in the general setting for $n = 1$; the conjugate Hardy space defined as the domain of the Hilbert transform on $L^1(\mathbb{R}, X)$ with the graph norm is always contained in the atomic Hardy space and agrees with it exactly when X is a UMD-space. For $n > 1$, we can replace Hilbert transform by Riesz transform and the condition of UMD-space for X by the boundedness of Riesz transform in $L^p(\mathbb{R}^n)$-space.

It is well-known fact that an integral operator satisfying the standard estimates and bounded on $L^p(\mathbb{R}^n, X)$ is also bounded from $H^1(\mathbb{R}^n, X)$ to $L^1(\mathbb{R}^n, X)$. The square function description of $H^1(\mathbb{R}^n, X)$ that we have in mind involves the wave expansion of a function in $H^1(\mathbb{R}^n)$. Recall the definition of the wavelet [10–12]. Let $\{\varphi^l : l = 1, 2, \ldots, 2^n - 1\}$ be a set of functions belonging to $L^2(\mathbb{R}^n)$.

Consider

$$\begin{aligned} \varphi_{j,k}^l(x) &= 2^{jn/2} \varphi^l\left(2^j x - k\right) \\ &= 2^{jn/2} \varphi^l\left(2^j x_1 - k_1, \ldots, 2^j x_n - k_n\right) \\ &\qquad (x = (x_1, \ldots, x_n) \in \mathbb{R}^n) \end{aligned} \tag{24}$$

for each $l = 1, 2, \ldots, 2^n - 1$, $j \in \mathbb{Z}$, and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. The sequence $\{\varphi^l : l = 1, 2, \ldots, 2^n - 1\}$ is a wavelet set if $\{\varphi_{j,k}^l : l = 1, 2, \ldots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ forms an orthonormal basis in $L^2(\mathbb{R}^n)$. Then $\{\varphi_{j,k}^l : l = 1, 2, \ldots, 2^n - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ is a wavelet basis in $L^2(\mathbb{R}^n)$ and each φ^l is a wavelet. The basis is called r-regular if $|\partial^\alpha \varphi^l(x)| \le C_m(1 + |x|)^{-m}$ and $\int x^\alpha \varphi^l(x) dx = 0$ for all $|\alpha| \le r$, all $m \in N$, and $\{l = 1, 2, \ldots, 2^n - 1\}$.

The aim of the present paper is to study that reasonable nice wavelets form unconditional bases in function space other than $L^2(\mathbb{R}^n)$ and characterization of convergence of wavelets series in $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$ has been obtained. For this purpose it will be enough to assume that $(\varphi_A)_{A \in \Lambda}$, where Λ is the set of dyadic cube of the form

$$Q_{j,k} = \Pi_{i=1}^n \left[2^{-j} k_i, 2^{-j}(k_i + 1)\right], \tag{25}$$

is an orthonormal wavelet basis such that $\varphi \in \mathbb{C}^1(\mathbb{R}^n)$ and

$$|\varphi(x)| + |\varphi'(x)| \le C(1 + |x|)^{-n-2}. \tag{26}$$

This is certainly not the weakest possible assumption on φ for somewhat less restrictive; see [10] (where the class $\mathbb{R}^0(\mathbb{R}^n)$ is introduced).

3. Main Results

Theorem 5. *Let $1 < p < \infty$ and let φ_A be a wavelet such that $\varphi \in \mathbb{C}^1(\mathbb{R}^n)$ and satisfies condition (26). Then the system $(\varphi_A)_{A\in\Lambda}$ is an unconditional basis for $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$.*

Proof. For arbitrary family of $\nu_A \in \mathbb{C}$, $A \in \Lambda$ such that $|\nu_A| \leq 1$, we define T_{ν_A} on the orthonormal basis by

$$T_{\nu_A} : L^2(\mathbb{R}^n, X) \longrightarrow L^2(\mathbb{R}^n, X), \qquad T_{\nu_A} : \varphi_A \longrightarrow \nu_A \varphi_A. \tag{27}$$

Here T_{ν_A} is not necessarily unitary, but it is bounded and

$$\left\| T_{\nu_A} \right\|_{L^2 \to L^2} \leq 1. \tag{28}$$

Now we can write

$$\begin{aligned} \left(T_{\nu_A} f\right)(x) &= \sum_{A\in\Lambda} \nu_A \langle f, \varphi_A \rangle \varphi_A(x) \\ &= \int_{\mathbb{R}^n} k_{\nu_A}(x, y) f(y)\, dy, \end{aligned} \tag{29}$$

where

$$\begin{aligned} k_{\nu_A}(x, y) &= \sum_{A\in f} \nu_A \varphi_A(x) \overline{\varphi_A(y)} \\ &= \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} \nu_{j,k} 2^{nj} \varphi\left(2^j x - k\right) \overline{\varphi\left(2^j y - k\right)}, \end{aligned} \tag{30}$$

where $F \subset \Lambda$ is any finite set.

We claim that T_{ν_A} are Calderón-Zygmund operators with constants uniform in ν_A and for that we have to show that k_{ν_A} satisfy standard estimates uniformly in ν_A.

Consider

$$\begin{aligned} \left|k_{\nu_A}(x, y)\right| &\leq \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} 2^{nj} \left(1 + \left|2^j x - k\right|\right)^{-n-2} \\ &\quad \times \left(1 + \left|2^j y - k\right|\right)^{-n-2} \\ &\leq \sum_{j\in\mathbb{Z}} C_n 2^{nj} \left(1 + 2^j |x - y|\right)^{-n-2} \\ &\leq C_n |x - y|^{-n}. \end{aligned} \tag{31}$$

Also for derivatives

$$\begin{aligned} &\nabla_x k_{\nu_A}(x, y) \\ &\quad = \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} \nu_{j,k} 2^{nj+2j} \varphi'\left(2^j x - k\right) \overline{\varphi(2^j y - k)} \\ &\left|\nabla_x k_{\nu_A}(x, y)\right| \\ &\quad \leq \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} 2^{nj+2j} \left(1 + \left|2^j x - k\right|\right)^{-n-2} \left(1 + \left|2^j y - k\right|\right)^{-n-2} \\ &\quad \leq \sum_{j\in\mathbb{Z}} C_n 2^{nj+2j} \left(1 + 2^j |x - y|\right)^{-n-2} \\ &\quad \leq C_n |x - y|^{-n-1}. \end{aligned} \tag{32}$$

So we have

$$\left|\nabla_x k_{\nu_A}(x, y)\right| + \left|\nabla_y k_{\nu_A}(x, y)\right| \leq C_n |x - y|^{-n-1}. \tag{33}$$

Condition $\int_{\mathbb{R}^n} f = 0 \Rightarrow \int_{\mathbb{R}^n} T_{\nu_A} f = 0$ follows from $\int_{\mathbb{R}^n} k_{\nu_A}(x, y) dx = 0$, which in turn follows from $\int_{\mathbb{R}^n} \varphi(x) = 0$. Therefore from Proposition 4, we know that operators T_{ν_A} extend continuously to $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$, and we get inequalities

$$\begin{aligned} \left\| T_{\nu_A} f \right\|_{L^p(\mathbb{R}^n, X)} &\leq C_{n,p} \|f\|_{L^p(\mathbb{R}^n, X)}, \\ \left\| T_{\nu_A} f \right\|_{H^1(\mathbb{R}^n, X)} &\leq C_n \|f\|_{H^1(\mathbb{R}^n, X)} \end{aligned} \tag{34}$$

and particularly for $f = \sum_{A\in f} C_A \varphi_A$, F is a finite subset of Λ, $C_A \in C$, can be written as

$$\left\| \sum_{A\in F} \nu_A C_A \varphi_A \right\|_{L^p(\mathbb{R}^n, X)} \leq C_{n,p} \left\| \sum_{A\in F} C_A \varphi_A \right\|_{L^p(\mathbb{R}^n, X)}, \tag{35}$$

$$\left\| \sum_{A\in F} \nu_A C_A \varphi_A \right\|_{H^1(\mathbb{R}^n, X)} \leq C_n \left\| \sum_{A\in F} C_A \varphi_A \right\|_{H^1(\mathbb{R}^n, X)}. \tag{36}$$

These inequalities imply that $(\varphi_A)_{A\in\Lambda}$ is an unconditional basis for

$$\begin{aligned} &\overline{\operatorname{span}}_{L^p(\mathbb{R}^n, X)} \left(\{\varphi_A : A \in \Lambda\}\right), \\ &\overline{\operatorname{span}}_{H^1(\mathbb{R}^n, X)} \left(\{\varphi_A : A \in \Lambda\}\right). \end{aligned} \tag{37}$$

However, it is still not clear that $(\varphi_A)_{A\in\Lambda}$ is complete in $L^p(\mathbb{R}^n, X)$ and $H^1(\mathbb{R}^n, X)$.

In particular, if we take $(\varepsilon_A)_{A\in\Lambda} \in \{\pm 1\}^\Lambda$, then (34) and $T^2_{\varepsilon_A}$ imply that

$$c_{n,p} \|f\|_{L^p(\mathbb{R}^n, X)} \leq \left\| T_{\varepsilon_A} f \right\|_{L^p(\mathbb{R}^n, X)} \leq C_{n,p} \|f\|_{L^p(\mathbb{R}^n, X)}, \tag{38}$$

$$c_n \|f\|_{H^1(\mathbb{R}^n, X)} \leq \left\| T_{\varepsilon_A} f \right\|_{H^1(\mathbb{R}^n, X)} \leq C_n \|f\|_{H^1(\mathbb{R}^n, X)}. \tag{39}$$

Let us assume that $\Omega = \{\pm 1\}^{\Lambda}$ with the product probability measure. First we consider the function of the form $f = \sum_{A\in F} C_A\varphi_A$ for some finite $F \subset \Lambda$ and $C_A \in \mathbb{C}$. Integrating (38) over Ω, interchanging the order of integration, and using Khintchine's inequality, we get

$$\begin{aligned}
&\left(\int_\Omega \left\|T_{\varepsilon_A} f\right\|^p_{L^p(\mathbb{R}^n,X)} d_{\varepsilon_A}\right)^{1/p} \\
&\quad\times \left\|\left(\int_\Omega \left|\sum_{A\in F} \varepsilon_A C_A \varphi_A\right|^p d_{\varepsilon_A}\right)^{1/p}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\Longrightarrow c_{n,p}\left\|\left(\sum_{A\in F}|C_A|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\le \left\|\left(\int_\Omega \left|\sum_{A\in F} \varepsilon_A C_A \varphi_A\right|^p d_{\varepsilon_A}\right)^{1/p}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\le C_{n,p}\left\|\left(\sum_{A\in F}|C_A|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)}.
\end{aligned} \tag{40}$$

In view of (38), we obtain

$$\begin{aligned}
c_{n,p}\|f\|_{L^p(\mathbb{R}^n,X)} &\le \left\|\left(\sum_{A\in F}|C_A|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\le C_{n,p}\|f\|_{L^p(\mathbb{R}^n,X)}.
\end{aligned} \tag{41}$$

Similarly, we get

$$\begin{aligned}
c_n\|f\|_{H^1(\mathbb{R}^n,X)} &\le \left\|\left(\sum_{A\in F}|C_A|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^1(\mathbb{R}^n,X)} \\
&\quad+ \left\|\left(\sum_{A\in F}|C_A|^2|\mathbb{R}_j\varphi_A|^2\right)^{1/2}\right\|_{L^1(\mathbb{R}^n,X)} \\
&\le C_n\|f\|_{H^1(\mathbb{R}^n,X)}.
\end{aligned} \tag{42}$$

Now we take $f \in L^p(\mathbb{R}^n, X)$ for some F finite subset of Λ and define $(\nu_A)_{A\in\Lambda}$ by $\nu_A = 1$ if $A \in F$ and by $\nu_A = 0$ if $A \notin F$, so that $T_{\nu_A} f = \sum_{A\in F}\langle f, \varphi_A\rangle\varphi_A$. Applying estimate (41) to $T_{\nu_A} f$ and using (34), we get

$$\begin{aligned}
c_{n,p}\left\|T_{\nu_A} f\right\|_{L^p(\mathbb{R}^n,X)} &\le \left\|\left(\sum_{A\in F}|\langle f,\varphi_A\rangle|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\le C_{n,p}\left\|T_{\nu_A} f\right\|_{L^p(\mathbb{R}^n,X)}, \\
\left\|T_{\nu_A} f\right\|_{L^p(\mathbb{R}^n,X)} &\le C_{n,p}\|f\|_{L^p(\mathbb{R}^n,X)}.
\end{aligned} \tag{43}$$

By virtue of monotone convergence theorem as finite sets F exhaust Λ, we get

$$\left\|\left(\sum_{A\in\Lambda}|\langle f,\varphi_A\rangle|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)} \le C_{n,p}\|f\|_{L^p(\mathbb{R}^n,X)}. \tag{44}$$

In order to prove the completeness of $(\varphi_A)_{A\in\Lambda}$ in $L^p(\mathbb{R}^n, X)$, consider $f \in L^p(\mathbb{R}^n, X) \cap L^2(\mathbb{R}^n, X)$ such that $f = \sum_{i=1(j,k_i)\in F}^{\infty}\langle f, \varphi_{j,k_i}\rangle\varphi_{j,k_i}$ with convergence in $L^2(\mathbb{R}^n, X)$ and there exists a subsequence of partial sums $(\sum_{i=1,(j,k_i)\in F}^{N_m}\langle f, \varphi_{j,k_i}\rangle\varphi_{j,k_i})_{m\in N}$ that converge almost everywhere on $\mathbb{R}^n$. By virtue of Fatou's lemma and (41) we obtain

$$\begin{aligned}
&\left\|f - \sum_{i=1,(j,k_i)\in F}^{N}\left\langle f,\varphi_{j,k_i}\right\rangle\varphi_{j,k_i}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\quad\le \liminf_{m\to\infty}\left\|\sum_{i=1,(j,k_i)\in F}^{N_m}\left\langle f,\varphi_{j,k_i}\right\rangle\varphi_{j,k_i}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\quad\le C_{n,p}\left\|\left(\sum_{i=N+1,(j,k_i)\in F}^{\infty}\left|\left\langle f,\varphi_{j,k_i}\right\rangle\varphi_{j,k_i}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)}.
\end{aligned} \tag{45}$$

In view of (44) we have

$$\left\|\left(\sum_{A\in F}|\langle f,\varphi_A\rangle|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)} < \infty. \tag{46}$$

Using dominated convergence theorem, we obtain

$$f = \sum_{i=1,(j,k_i)\in F}^{\infty}\left\langle f,\varphi_{j,k_i}\right\rangle\varphi_{j,k_i} \tag{47}$$

with convergence in $L^p(\mathbb{R}^n, X)$. Therefore $\overline{\operatorname{span}}_{L^p(\mathbb{R}^n,X)}(\{\varphi_A : A \in \Lambda\}) \supset L^p(\mathbb{R}^n, X)\cap L^2(\mathbb{R}^n, X)$ since $L^p(\mathbb{R}^n, X)\cap L^2(\mathbb{R}^n, X)$ is dense in $L^p(\mathbb{R}^n, X)$. These facts lead to the conclusion that $(\varphi_A)_{A\in\Lambda}$ is complete in $L^p(\mathbb{R}^n, X)$. By virtue of (35) this proves that $(\varphi_A)_{A\in\Lambda}$ is an unconditional basis for $L^p(\mathbb{R}^n, X)$.

Similarly we can prove the result for $H^1(\mathbb{R}^n, X)$ using (42). Hence the proof is completed. □

Theorem 6. *Let $1 < p < \infty$ and let φ_A be a wavelet such that $\varphi \in \mathbb{C}^1(\mathbb{R}^n)$ and satisfies condition* (26). *Then for $f \in L^p(\mathbb{R}^n, X)$ one has*

$$\begin{aligned}
c_{n,p}\|f\|_{L^p(\mathbb{R}^n,X)} &\le \left\|\left(\sum_{A\in\Lambda}|\langle f,\varphi_A\rangle|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n,X)} \\
&\le C_{n,p}\|f\|_{L^p(\mathbb{R}^n,X)}
\end{aligned} \tag{48}$$

and for $f \in H^1(\mathbb{R}^n, X)$ one has

$$\begin{aligned}
c_{n,1}\|f\|_{H^1(\mathbb{R}^n,X)} &\le \left\|\left(\sum_{A\in\Lambda}|\langle f,\varphi_A\rangle|^2|\varphi_A|^2\right)^{1/2}\right\|_{L^1(\mathbb{R}^n,X)} \\
&\le C_{n,1}\|f\|_{H^1(\mathbb{R}^n,X)}.
\end{aligned} \tag{49}$$

Proof. We have

$$\begin{aligned} |\langle f, \varphi_A \rangle| &\le \|f\|_{L^p(\mathbb{R}^n,X)} \|\varphi_A\|_{L^{p'}(\mathbb{R}^n,X)}, \\ |\langle f, \varphi_A \rangle| &\le \|f\|_{H^1(\mathbb{R}^n,X)} \|\varphi_A\|_{\mathrm{BMO}}, \end{aligned} \tag{50}$$

where the space $\mathrm{BMO}(\mathbb{R}^n)$ is of functions with bounded mean oscillation. $H^1(\mathbb{R}^n)$ is the predual of $\mathrm{BMO}(\mathbb{R}^n)$. We see that $\{\langle \cdot, \varphi_A \rangle : A \in \Lambda\}$ defines continuous linear functionals on $L^p(\mathbb{R}^n, X)$, $1 < p < \infty$, and $H^1(\mathbb{R}^n, X)$. If $f = \sum_{A\in\Lambda} C_A \varphi_A$ unconditionally in $L^p(\mathbb{R}^n, X)$ or $H^1(\mathbb{R}^n, X)$, then for every $A' \in \Lambda$ by continuity of $\langle \cdot, \varphi_{A'}, \rangle$

$$\langle f, \varphi_{A'} \rangle = \sum_{A\in\Lambda} C_A \langle \varphi_A, \varphi_{A'} \rangle = \sum_{A\in\Lambda} C_A \delta_{A,A'} = C_{A'}. \tag{51}$$

Now for every $f \in L^p(\mathbb{R}^n, X)$ we have $f = \sum_{i=1,(j,k_i)\in F}^{\infty} \langle f, \varphi_{j,k_i} \rangle \varphi_{j,k_i}$, with convergence in $L^p(\mathbb{R}^n, X)$. Using (41), we obtain

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n,X)} &= \lim_{n\to\infty} \left\| \sum_{i=1,(j,k_i)\in F}^{N} \langle f, \varphi_{j,k_i} \rangle \varphi_{j,k_i} \right\|_{L^p(\mathbb{R}^n,X)} \\ &\le C_{n,p} \left\| \left(\sum_{A\in\Lambda} |\langle f, \varphi_A \rangle|^2 |\varphi_A|^2 \right) \right\|_{L^p(\mathbb{R}^n,X)}. \end{aligned} \tag{52}$$

Combining (44) and (52), we get (48). Similarly we would handle the case for $H^1(\mathbb{R}^n, X)$. Hence the proof is completed. □

Corollary 7. *For every family of scalars $(C_A)_{A\in\Lambda}$ one has that*

$$\begin{aligned} &\sum_{A\in\Lambda} C_A \varphi_A \text{ converges unconditionally in } L^p(\mathbb{R}^n, X) \\ &\quad \Longleftrightarrow \left\| \left(\sum_{A\in\Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n,X)} < \infty, \\ &\sum_{A\in\Lambda} C_A \varphi_A \text{ converges unconditionally in } H^1(\mathbb{R}^n, X) \\ &\quad \Longleftrightarrow \left\| \left(\sum_{A\in\Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^n,X)} < \infty. \end{aligned} \tag{53}$$

Proof. Suppose that $(C_A)_{A\in\Lambda}$ is a family of scalars such that $\|(\sum_{A\in\Lambda} |C_A|^2 |\varphi_A|^2)^{1/2}\|_{L^p(\mathbb{R}^n,X)} < \infty$. For every $\varepsilon > 0$ we can find F_0 finite subset of Λ such that $\|(\sum_{A\in\Lambda\setminus F_0} |C_A|^2 |\varphi_A|^2)^{1/2}\|_{L^p(\mathbb{R}^n,X)} \le \varepsilon$.

Then for every F finite subset of $\Lambda \setminus F_0$, by (44), we have

$$\left\| \sum_{A\in F} C_A \varphi_A \right\|_{L^p(\mathbb{R}^n,X)} \le C_{n,p} \varepsilon. \tag{54}$$

This implies that the series $\sum_{A\in\Lambda} C_A \varphi_A$ converges unconditionally in $L^p(\mathbb{R}^n, X)$ to the same L^p-function. Moreover, we have proved that

$$\begin{aligned} &\sum_{A\in\Lambda} C_A \varphi_A \text{ converges unconditionally in} \\ &\quad L^p(\mathbb{R}^n, X) \Longleftrightarrow \left\| \left(\sum_{A\in\Lambda} |C_A|^2 |\varphi_A|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n,X)} < \infty. \end{aligned} \tag{55}$$

This gives the characterization of the convergence of wavelet series in $L^p(\mathbb{R}^n, X)$. Similarly we would characterize convergence in $H^1(\mathbb{R}^n, X)$.

Hence the proof is completed. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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