Hindawi Publishing Corporation
ISRN Applied Mathematics
Volume 2014, Article ID 985782, 8 pages
http://dx.doi.org/10.1155/2014/985782

Research Article

Some Remarks on the Mathieu Series

Robert Frontczak

Landesbank Baden-Württemberg (LBBW), Am Hauptbahnhof 2, 70173 Stuttgart, Germany

Correspondence should be addressed to Robert Frontczak; robert.frontczak@lbbw.de

Received 11 December 2013; Accepted 29 January 2014; Published 13 March 2014

Academic Editors: G. Fikioris and C.-H. Lien

The object of this note is to present new expressions for the classical Mathieu series in terms of hyperbolic functions. The derivation is based on elementary arguments concerning the integral representation of the series. The results are used afterwards to prove, among others, a new relationship between the Mathieu series and its alternating companion. A recursion formula for the Mathieu series is also presented. As a byproduct, some closed-form evaluations of integrals involving hyperbolic functions are inferred.

1. Introduction

The infinite series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0, \tag{1}$$

is called a Mathieu series. It was introduced in 1890 by É. L. Mathieu (1835–1890) who studied various problems in mathematical physics. Since its introduction the series $S(r)$ has been studied intensively. Mathieu himself conjectured that $S(r) < 1/r^2$. The conjecture was proved in 1952 by Berg in [1]. Nowadays, the mathematical literature provides a range of papers dealing with inequalities for the series $S(r)$. In 1957 Makai [2] derived the double inequality

$$\frac{1}{r^2+1/2} < S(r) < \frac{1}{r^2}. \tag{2}$$

More recently, Alzer et al. proved in [3] that

$$\frac{1}{r^2+1/(2\zeta(3))} < S(r) < \frac{1}{r^2+1/6}. \tag{3}$$

Here, as usual, $\zeta(s)$ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $\mathrm{Re}(s) > 1$. The constants $1/(2\zeta(3))$ and $1/6$ are the best possible. Other lower and upper bound estimates for the Mathieu series can be found in the articles of Qi et al. [4] and Hoorfar and Qi [5].

An integral representation for the Mathieu series (1) is given by

$$S(r) = \frac{1}{r}\int_0^{\infty} \frac{x}{e^x-1} \sin(rx)\, dx. \tag{4}$$

The integral representation was used by Elbert in [6] to derive the asymptotic expansion of $S(r)$:

$$S(r) = \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{r^{2k+2}} = \frac{1}{r^2} - \frac{1}{6r^4} \pm \cdots \quad (r \longrightarrow \infty), \tag{5}$$

where B_{2k} denote the even indexed Bernoulli numbers defined by the generating function

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}, \quad |x| < 2\pi. \tag{6}$$

See also [7] for a derivation.

The Mathieu series admits various generalizations that have been introduced and investigated intensively in recent years. The generalizations include the alternating Mathieu series, the m-fold generalized Mathieu series, Mathieu a-series, and Mathieu (a,λ)-series [8–11]. The generalizations recapture the classical Mathieu series as a special case. On the other hand, the alternating Mathieu series, although connected to its classical companion, is a variant that allows a separate study. It was introduced by Pogány et al. in [11] by the equation

$$S^*(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2}, \quad r > 0. \tag{7}$$

It possesses the integral representation

$$S^*(r) = \frac{1}{r} \int_0^{\infty} \frac{x}{e^x + 1} \sin(rx)\, dx. \tag{8}$$

Recently derived bounding inequalities for alternating Mathieu-type series can be found in the paper of Pogány and Tomovski [12]. The latest research article on integral forms for Mathieu-type series is the paper of Milovanović and Pogány [13]. The authors present a systematic treatment of the subject based on contour integration. Among others, the following new integral representation for $S(r)$ is derived [13, Corollary 2.2]:

$$S(r) = \pi \int_0^{\infty} \frac{r^2 - x^2 + (1/4)}{(x^2 - r^2 + (1/4))^2 + r^2} \cdot \frac{dx}{\cosh^2(\pi x)}. \tag{9}$$

In this note we restrict the attention to the classical Mathieu series. Starting with the integral form (4) new representations for $S(r)$ are derived. The derivation is based on elementary arguments concerning the integrand combined with related integral identities. The results are used afterwards to establish interesting properties of the series. Among others, a new relationship between the Mathieu series and its alternating variant is derived. A recursion formula for the Mathieu series is also presented. As a byproduct, some closed-form evaluations of integrals involving hyperbolic functions are inferred. Finally, a new proof is given for an exact evaluation of an infinite series related to $S(r)$.

2. Main Results

In what follows we will use the following integral identities. The identities are well known and are stated without proof (see [14] for a reference).

Lemma 1. *For $r > 0$ and $\mathrm{Re}(\beta) > 0$ it holds that*

$$\int_0^{\infty} \frac{x}{e^{\beta x} - 1} \cos(rx)\, dx = \frac{1}{2r^2} - \frac{\pi^2}{2\beta^2} \operatorname{csch}^2\left(\frac{r\pi}{\beta}\right), \tag{10}$$

where $\operatorname{csch}(x)$ *denotes the hyperbolic cosecant of x defined by*

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}. \tag{11}$$

Similarly, for $r > 0$ and $n \geqslant 0$ an integer

$$\int_0^{\infty} \frac{x^{2n}}{e^x - 1} \sin(rx)\, dx = (-1)^n \frac{\partial^{2n}}{\partial r^{2n}} \left(\frac{\pi}{2} \coth(r\pi) - \frac{1}{2r}\right), \tag{12}$$

where $\coth(x)$ *denotes the hyperbolic cotangent of x defined by*

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{2x} + 1}{e^{2x} - 1}. \tag{13}$$

Finally, for $r > 0$, $\mathrm{Re}(\beta) > 0$, and $n \geqslant 0$ an integer

$$\int_0^{\infty} x^n e^{-\beta x} \cos(rx)\, dx = (-1)^n \frac{\partial^n}{\partial \beta^n} \left(\frac{\beta}{\beta^2 + r^2}\right). \tag{14}$$

The main theorem of this note is stated next. It gives three expressions for the Mathieu series in a semi-integral form.

Theorem 2. *The Mathieu series $S(r)$ has the following representations*

$$\begin{aligned} S(r) = {} & \frac{\pi}{r} \coth(\pi r) - \left(\frac{\pi}{\sinh(\pi r)}\right)^2 \\ & - \frac{1}{r} \int_0^{\infty} \frac{\cosh(x) + 1}{\sinh(x)} g(x) \sin(rx)\, dx, \end{aligned} \tag{15}$$

$$\begin{aligned} S(r) = {} & \frac{1}{r^2} - \frac{1}{4r^4} + \left(\frac{\pi}{2r \sinh(\pi r)}\right)^2 \\ & + \frac{1}{r^2} \int_0^{\infty} \left(\frac{1}{x} - \frac{1}{2} \frac{\cosh(x) + 1}{\sinh(x)}\right) g(x) \cos(rx)\, dx, \end{aligned} \tag{16}$$

$$\begin{aligned} S(r) = {} & \frac{1}{r^2} - \frac{1}{r^4} + \frac{\pi}{2r^3} \coth(\pi r) \\ & + \frac{1}{2}\left(\frac{\pi}{r \sinh(\pi r)}\right)^2 + \frac{2}{r^3} \int_0^{\infty} \frac{g'(x)}{e^x - 1} \sin(rx)\, dx, \end{aligned} \tag{17}$$

where $\sinh(x)$ *and* $\cosh(x)$ *denote the hyperbolic sine and cosine functions, respectively,* $g(x) = x/(e^x - 1)$, *and* $g'(x)$ *denotes its first derivative.*

Proof. Let $g(x) = x/(e^x - 1)$. The main argument in the proof is the observation that $g(x)$ satisfies the nonlinear differential equation

$$g'(x) = -g(x) + \frac{g(x)}{x} - \frac{g^2(x)}{x}, \tag{18}$$

which may be also written in the form

$$g(x) = \frac{g(x)}{x} - \frac{g^2(x)}{x} - g'(x). \tag{19}$$

Inserting the relation in (4) and using (12) with $n = 0$ give

$$\begin{aligned} S(r) = {} & -\frac{1}{2r^2} + \frac{\pi}{2r} \coth(\pi r) \\ & - \frac{1}{r} \int_0^{\infty} \left(\frac{g^2(x)}{x} + g'(x)\right) \sin(rx)\, dx. \end{aligned} \tag{20}$$

Since

$$\frac{g^2(x)}{x} = -\frac{1}{2} \left(g(x) - g(x) \coth\left(\frac{x}{2}\right)\right), \tag{21}$$

we see that

$$S(r) = -\frac{1}{r^2} + \frac{\pi}{r}\coth(\pi r) - \frac{1}{r}\int_0^\infty \left(g(x)\coth\left(\frac{x}{2}\right) + 2g'(x)\right)\sin(rx)\,dx. \tag{22}$$

Integration by parts gives

$$\int_0^\infty g'(x)\sin(rx)\,dx = -r\int_0^\infty g(x)\cos(rx)\,dx, \tag{23}$$

which is easily evaluated using (10) with $\beta = 1$. Finally, the elementary relation

$$\coth\left(\frac{x}{2}\right) = \frac{1}{\tanh(x/2)} = \frac{\cosh(x)+1}{\sinh(x)} \tag{24}$$

establishes (15). Integrating (4) by parts results in

$$S(r) = \frac{1}{r^2} + \frac{1}{r^2}\int_0^\infty g'(x)\cos(rx)\,dx. \tag{25}$$

Equations (18) and (10) give

$$S(r) = \frac{1}{r^2} - \frac{1}{2r^4} + \frac{1}{2}\left(\frac{\pi}{r\sinh(\pi r)}\right)^2 + \frac{1}{r^2}\int_0^\infty \left(\frac{g(x)}{x} - \frac{g^2(x)}{x}\right)\cos(rx)\,dx. \tag{26}$$

Let $G(r)$ denote the last component of (26); that is,

$$G(r) = \frac{1}{r^2}\int_0^\infty \left(\frac{g(x)}{x} - \frac{g^2(x)}{x}\right)\cos(rx)\,dx. \tag{27}$$

Then, $G(r)$ can be evaluated in two different ways. First, the relation from (21) in combination with (10) shows that

$$G(r) = \frac{1}{4r^4} - \left(\frac{\pi}{2r\sinh(\pi r)}\right)^2 + \frac{1}{r^2}\int_0^\infty \left(\frac{1}{x} - \frac{1}{2}\frac{\cosh(x)+1}{\sinh(x)}\right) g(x)\cos(rx)\,dx. \tag{28}$$

This proves (16). Finally, by the rule of l'Hospital,

$$\lim_{x\to 0}\left(\frac{g(x)}{x} - \frac{g^2(x)}{x}\right) = \lim_{x\to 0}\frac{e^x - 1 - x}{(e^x-1)^2} = \frac{1}{2}, \qquad \lim_{x\to\infty}\left(\frac{g(x)}{x} - \frac{g^2(x)}{x}\right) = 0. \tag{29}$$

Splitting the integrand of $G(r)$ into two parts and integrating by parts in each case, we obtain

$$G(r) = \frac{1}{r^3}\int_0^\infty \left(\frac{e^x+1}{(e^x-1)^2} - \frac{2xe^x}{(e^x-1)^3}\right)\sin(rx)\,dx. \tag{30}$$

In the last equation we have used the fact that

$$\lim_{x\to 0}\frac{\sin(rx)}{e^x-1} = \lim_{x\to 0}\frac{x\sin(rx)}{(e^x-1)^2} = r. \tag{31}$$

Since

$$\frac{e^x+1}{(e^x-1)^2} = \frac{1}{e^x-1} + \frac{2}{(e^x-1)^2}, \tag{32}$$

$G(r)$ is equal to

$$G(r) = \frac{1}{r^3}\int_0^\infty \frac{\sin(rx)}{e^x-1}dx + \frac{1}{r^3}\int_0^\infty \left(\frac{2}{(e^x-1)^2} - \frac{2xe^x}{(e^x-1)^3}\right)\sin(rx)\,dx. \tag{33}$$

For the first integral we apply once more the result from (12) with $n = 0$. Direct calculation shows that the expression in brackets under the second integral sign is equal to $2g'(x)/(e^x-1)$. This completes the proof. □

The elementary relation in (18) seems to have been overseen in the literature. It allows to express $S(r)$ in terms of the hyperbolic sine and cosine functions, respectively. The representations may turn out to be useful to study the properties of $S(r)$. For the remainder of this note, however, we will mainly work with (15). The equation will be used to infer interesting consequences, which we are going to present immediately. Additionally, in Section 3 it will be outlined that (18) is also useful to study some topics that are related to the evaluation of $S(r)$.

Concerning the integrands in Theorem 2 we make the following observations. Firstly, standard arguments show that $(\cosh(x)+1)/\sinh(x)g(x) > 0$ for $x > 0$ and

$$\lim_{x\to 0}\frac{\cosh(x)+1}{\sinh(x)}g(x)\sin(rx) = 2r. \tag{34}$$

Secondly, $1/x - 1/2(\cosh(x)+1)/\sinh(x)g(x) < 0$ for $x > 0$ and

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{2}\frac{\cosh(x)+1}{\sinh(x)}\right) g(x)\cos(rx) = 0. \tag{35}$$

Finally, since $e^x - 1 < xe^x$ for $x > 0$, it is clear that $g'(x)/(e^x - 1) < 0$ and

$$\lim_{x\to 0}\frac{g'(x)\sin(rx)}{e^x-1} = -\frac{r}{2}. \tag{36}$$

Figure 1 illustrates the results from Theorem 2 numerically. $S(r)$ is plotted together with (15)–(17), where the new representations are decomposed into nonintegral and integral parts (termed first and second parts, resp.). More precisely, this means that (15) is decomposed in the following manner:

$$\text{First Part} = \frac{\pi}{r}\coth(\pi r) - \left(\frac{\pi}{\sinh(\pi r)}\right)^2, \qquad \text{Second Part} = -\frac{1}{r}\int_0^\infty \frac{\cosh(x)+1}{\sinh(x)}g(x)\sin(rx)\,dx. \tag{37}$$

Equations (16) and (17) are decomposed analogously.

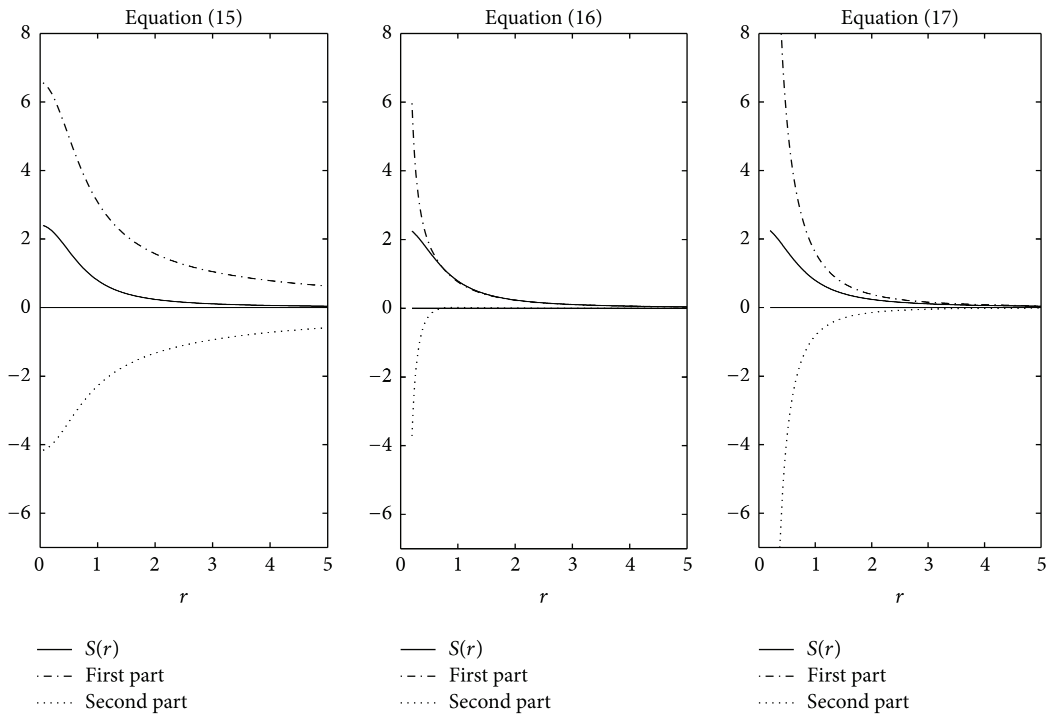

Figure 1: Numerical (graphical) illustrations of the results from Theorem 2.

A first interesting consequence of the theorem is the following closed-form evaluation of an integral.

Corollary 3. *Let $g(x) = x/(e^x - 1)$. Then, it holds that*

$$\int_0^\infty \frac{\cosh(x)+1}{\sinh(x)} g(x)\, x\, dx = 2(2\zeta(2) - \zeta(3)) \approx 4.1756. \tag{38}$$

Proof. From (1) it is obvious that $\lim_{r\to 0} S(r) = 2\zeta(3)$. Further, from the identity for Bernoulli numbers (see [14])

$$\int_0^\infty \frac{x^{2n-1}}{e^{px}-1} dx = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n}, \quad n = 1,2,3,\ldots, \tag{39}$$

and the fact that $B_2 = 1/6$, we easily deduce that

$$\begin{aligned}
&\lim_{r\to 0} 2\int_0^\infty g(x)\cos(rx)\, dx \\
&\qquad = \lim_{r\to 0}\left(\frac{1}{r^2} - \left(\frac{\pi}{\sinh(\pi r)}\right)^2\right) = 2\zeta(2), \\
&\qquad \lim_{r\to 0}\left(\frac{\pi}{r}\coth(\pi r) - \frac{1}{r^2}\right) = 2\zeta(2).
\end{aligned} \tag{40}$$

Hence,

$$\lim_{r\to 0}\left(\frac{\pi}{r}\coth(\pi r) - \left(\frac{\pi}{\sinh(\pi r)}\right)^2\right) = 4\zeta(2). \tag{41}$$

The assertion follows from (15). □

It is clear that a successive repetition of integration by parts in Theorem 2 will produce a range of other expressions for $S(r)$. Among others, from (15) and the fact that

$$\tanh\left(\frac{x}{2}\right) = \frac{\cosh(x)-1}{\sinh(x)} \tag{42}$$

it is fairly easy to produce the following characterization:

$$\begin{aligned}
S(r) &= \frac{\pi}{r}\coth(\pi r) - \left(\frac{\pi}{\sinh(\pi r)}\right)^2 \\
&\quad + \int_0^\infty \ln(\cosh(x)-1)\, g(x)\cos(rx)\, dx \\
&\quad + \frac{1}{r}\int_0^\infty \ln(\cosh(x)-1)\, g'(x)\sin(rx)\, dx.
\end{aligned} \tag{43}$$

In the above characterization the sine and cosine functions appear simultaneously as integrands. Now, in view of the previous proof and (18) it is straightforward to get

$$\begin{aligned}
&\int_0^\infty \ln(\cosh(x)-1)\left(2g(x) - g(x)\, g(-x)\right) dx \\
&\qquad = 2(\zeta(3) - 2\zeta(2)),
\end{aligned} \tag{44}$$

where we have used the relation $x + g(x) = g(-x)$. Furthermore, since direct computation verifies that

$$g(x)\, g(-x) = \frac{1}{2}x^2\frac{1}{\cosh(x)-1} \tag{45}$$

the last integral may be written in the form

$$\int_0^\infty \ln(\cosh(x)-1)\, g(x)\, dx = \zeta(3) - 2\zeta(2) + \int_0^\infty \left(\frac{x}{2}\right)^2 \frac{\ln(\cosh(x)-1)}{\cosh(x)-1} dx. \tag{46}$$

A further interesting consequence of the previous theorem is the following new integral-type representation for the alternating Mathieu series $S^*(r)$.

Proposition 4. *The alternating Mathieu series $S^*(r)$ may be represented as*

$$S^*(r) = -S(r) + \frac{\pi}{2}\frac{\cosh(\pi r)-1}{\sinh(\pi r)}\left(\frac{1}{r} + \frac{\pi}{\sinh(\pi r)}\right) + J(r), \tag{47}$$

with

$$J(r) = \frac{1}{r}\int_0^\infty \left(\frac{1}{\cosh(x)+1} - \frac{1}{\sinh(x)}\right) g(x)\sin(rx)\, dx. \tag{48}$$

Proof. We start with the identity

$$S^*(r) = S(r) - \frac{1}{4} S\left(\frac{r}{2}\right). \tag{49}$$

In view of (15) and applying the relation

$$\coth\left(\frac{\pi r}{2}\right) = \coth(\pi r) + \frac{1}{\sinh(\pi r)}, \tag{50}$$

we obtain

$$S^*(r) = \frac{\pi}{2r}\coth(\pi r) - \frac{\pi}{2r}\frac{1}{\sinh(\pi r)} + \left(\frac{\pi}{2\sinh((1/2)\pi r)}\right)^2 - \left(\frac{\pi}{\sinh(\pi r)}\right)^2 + I(r), \tag{51}$$

where

$$I(r) = -\frac{1}{r}\int_0^\infty \frac{\cosh(x)+1}{\sinh(x)} g(x)\sin(rx)\, dx + \frac{1}{2r}\int_0^\infty \frac{\cosh(x)+1}{\sinh(x)} g(x)\sin\left(\frac{1}{2}rx\right) dx. \tag{52}$$

For the second integral, a change of variable ($y = 1/2x$) is used in combination with the multiple-angle formulas for the hyperbolic sine and cosine functions, respectively, to get

$$\frac{\cosh(2x)+1}{\sinh(2x)} = \coth(x), \tag{53}$$

and hence

$$I(r) = -\frac{1}{r}\int_0^\infty \left((g(x) - g(2x))\coth(x) + \frac{g(x)}{\sinh(x)}\right)\sin(rx)\, dx. \tag{54}$$

Next, since

$$g(2x) = \frac{2g(x)}{e^x+1}, \tag{55}$$

$I(r)$ can be simplified to

$$I(r) = -\frac{1}{r}\int_0^\infty \left(\coth(x)\coth\left(\frac{x}{2}\right) + \frac{1}{\sinh(x)}\right) g(x)\sin(rx)\, dx. \tag{56}$$

The relation

$$\coth(x)\coth\left(\frac{x}{2}\right) = \frac{\cosh(x)}{\cosh(x)+1} = 1 - \frac{1}{\cosh(x)+1} \tag{57}$$

shows that $I(r) = -S(r) + J(r)$. The final formula follows from

$$\left(\frac{\pi}{2\sinh((1/2)\pi r)}\right)^2 - \left(\frac{\pi}{\sinh(\pi r)}\right)^2 = \frac{\pi^2}{\sinh^2(\pi r)}\left(\cosh^2\left(\frac{1}{2}\pi r\right) - 1\right) \tag{58}$$

and the half-angle formula

$$\cosh\left(\frac{1}{2}x\right) = \sqrt{\frac{\cosh(x)+1}{2}}. \tag{59}$$

□

Corollary 5. *It holds that*

$$\int_0^\infty \left(\frac{1}{\cosh(x)+1} - \frac{1}{\sinh(x)}\right) g(x)\, x\, dx = \frac{7}{2}\zeta(3) - 3\zeta(2) \approx -0.7276, \tag{60}$$

$$\int_0^\infty \left(\frac{1}{\cosh(x)+1} + \coth(x)\right) g(x)\, x\, dx = \frac{3}{2}\zeta(3) + \zeta(2) \approx 3.4480. \tag{61}$$

Proof. We have $\lim_{r\to 0}(S^*(r) + S(r)) = 7/2\zeta(3)$. Additionally, using similar arguments as in Corollary 3, it can be shown that

$$\lim_{r\to 0}\left(\frac{\pi}{2}\frac{\cosh(\pi r)-1}{\sinh(\pi r)}\left(\frac{1}{r} + \frac{\pi}{\sinh(\pi r)}\right)\right) = 3\zeta(2). \tag{62}$$

This proves the first expression. The second follows from combining (38) and (60). □

The next assertion provides a recursion formula for the Mathieu series. Surprisingly, it is also a consequence of Proposition 4.

Corollary 6. *For $r > 0$ define the function $f(r)$ as*

$$f(r) = \frac{\pi}{2}\frac{\cosh(\pi r)-1}{\sinh(\pi r)}\left(\frac{1}{r} + \frac{\pi}{\sinh(\pi r)}\right) + J(r), \tag{63}$$

where $J(r)$ is given in (48). Then, for $n \geqslant 1$ one has the recursion formula

$$S\left(\frac{r}{2^n}\right) = 2^{3n} S(r) - \sum_{i=0}^{n-1} 2^{3i+2} f\left(\frac{r}{2^{n-(i+1)}}\right). \tag{64}$$

Proof. Combine (47) and (49) to get

$$S\left(\frac{r}{2}\right) = 8S(r) - 4f(r). \tag{65}$$

Successive repetition of the identity establishes the stated formula. □

3. Evaluation of a Related Series

It is interesting to mention that (18) may be used to prove a result for a closed-form evaluation of an infinite series that is related to $S(r)$. More precisely, if we define the function $T(r)$ for $r \in \mathbb{R}$ by the infinite series

$$T(r) = \sum_{n=1}^{\infty} \frac{n^2 - r^2}{(n^2 + r^2)^2}, \qquad T(0) = \zeta(2) = \frac{\pi^2}{6}, \tag{66}$$

then $S(r)$ and $T(r)$ are connected via the following relationship. Consider the (complex) function

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} = \left(\frac{d}{dz}\right)^2 \ln \Gamma(z), \quad z \neq 0, -1, -2, \ldots \tag{67}$$

with $\Gamma(z)$ being the Gamma function. Note first that

$$\sum_{n=1}^{\infty} \frac{1}{(n+ir)^2} = \sum_{n=1}^{\infty} \frac{(n-ir)^2}{(n^2+r^2)^2} = T(r) - irS(r). \tag{68}$$

Comparing the two equations we see that

$$T(r) - irS(r) = g(1+ir) = g(ir) + \frac{1}{r^2}. \tag{69}$$

The function $T(r)$ admits an exact evaluation (see [7]) for which we can provide a new elementary proof applying the analytical structure of $S(r)$. However, in contrary to the pervious section where we have focused on integral-type expressions, we change the point of view and work with summation-type representations of $S(r)$.

Proposition 7. *The function $T(r)$ can be evaluated exactly as*

$$T(r) = \frac{1}{2r^2}\left(1 - \left(\frac{\pi r}{\sinh(\pi r)}\right)^2\right), \quad r \neq 0. \tag{70}$$

Proof. Once more, let $g(x) = x/(e^x - 1)$. We start with (18) and (26). Since

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx}, \quad x > 0, \tag{71}$$

we have

$$\frac{1}{(e^x-1)^2} = \sum_{n=1}^{\infty} n e^{-(n+1)x}, \quad x > 0,$$

$$\frac{g(x)}{x} - \frac{g^2(x)}{x} = \sum_{n=1}^{\infty} n e^{-nx} - \sum_{n=1}^{\infty} n e^{-(n+1)x} - x \sum_{n=1}^{\infty} n e^{-(n+1)x}. \tag{72}$$

Thus, after interchanging summation and integration and applying (14)

$$\int_0^{\infty} \left(\frac{g(x)}{x} - \frac{g^2(x)}{x}\right) \cos(rx)\, dx = C, \tag{73}$$

where

$$C = \sum_{n=1}^{\infty} \frac{n^2}{n^2+r^2} - \sum_{n=1}^{\infty} \frac{n(n+1)}{(n+1)^2+r^2} - \sum_{n=1}^{\infty} n \frac{(n+1)^2 - r^2}{\left((n+1)^2+r^2\right)^2}. \tag{74}$$

This gives

$$S(r) = \sum_{n=1}^{\infty} \frac{n}{\left((n+1)^2+r^2\right)^2} + \frac{1}{r^2} - \frac{1}{2r^4} + \frac{1}{2}\left(\frac{\pi}{r\sinh(\pi r)}\right)^2 + \frac{1}{r^2}\sum_{n=1}^{\infty}\left(\frac{n^2}{n^2+r^2} - \frac{n(n+1)}{(n+1)^2+r^2} - \frac{n(n+1)^2}{\left((n+1)^2+r^2\right)^2}\right). \tag{75}$$

From the identity

$$\sum_{n=1}^{\infty} \frac{n}{\left((n+1)^2+r^2\right)^2} = \frac{1}{2}S(r) - \sum_{n=1}^{\infty} \frac{1}{(n^2+r^2)^2} \tag{76}$$

we arrive at

$$S(r) = -\sum_{n=1}^{\infty} \frac{2}{(n^2+r^2)^2} + \frac{2}{r^2} - \frac{1}{r^4} + \left(\frac{\pi}{r\sinh(\pi r)}\right)^2 + \frac{2}{r^2}\sum_{n=1}^{\infty}\left(\frac{n^2}{n^2+r^2} - \frac{n(n+1)}{(n+1)^2+r^2} - \frac{n(n+1)^2}{\left((n+1)^2+r^2\right)^2}\right). \tag{77}$$

Next, we simplify the sums in the above equation for $S(r)$ further. Using the fact that $n(n+1) = (n+1)^2 - (n+1)$ it is straightforward to show that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{n^2}{n^2+r^2} - \frac{n(n+1)}{(n+1)^2+r^2}\right) &= r^2\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2+r^2} - \frac{1}{n^2+r^2}\right) \\ &\quad + \sum_{n=1}^{\infty}\frac{n+1}{(n+1)^2+r^2} \\ &= -\frac{r^2}{1+r^2} + \sum_{n=1}^{\infty}\frac{n+1}{(n+1)^2+r^2}.\end{aligned} \tag{78}$$

The last identity is true since the first sum on the right-hand side of the equality telescopes. This gives

$$\begin{aligned}S(r) &= -\frac{2}{1+r^2} - \sum_{n=1}^{\infty}\frac{2}{(n^2+r^2)^2} + \frac{2}{r^2} - \frac{1}{r^4} \\ &\quad + \left(\frac{\pi}{r\sinh(\pi r)}\right)^2 - \frac{2}{r^2(1+r^2)} \\ &\quad + \frac{2}{r^2}\sum_{n=1}^{\infty}\left(\frac{n}{n^2+r^2} - \frac{n(n+1)^2}{\left((n+1)^2+r^2\right)^2}\right),\end{aligned} \tag{79}$$

which may be written as

$$\begin{aligned}S(r) &= \sum_{n=1}^{\infty}\frac{-2}{(n^2+r^2)^2} + \frac{2}{r^2}\left(-\frac{1}{2r^2} + \frac{1}{2}\left(\frac{\pi}{\sinh(\pi r)}\right)^2\right) \\ &\quad + \frac{2}{r^2}\sum_{n=1}^{\infty}\left(\frac{n}{n^2+r^2} - \frac{n(n+1)^2}{\left((n+1)^2+r^2\right)^2}\right).\end{aligned} \tag{80}$$

Finally, notice that

$$\frac{2}{r^2}\sum_{n=1}^{\infty}\left(\frac{n}{n^2+r^2} - \frac{n(n+1)^2}{\left((n+1)^2+r^2\right)^2}\right) = \frac{2}{r^2}\sum_{n=1}^{\infty}\frac{nr^2+n^2}{(n^2+r^2)^2}. \tag{81}$$

This leads to canceling out $S(r)$ in (80) and the proof is complete. □

Remark 8. Using the further observation that $g^2(x)/x$ may be expressed as

$$\begin{aligned}\frac{g^2(x)}{x} &= xe^{-2x}\left(1 + \frac{g(x)}{x} + e^x\left(\frac{g(x)}{x}\right)^2\right) \\ &= \frac{xe^{-x}}{e^x-1} + \frac{xe^{-x}}{(e^x-1)^2}\end{aligned} \tag{82}$$

it is possible to derive the following summation-type form of $S(r)$:

$$\begin{aligned}S(r) &= \frac{-\left(1+r^4\right)}{\left(r^2\left(1+r^2\right)\right)^2} + \left(\frac{\pi}{r\sinh(\pi r)}\right)^2 \\ &\quad + \frac{2}{r^2}\sum_{n=1}^{\infty}\left(\frac{n-1}{n^2+r^2} - \frac{n(n+2)^2}{\left((n+2)^2+r^2\right)^2}\right).\end{aligned} \tag{83}$$

4. Conclusion

In this paper new expressions for the Mathieu series were derived in terms of hyperbolic functions. To derive the new identities, a basic property of the function $x/(e^x-1)$ was utilized. Using a particular identity it was possible to prove a new relationship between the Mathieu series and the alternating variant. Also, a new recursion formula and some interesting closed-form evaluations of definite integrals involving hyperbolic functions were established. Finally, a new elementary proof for an evaluation of an infinite series related to the Mathieu series was presented.

It would be interesting to know whether the new identities can be used to derive new (double) inequalities for the series.

Disclaimer

The statements and conclusions made in this paper are entirely those of the author. They do not necessarily reflect the views of LBBW.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author thanks two anonymous referees for a careful reading of the paper and valuable comments that helped to improve the content of the paper.

References

[1] L. Berg, "Über eine Abschätzung von Mathieu," *Mathematische Nachrichten*, vol. 7, pp. 257–259, 1952 (German).

[2] E. Makai, "On the inequality of Mathieu," *Publicationes Mathematicae Debrecen*, vol. 5, pp. 204–205, 1957.

[3] H. Alzer, J. L. Brenner, and O. G. Ruehr, "On Mathieu's inequality," *Journal of Mathematical Analysis and Applications*, vol. 218, no. 2, pp. 607–610, 1998.

[4] F. Qi, C.-P. Chen, and B.-N. Guo, "Notes on double inequalities of Mathieu's series," *International Journal of Mathematics and Mathematical Sciences*, no. 16, pp. 2547–2554, 2005.

[5] A. Hoorfar and F. Qi, "Some new bounds for Mathieu's series," *Abstract and Applied Analysis*, vol. 2007, Article ID 94854, 10 pages, 2007.

[6] Á. Elbert, "Asymptotic expansion and continued fraction for Mathieu's series," *Periodica Mathematica Hungarica*, vol. 13, no. 1, pp. 1–8, 1982.

[7] D. C. Russell, "A note on Mathieu's inequality," *Aequationes Mathematicae*, vol. 36, no. 2-3, pp. 294–302, 1988.

[8] P. Cerone and C. T. Lenard, "On integral forms of generalised Mathieu series," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 5, 2003.

[9] H. M. Srivastava and Ž. Tomovski, "Some problems and solutions involving Mathieu's series and its generalizations," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 2, 2004.

[10] T. K. Pogány and Ž. Tomovski, "On multiple generalized Mathieu series," *Integral Transforms and Special Functions*, vol. 17, no. 4, pp. 285–293, 2006.

[11] T. K. Pogány, H. M. Srivastava, and Ž. Tomovski, "Some families of Mathieu **a**-series and alternating Mathieu **a**-series," *Applied Mathematics and Computation*, vol. 173, no. 1, pp. 69–108, 2006.

[12] T. K. Pogány and Ž. Tomovski, "Bounds improvement for alternating Mathieu type series," *Journal of Mathematical Inequalities*, vol. 4, no. 3, pp. 315–324, 2010.

[13] G. V. Milovanović and T. K. Pogány, "New integral forms of generalized Mathieu series and related applications," *Applicable Analysis and Discrete Mathematics*, vol. 7, no. 1, pp. 180–192, 2013.

[14] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Elsevier Academic Press, Amsterdam, The Netherlands, 7th edition, 2007.