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Research Article

Common Fixed Point Theorems of Greguš Type (ϕ, ψ)-Weak Contraction for R-Weakly Commuting Mappings in 2-Metric Spaces

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The main purpose of this paper is to establish a common fixed point theorem for set valued mappings in 2-metric spaces by generalizing a theorem of Abd EL-Monsef et al. (2009) and Murthy and Tas (2009) by using (ϕ, ψ)-weak contraction in view of Greguš type condition for set valued mappings using R-weakly commuting maps.

1. Introduction and Preliminaries

The weak contraction condition in Hilbert Space was introduced by Alber and Guerre-Delabriere [1]. Later, Rhoades [2] has noticed that the results of Alber and Guerre-Delabriere [1] in Hilbert Spaces are also true in a complete metric space.

Rhoades [2] established a fixed point theorem in a complete metric space by using the following contraction condition.

Let $T : X \to X$ which satisfies the following condition:

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1}$$

where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous ad nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark 1. In this above result, if $\varphi(t) = (1 - k)t$, where $k \in (0, 1)$, then we obtain condition (1) of Banach.

In the recent years, Dutta and Choudhury [3], Zhang and Song [4], and Đorić [5] have given the results in (ϕ, ψ)-weak contractive mapping.

The concept of 2-metric space is a natural generalization of the metric space. The concept of 2-metric spaces has been investigated initially in a series of papers (see Gahler [6–8]) and has been developed extensively by Gahler and many others. Gahler defined a 2-metric space as follows.

Definition 2 (see [6]). A 2-metric space on a set X with at least three points is nonnegative real-valued mapping $d : X \times X \times X \to R^+$ satisfying the following conditions:

(1) For two distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(2) $d(x, y, z) = 0$ if at least two of x, y, and z are equal.

(3) $d(x, y, z) = d(x, z, y) = d(y, x, z)$.

(4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all x, y, z, and u in X.

The function d is called a 2-metric for the space X and the pair (X, d) is then called a 2-metric space.

Geometrically, the value of a 2-metric $d(x, y, z)$ represents the area of a triangle with vertices x, y, and c.

After this, a number of fixed point theorems have been proved for 2-metric spaces by introducing compatible mappings, commuting and weakly commuting mappings. There were some generalizations of metric such as a 2-metric, a D-metric, a G-metric, a cone metric, and a complex-valued metric. Note that a 2-metric is not a continuous function of its variables, whereas an ordinary metric is. This led Dhage to introduce the notion of a D-metric in [9]. But, in 2003, Mustafa and Sims [10] demonstrated that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect. After that, in 2006, Mustafa and Sims [11] introduced the notion of G-metric spaces. Only a 2-metric space has not been known to be topologically equivalent to an ordinary metric. Then, there was no easy relationship between results obtained in 2-metric spaces and metric spaces. In particular, the fixed point theorems on 2-metric spaces and metric spaces may be unrelated easily. For more fixed point theorems on 2-metric spaces, the researchers may refer to [12–15].

Throughout this paper, (X, d) is for a 2-metric space and $B(X)$ is the class of all nonempty bounded subsets of X.

Definition 3 (see [15]). A sequence $\{x_n\}$ in (X, d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x, c) = 0$ for all $c \in X$. The point x is called the limit of the sequence $\{x_n\}$ in X.

Definition 4 (see [15]). A sequence $\{x_n\}$ in (X, d) is said to be Cauchy sequence if $\lim_{m,n\to\infty} d(x_m, x_n, c) = 0$, for all $c \in X$.

Definition 5 (see [15]). The space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Let A, B, and C be nonempty sets in $B(X)$. Let $\delta(A, B, C)$ and $D(A, B, C)$ be the functions defined by

$$\delta(A,B,C) = \sup\{d(a,b,c) : a \in A,\ b \in B,\ c \in C\},$$
$$D(A,B,C) = \inf\{d(a,b,c) : a \in A,\ b \in B,\ c \in C\}. \tag{2}$$

If A is a singleton set, then $\delta(A, B, C) = \delta(a, B, C)$. In case B and C are also singleton sets, then

$$\delta(A,B,C) = D(A,B,C) = d(a,b,c) \tag{3}$$

for every $A = \{a\}$, $B = \{b\}$, and $C = \{c\}$. From the definition of δ, we can say that

$$\delta(A,B,C) = \delta(A,C,B) = \delta(C,A,B) = \delta(B,C,A)$$
$$= \delta(C,B,A) = \delta(B,A,C) \geq 0. \tag{4}$$

Also,

$$\delta(A,B,C) \leq \delta(A,B,E) + \delta(A,E,C) + \delta(E,B,C), \tag{5}$$

for all $A, B, C, E \in B(X)$. Let us note that $\delta(A, B, C) = 0$ if at least two of A, B, and C are equal singleton sets.

Definition 6 (see [15]). A sequence $\{A_n\}_{n=1}^{\infty}$ of subset of a 2-metric space (X, d) is said to be convergent to a subset A of X if,

(1) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} d(a_n, a, c) = 0$;

(2) given $\epsilon > 0$, there exists a positive integer n_0 such that $A_n \subset A_\epsilon$ for $n > n_0$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

Definition 7 (see [16]). Let $G : X \to X$ and $F : X \to B(X)$. Then, the pair $\{G, F\}$ is said to be weakly commuting if $GF(X) \in B(X)$ and

$$\delta(FGx, GFx, C)$$
$$\leq \max\{\delta(Gx, Fx, C), \delta(GFx, GFx, C)\} \tag{6}$$

for every $x \in X$, and $C \in B(X)$.

Definition 8 (see [16]). Let $G : X \to X$ and $F : X \to B(X)$. Then, the pair $\{G, F\}$ is said to be R-weakly commuting if

$$\delta(FGx, GFx, C)$$
$$\leq R \cdot \max\{\delta(Gx, Fx, C), \delta(GFx, GFx, C)\} \tag{7}$$

for every $x \in X$, and $C \in B(X)$ and $R > 0$.

Remark 9 (see [16]). If F is a single valued function, then Definitions 7 and 8 reduce to the following:

$$\delta(FGx, GFx, C) = d(FGx, GFx, C) \leq d(Gx, Fx, C)$$
$$= \delta(Gx, Fx, C), \tag{8}$$

$$\delta(FGx, GFx, C) = d(FGx, GFx, C)$$
$$\leq R \cdot d(Gx, Fx, C) \tag{9}$$
$$= R \cdot \delta(Gx, Fx, C),$$

respectively.

Common fixed points of Greguš type [17] have been proved by Diviccaro et al. [18], Fisher and Sessa [19], Mukherjee and Verma [20], Murthy et al. [21], and Singh et al. [14] under weaker conditions. Later, Murthy and Tas [16] generalized and extended the results of Singh et al. [14] and proved a theorem for set valued mapping in 2-metric space.

In this paper, we generalize the results of Abd EL-Monsef et al. [15] and Murthy and Tas [16] by using (ϕ, ψ)-weak contraction with Greguš type condition in 2-metric spaces for set valued mapping for R-weakly commuting maps.

2. Main Results

Let S and T be mapping of 2-metric space (X,d) into itself and $A,B:X\to B(X)$ are two set valued mappings satisfying the following condition:

$$\bigcup A(X)\subset T(X),\qquad \bigcup B(X)\subset S(X), \tag{10}$$

for every $x,y\in X$, $C\in B(X)$, and $p>0$,

$$\psi\left(\delta^{p}\left(Ax,By,C\right)\right)\leq\psi\left(M\left(x,y,C\right)\right)-\phi\left(M\left(x,y,C\right)\right), \tag{11}$$

where

$$\begin{aligned}M\left(x,y,C\right)&=a\delta^{p}\left(Sx,Ty,C\right)+\left(1-a\right)\\&\cdot\max\left\{\delta^{p}\left(Ax,Sx,C\right),\delta^{p}\left(By,Ty,C\right),\right.\\&\left.bD^{p}\left(Sx,By,C\right)+cD^{p}\left(Ty,Ax,C\right)\right\},\end{aligned} \tag{12}$$

$a\in(0,1)$, $0\leq b+c\leq 1/2$, $c\geq 0$, and

(1) $\psi:[0,\infty)\to[0,\infty)$ is continuous monotone nondecreasing function with $\psi(t)=0$ if and only if $t=0$;

(2) $\phi:[0,\infty)\to[0,\infty)$ is lower semicontinuous, monotone decreasing function with $\phi(t)=0$ if and only if $t=0$ and $\phi(t)>0$ for all $t\in(0,\infty)$.

Let x_0 be an arbitrary point in X. Since $\bigcup A(X)\subset T(X)$, then $\exists$ a point $x_1\in X$ such that $Tx_1\in Ax_0=y_0$. Now again, since $\bigcup B(X)\subset S(X)$, for the point $x_1\in X$. we can find a point $x_2\in X$ such that $Sx_1\in Bx_0=y_0$ and so on. Inductively, we can construct a sequence $\{x_n\}$ in X such that

$$\begin{aligned}Tx_{n+1}&\in Ax_n=y_n,\quad\text{when } n \text{ is even},\\Sx_{n+1}&\in Bx_n=y_n,\quad\text{when } n \text{ is odd}.\end{aligned} \tag{13}$$

Now we need to prove the following lemma for our main theorem.

Lemma 10. *Let (X,d) be 2-metric space. Let S and T be self-maps of X and $A,B:X\to B(X)$ satisfying conditions (10) and (11). Then, for every $n\in N$, one has*

$$\lim_{n\to\infty}\delta^{p}\left(y_n,y_{n+1},y_{n+2}\right)=0. \tag{14}$$

Proof. Since

$$\delta^{p}\left(y_{2n+2},y_{2n+1},y_{2n}\right)=\delta^{p}\left(Ax_{2n+2},Bx_{2n+1},y_{2n}\right), \tag{15}$$

we have

$$\begin{aligned}&\psi\left(\delta^{p}\left(y_{2n+2},y_{2n+1},y_{2n}\right)\right)\leq\psi\left[a\delta^{p}\left(Sx_{2n+2},Tx_{2n+1},\right.\right.\\&\left.y_{2n}\right)+\left(1-a\right)\max\left\{\delta^{p}\left(Sx_{2n+2},Ax_{2n+2},y_{2n}\right),\right.\\&\delta^{p}\left(Bx_{2n+1},Tx_{2n+2},y_{2n}\right),bD^{p}\left(Sx_{2n+2},Bx_{2n+1},y_{2n}\right)\\&\left.\left.+cD^{p}\left(Tx_{2n+1},Ax_{2n+2},y_{2n}\right)\right\}\right]-\phi\left[a\delta^{p}\left(Sx_{2n+2},\right.\right.\\&\left.Tx_{2n+1},y_{2n}\right)+\left(1-a\right)\\&\cdot\max\left\{\delta^{p}\left(Ax_{2n+2},Sx_{2n+2},y_{2n}\right),\right.\\&\delta^{p}\left(Bx_{2n+1},Tx_{2n+2},y_{2n}\right),bD^{p}\left(Sx_{2n+2},Bx_{2n+1},y_{2n}\right)\\&\left.\left.+cD^{p}\left(Tx_{2n+1},Ax_{2n+2},y_{2n}\right)\right\}\right]\leq\psi\left[a\delta^{p}\left(Sx_{2n+2},\right.\right.\\&\left.Tx_{2n+1},y_{2n}\right)+\left(1-a\right)\\&\cdot\max\left\{\delta^{p}\left(Sx_{2n+2},Ax_{2n+2},y_{2n}\right),\right.\\&\delta^{p}\left(Bx_{2n+1},Tx_{2n+2},y_{2n}\right),b\delta^{p}\left(Sx_{2n+2},Bx_{2n+1},y_{2n}\right)\\&\left.\left.+c\delta^{p}\left(Tx_{2n+1},Ax_{2n+2},y_{2n}\right)\right\}\right]-\phi\left[a\delta^{p}\left(Sx_{2n+2},\right.\right.\\&\left.Tx_{2n+1},y_{2n}\right)+\left(1-a\right)\\&\cdot\max\left\{\delta^{p}\left(Ax_{2n+2},Sx_{2n+2},y_{2n}\right),\right.\\&\delta^{p}\left(Bx_{2n+1},Tx_{2n+2},y_{2n}\right),b\delta^{p}\left(Sx_{2n+2},Bx_{2n+1},y_{2n}\right)\\&\left.\left.+c\delta^{p}\left(Tx_{2n+1},Ax_{2n+2},y_{2n}\right)\right\}\right]\leq\psi\left[a\delta^{p}\left(y_{2n+1},\right.\right.\\&\left.y_{2n},y_{2n}\right)+\left(1-a\right)\max\left\{\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right),\right.\\&\delta^{p}\left(y_{2n+1},y_{2n+1},y_{2n}\right),b\delta^{p}\left(y_{2n+1},y_{2n+1},y_{2n}\right)\\&\left.\left.+c\delta^{p}\left(y_{2n},y_{2n+2},y_{2n}\right)\right\}\right]-\phi\left[a\delta^{p}\left(y_{2n+1},y_{2n},y_{2n}\right)\right.\\&+\left(1-a\right)\max\left\{\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right),\right.\\&\delta^{p}\left(y_{2n+1},y_{2n+1},y_{2n}\right),b\delta^{p}\left(y_{2n+1},y_{2n+1},y_{2n}\right)\\&\left.\left.+c\delta^{p}\left(y_{2n},y_{2n+2},y_{2n}\right)\right\}\right]\leq\psi\left[\left(1-a\right)\delta^{p}\left(y_{2n+1},\right.\right.\\&\left.\left.y_{2n+2},y_{2n}\right)\right]-\phi\left[\left(1-a\right)\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\right].\end{aligned} \tag{16}$$

Since ψ is nondecreasing function, we can write

$$\begin{aligned}&\left(1-a\right)\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\leq\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\\&\quad\Longrightarrow\psi\left(\left(1-a\right)\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\right)\\&\quad\leq\psi\left(\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\right).\end{aligned} \tag{17}$$

Hence, we can write

$$\begin{aligned}&\psi\left(\delta^{p}\left(y_{2n+2},y_{2n+1},y_{2n}\right)\right)\\&\quad\leq\psi\left[\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\right]\\&\qquad-\phi\left[\left(1-a\right)\delta^{p}\left(y_{2n+1},y_{2n+2},y_{2n}\right)\right],\end{aligned} \tag{18}$$

a contradiction as $\phi(t)>0$ for each $t\in(0,\infty)$.

Consider

$$\delta^p(y_{2n+3}, y_{2n+2}, y_{2n+1}) = \delta^p(Ax_{2n+3}, Bx_{2n+2}, y_{2n+1}). \quad (19)$$

We have

$$\begin{aligned}
&\psi(\delta^p(y_{2n+3}, y_{2n+2}, y_{2n+1})) \le \psi[a\delta^p(Sx_{2n+3}, Tx_{2n+2}, \\
&\quad y_{2n+1}) + (1-a)\max\{\delta^p(Ax_{2n+3}, Sx_{2n+3}, y_{2n+1}), \\
&\quad \delta^p(Bx_{2n+2}, Tx_{2n+3}, y_{2n+1}), \\
&\quad bD^p(Sx_{2n+3}, Bx_{2n+2}, y_{2n+1}) \\
&\quad + cD^p(Tx_{2n+2}, Ax_{2n+3}, y_{2n+1})\}] - \phi[a\delta^p(Sx_{2n+3}, \\
&\quad Tx_{2n+2}, y_{2n+1}) + (1-a) \\
&\quad \cdot \max\{\delta^p(Ax_{2n+3}, Sx_{2n+3}, y_{2n+1}), \\
&\quad \delta^p(Bx_{2n+2}, Tx_{2n+3}, y_{2n+1}), \\
&\quad bD^p(Sx_{2n+3}, Bx_{2n+2}, y_{2n+1}) \\
&\quad + cD^p(Tx_{2n+2}, Ax_{2n+3}, y_{2n+1})\}] \\
&\quad \le \psi[a\delta^p(Sx_{2n+3}, Tx_{2n+2}, y_{2n+1}) + (1-a) \\
&\quad \cdot \max\{\delta^p(Ax_{2n+3}, Sx_{2n+3}, y_{2n+1}), \\
&\quad \delta^p(Bx_{2n+2}, Tx_{2n+3}, y_{2n+1}), \\
&\quad b\delta^p(Sx_{2n+3}, Bx_{2n+2}, y_{2n+1}) \\
&\quad + c\delta^p(Tx_{2n+2}, Ax_{2n+3}, y_{2n+1})\}] - \phi[a\delta^p(Sx_{2n+3}, \\
&\quad Tx_{2n+2}, y_{2n+1}) + (1-a) \\
&\quad \cdot \max\{\delta^p(Ax_{2n+3}, Sx_{2n+3}, y_{2n+1}), \\
&\quad \delta^p(Bx_{2n+2}, Tx_{2n+3}, y_{2n+1}), \\
&\quad b\delta^p(Sx_{2n+3}, Bx_{2n+2}, y_{2n+1}) \\
&\quad + c\delta^p(Tx_{2n+2}, Ax_{2n+3}, y_{2n+1})\}] \le \psi[a\delta^p(y_{2n+2}, \\
&\quad y_{2n+1}, y_{2n+1}) + (1-a) \\
&\quad \cdot \max\{\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1}), \\
&\quad \delta^p(y_{2n+2}, y_{2n+2}, y_{2n+1}), b\delta^p(y_{2n+2}, y_{2n+2}, y_{2n+1}) \\
&\quad + c\delta^p(y_{2n+1}, y_{2n+3}, y_{2n+1})\}] - \phi[a\delta^p(y_{2n+2}, \\
&\quad y_{2n+1}, y_{2n+1}) + (1-a) \\
&\quad \cdot \max\{\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1}), \\
&\quad \delta^p(y_{2n+2}, y_{2n+2}, y_{2n+1}), b\delta^p(y_{2n+2}, y_{2n+2}, y_{2n+1}) \\
&\quad + c\delta^p(y_{2n+1}, y_{2n+3}, y_{2n+1})\}] \le \psi[(1-a) \\
&\quad \cdot \delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1})] - \phi[(1-a)\delta^p(y_{2n+2}, \\
&\quad y_{2n+3}, y_{2n+1})].
\end{aligned} \quad (20)$$

Since ψ is nondecreasing function,

$$\begin{aligned}
&(1-a)\delta^p(y_{2n+1}, y_{2n+3}, y_{2n+1}) \\
&\quad \le \delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1}) \\
&\quad \Longrightarrow \psi((1-a)\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1})) \\
&\quad \le \psi(\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1})).
\end{aligned} \quad (21)$$

From the above conditions, we have

$$\begin{aligned}
&\psi(\delta^p(y_{2n+3}, y_{2n+2}, y_{2n+1})) \\
&\quad \le \psi[\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1})] \\
&\qquad - \phi[(1-a)\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1})],
\end{aligned} \quad (22)$$

a contradiction. Hence, we have

$$\lim_{n\to\infty} \delta^p(y_n, y_{n+1}, y_{n+2}) = 0. \quad (23)$$

Now we are ready to prove a common fixed point theorem by using the concept of R-weakly commuting maps theorem as follows. □

Theorem 11. *Let S and T be mapping of 2-metric space (X, d) into itself and $A, B : X \to B(X)$ are two set valued mappings satisfying conditions (10), (11), and (12) and the following:*

(a) *$S(X)$ or $T(X)$ is a complete subspace of X.*

(b) *The pair $\{A, S\}$ and $\{B, T\}$ are R-weakly commuting.*

Then A, B, S, and T have unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $\bigcup A(X) \subset T(X)$, then there exists a point $x_1 \in X$ such that $Tx_1 \in Ax_0 = y_0$. Now again, since $\bigcup B(X) \subset S(X)$, for the point $x_1 \in X$. we can find a point $x_2 \in X$ such that $Sx_1 \in Bx_0 = y_0$ and so on. Inductively, we can construct a sequence $\{x_n\}$ in X such that

$$\begin{aligned}
Tx_{n+1} &\in Ax_n = y_n, \quad \text{when } n \text{ is even}, \\
Sx_{n+1} &\in Bx_n = y_n, \quad \text{when } n \text{ is odd}.
\end{aligned} \quad (24)$$

Firstly, we have to prove that

$$\lim_{n\to\infty} \delta^p(y_n, y_{n+1}, C) = 0. \quad (25)$$

For this, assume $V_n = \delta(y_n, y_{n+1}, C)$ for $n = 0, 1, 2, \ldots$ by using (11):

$$\begin{aligned}
\psi(V_{2n}^p) &= \psi(\delta^p(y_{2n}, y_{2n+1}, C)) \\
&= \psi(\delta^p(Ax_{2n}, Bx_{2n+1}, C)) \\
&\le \psi(M(x_{2n}, x_{2n+1}, C)) \\
&\quad - \phi(M(x_{2n}, x_{2n+1}, C)),
\end{aligned} \quad (26)$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}, C) &= a\delta^p(Sx_{2n}, Tx_{2n+1}, C) + (1-a) \\ &\cdot \max\{\delta^p(Ax_{2n}, Sx_{2n}, C), \delta^p(Bx_{2n+1}, Tx_{2n+1}, C), \\ &bD^p(Sx_{2n}, Bx_{2n+1}, C) + cD^p(Tx_{2n+1}, Ax_{2n}, C)\} \\ &= a\delta^p(y_{2n-1}, y_{2n}, C) + (1-a) \\ &\cdot \max\{\delta^p(y_{2n}, y_{2n-1}, C), \delta^p(y_{2n+1}, y_{2n}, C), \\ &bD^p(y_{2n-1}, y_{2n+1}, C) + cD^p(y_{2n}, y_{2n}, C)\} \\ &\le a\delta^p(y_{2n-1}, y_{2n}, C) + (1-a) \\ &\cdot \max\{\delta^p(y_{2n}, y_{2n-1}, C), \delta^p(y_{2n+1}, y_{2n}, C), \\ &b\delta^p(y_{2n-1}, y_{2n+1}, C) + c\delta^p(y_{2n}, y_{2n}, C)\} \\ &\le a\delta^p(y_{2n-1}, y_{2n}, C) + (1-a) \\ &\cdot \max\{\delta^p(y_{2n}, y_{2n-1}, C), \delta^p(y_{2n+1}, y_{2n}, C), \\ &b(\delta^p(y_{2n-1}, y_{2n+1}, y_{2n}) + \delta^p(y_{2n-1}, y_{2n}, C) \\ &+ \delta^p(y_{2n}, y_{2n+1}, C))\} = aV_{2n-1}^p + (1-a) \\ &\cdot \max\left\{V_{2n-1}^p, V_{2n}^p, b\left(V_{2n-1}^p + V_{2n}^p\right)\right\}. \end{aligned} \tag{27}$$

By (26), we get

$$\begin{aligned} \psi\left(V_{2n}^p\right) &\le \psi\left(aV_{2n-1}^p\right. \\ &\left.+ (1-a)\max\left\{V_{2n-1}^p, V_{2n}^p, b\left(V_{2n-1}^p + V_{2n}^p\right)\right\}\right) \\ &- \phi\left(aV_{2n-1}^p\right. \\ &\left.+ (1-a)\max\left\{V_{2n-1}^p, V_{2n}^p, b\left(V_{2n-1}^p + V_{2n}^p\right)\right\}\right). \end{aligned} \tag{28}$$

If we take $V_{2n-1} < V_{2n}$, in the above equation by using the property of ϕ and ψ function, we can write

$$\begin{aligned} \psi\left(V_{2n}^p\right) &\le \psi\left(aV_{2n-1}^p + (1-a)V_{2n}^p\right) \\ &- \phi\left(aV_{2n-1}^p + (1-a)V_{2n}^p\right). \end{aligned} \tag{29}$$

The above implies that

$$\psi\left(V_{2n}^p\right) \le \psi\left(V_{2n}^p\right) - \phi\left(V_{2n}^p\right), \tag{30}$$

a contradiction. So we obtain

$$V_{2n-1} \ge V_{2n}. \tag{31}$$

By using (28) and (31) and employing the properties of ϕ and ψ function, we may write

$$\begin{aligned} &\psi\left(V_{2n}^p\right) \\ &\quad\le \psi\left[aV_{2n-1}^p + (1-a)\max\left\{V_{2n-1}^p, 2bV_{2n-1}^p\right\}\right] \\ &\qquad- \phi\left[aV_{2n-1}^p + (1-a)\max\left\{V_{2n-1}^p, 2bV_{2n-1}^p\right\}\right] \\ &\quad\le \psi\left[aV_{2n-1}^p + (1-a)V_{2n-1}^p\right] \\ &\qquad- \phi\left[aV_{2n-1}^p + (1-a)V_{2n}^p\right], \end{aligned} \tag{32}$$

$$\psi\left(V_{2n}^p\right) \le \psi\left(V_{2n-1}^p\right) - \phi\left(V_{2n-1}^p\right). \tag{33}$$

Again, (31) implies that

$$\begin{aligned} V_{2n} \le V_{2n-1} \Longrightarrow &\ \delta^p(y_{2n}, y_{2n+1}, C) \\ &\le \delta^p(y_{2n-1}, y_{2n}, C). \end{aligned} \tag{34}$$

Therefore, $V_{2n} = \delta^p(y_{2n-1}, y_{2n}, C)$ is a monotone decreasing sequence of nonnegative real number. There exists a nonnegative real number $r > 0$ such that

$$\lim_{n\to\infty} \delta^p(y_{2n-1}, y_{2n}, C) = r. \tag{35}$$

Letting limit $n \to \infty$ in (33), we get

$$\psi(r) \le \psi(r) - \phi(r), \tag{36}$$

a contradiction with the property of ϕ and ψ function. This implies that $r = 0$. Thus, we have

$$\lim_{n\to\infty} \delta^p(y_{2n-1}, y_{2n}, C) = 0. \tag{37}$$

Now, repeating the above process by putting $x = x_{2n+1}$ and $y = x_{2n+2}$, we obtain

$$\lim_{n\to\infty} \delta^p(y_{2n}, y_{2n+1}, C) = 0. \tag{38}$$

Hence, for all $n \ge 0$, we can write

$$\lim_{n\to\infty} \delta^p(y_n, y_{n+1}, C) = 0. \tag{39}$$

Next, we will show that $\{y_n\}$ is a Cauchy sequence. If, otherwise, there exists $\epsilon > 0$ and sequence of natural numbers $\{m(k)\}$ and $\{n(k)\}$ such that, for every natural number k,

$$n(k) > m(k) > k, \tag{40}$$

$$\delta^p(y_{m(k)}, y_{n(k)}, C) \ge \epsilon, \tag{41}$$

corresponding to $m(k)$, we can choose $n(k)$ to be the smallest integer such that (41) is satisfied. Then, we have

$$\delta^p(y_{m(k)}, y_{n(k)-1}, C) \le \epsilon. \tag{42}$$

Putting $x = x_{n(k)}$ and $y = y_{m(k)}$ in (11), we get

$$\begin{aligned} \psi(\delta^p(y_{n(k)}, y_{m(k)}, C)) &= \psi(\delta^p(Ax_{n(k)}, Bx_{m(k)}, C)) \\ &\le \psi(M(x_{n(k)}, x_{m(k)}, C)) \\ &\quad- \phi(M(x_{n(k)}, x_{m(k)}, C)), \end{aligned} \tag{43}$$

where

$$\begin{aligned}
M(x_{n(k)}, x_{m(k)}, C) &= a\delta^p(Sx_{n(k)}, Tx_{m(k)}, C) + (1-a) \\
&\cdot \max\{\delta^p(Ax_{n(k)}, Sx_{n(k)}, C), \\
&\delta^p(Bx_{m(k)}, Tx_{m(k)}, C), bD^p(Sx_{n(k)}, Bx_{m(k)}, C) \\
&+ cD^p(Tx_{m(k)}, Ax_{n(k)}, C)\} \le a\delta^p(Sx_{n(k)}, Tx_{m(k)}, \\
&C) + (1-a)\max\{\delta^p(Ax_{n(k)}, Sx_{n(k)}, C), \\
&\delta^p(Bx_{m(k)}, Tx_{m(k)}, C), b\delta^p(Sx_{n(k)}, Bx_{m(k)}, C) \\
&+ c\delta^p(Tx_{m(k)}, Ax_{n(k)}, C)\} \le a\delta^p(y_{n(k)-1}, y_{m(k)-1}, \\
&C) + (1-a)\max\{\delta^p(y_{n(k)}, y_{n(k)-1}, C), \\
&\delta^p(y_{m(k)}, y_{m(k)-1}, C), b\delta^p(y_{n(k)-1}, y_{m(k)}, C) \\
&+ c\delta^p(y_{m(k)-1}, y_{n(k)}, C)\}.
\end{aligned} \tag{44}$$

Letting limit $n \to \infty$ in the above,

$$\begin{aligned}
&M(x_{n(k)}, x_{m(k)}, C) \\
&\quad = a\delta^p(y_{n(k)-1}, y_{m(k)-1}, C) \\
&\qquad + (1-a)\max\{0, 0, b\epsilon + c\delta^p(y_{m(k)-1}, y_{n(k)}, C)\}.
\end{aligned} \tag{45}$$

Now

$$\begin{aligned}
\psi(\delta^p(y_{n(k)}, y_{m(k)}, C)) &\le \psi[a\delta^p(y_{n(k)-1}, y_{m(k)-1}, C) \\
&+ (1-a)\{b\epsilon + c\delta^p(y_{m(k)-1}, y_{n(k)}, C)\}] \\
&- \phi[a\delta^p(y_{n(k)-1}, y_{m(k)-1}, C) \\
&+ (1-a)\{b\epsilon + c\delta^p(y_{m(k)-1}, y_{n(k)}, C)\}].
\end{aligned} \tag{46}$$

We have to show that

$$\begin{aligned}
\delta^p(y_{m(k)}, y_{n(k)}, C) &\longrightarrow \epsilon, \\
\delta^p(y_{m(k)-1}, y_{n(k)-1}, C) &\longrightarrow \epsilon, \\
\delta^p(y_{m(k)-1}, y_{n(k)}, C) &\longrightarrow \epsilon.
\end{aligned} \tag{47}$$

Now, using properties of 2-metric space, we get

$$\begin{aligned}
\delta^p(y_{m(k)}, y_{n(k)}, C) \le \delta^p(y_{m(k)}, y_{n(k)}, y_{n(k)-1}) \\
+ \delta^p(y_{m(k)}, y_{n(k)-1}, C) \\
+ \delta^p(y_{n(k)-1}, y_{n(k)}, C).
\end{aligned} \tag{48}$$

Letting $\lim_{n\to\infty}$, we get

$$\delta^p(y_{m(k)}, y_{n(k)}, C) \longrightarrow \epsilon. \tag{49}$$

By using properties of 2-metric space, we can write

$$\begin{aligned}
&|\delta^p(y_{n(k)}, C, y_{m(k)-1}) - \delta^p(y_{n(k)}, C, y_{m(k)})| \\
&\quad \le \delta^p(y_{m(k)-1}, y_{m(k)}, y_{n(k)}) \\
&\qquad + \delta^p(y_{m(k)-1}, y_{m(k)}, C).
\end{aligned} \tag{50}$$

Letting limit $n \to \infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty}(\delta^p(y_{n(k)}, C, y_{m(k)-1}) - \delta^p(y_{n(k)}, C, y_{m(k)})) = 0 \\
&\text{or } \delta^p(y_{n(k)}, y_{m(k)-1}, C) \longrightarrow \epsilon.
\end{aligned} \tag{51}$$

Again using properties of 2-metric space, we can write

$$\begin{aligned}
\delta^p(y_{m(k)}, y_{n(k)-1}, C) \le \delta^p(y_{m(k)}, y_{n(k)-1}, y_{n(k)}) \\
+ \delta^p(y_{m(k)}, y_{n(k)}, C) \\
+ \delta^p(y_{n(k)}, y_{n(k)-1}, C).
\end{aligned} \tag{52}$$

Letting limit $n \to \infty$, we have

$$\delta^p(y_{m(k)}, y_{n(k)-1}, C) \longrightarrow \epsilon. \tag{53}$$

Using (46), (49), (51), and (53), we have

$$\begin{aligned}
\psi(\epsilon) \le \psi(a\epsilon + (1-a)(b\epsilon + c\epsilon)) \\
- \phi(a\epsilon + (1-a)(b\epsilon + c\epsilon)).
\end{aligned} \tag{54}$$

Since ψ is nondecreasing function, $(a+(1-a)(b+c))\epsilon \le \epsilon \Rightarrow \psi((a+(1-a)(b+c))\epsilon) \le \psi(\epsilon)$.

Therefore,

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(a\epsilon + (1-a)(b\epsilon + c\epsilon)), \tag{55}$$

a contradiction with ϕ function; hence, $\{y_n\}$ is a Cauchy sequence.

Assume $T(X)$ is a complete subspace X. Since the sequence $\{x_n\}$ is Cauchy, then its subsequence Tx_{2n+1} is Cauchy and converges to a point z in $T(X)$. Since $T(X)$ is complete subspace of X, for some $u \in X$,

$$Tx_{2n+1} \longrightarrow z = Tu. \tag{56}$$

According to the construction of sequence, we can have

$$\delta(Sx_{2n+2}, Tx_{2n+1}, C) \le \delta(y_{2n+1}, y_{2n}, C). \tag{57}$$

Letting limit $n \to \infty$,

$$\begin{aligned}
&\lim_{n\to\infty}\delta(Sx_{2n+2}, Tx_{2n+1}, C) \le \lim_{n\to\infty}\delta(y_{2n+1}, y_{2n}, C) \\
&\quad = 0.
\end{aligned} \tag{58}$$

The above implies that

$$\delta(Sx_{2n+2}, Tx_{2n+1}, C) = 0. \tag{59}$$

Therefore, we get

$$\lim_{n\to\partial} Sx_{2n+2} = \lim_{n\to\partial} Tx_{2n+1} = z. \tag{60}$$

Similarly,

$$\delta(Ax_{2n+2}, Bx_{2n+1}, C) \le \delta(y_{2n+2}, y_{2n+1}, C). \tag{61}$$

Letting limit $n \to \infty$,

$$\lim_{n\to\infty} \delta\left(Ax_{2n+2}, Bx_{2n+1}, C\right) \leq \lim_{n\to\infty} \delta\left(y_{2n+2}, y_{2n+1}, C\right) = 0. \quad (62)$$

Therefore, we get

$$\lim_{n\to\partial} Ax_{2n+2} = \lim_{n\to\partial} Bx_{2n+1} = z. \quad (63)$$

Now we will show that u is a coincidence point of B and T. For $n = 0, 1, 2, 3, \ldots$ and putting $x = x_{2n}$ and $y = u$ in (11) and (12), we have

$$\psi\left(\delta^p\left(Ax_{2n}, Bu, C\right)\right) \leq \psi\left(M\left(x_{2n}, u, C\right)\right) - \phi\left(M\left(x_{2n}, u, C\right)\right), \quad (64)$$

where

$$\begin{aligned} M\left(x_{2n}, u, C\right) &= a\delta^p\left(Sx_{2n}, Tu, C\right) + (1-a) \\ &\cdot \max\left\{\delta^p\left(Ax_{2n}, Sx_{2n}, C\right), \delta^p\left(Bu, Tu, C\right),\right. \\ &\left. bD^p\left(Sx_{2n}, Bu, C\right) + cD^p\left(Tu, Ax_{2n}, C\right)\right\}, \\ M\left(x_{2n}, u, C\right) &= a\delta^p\left(Sx_{2n}, Tu, C\right) + (1-a) \\ &\cdot \max\left\{\delta^p\left(Ax_{2n}, Sx_{2n}, C\right), \delta^p\left(Bu, Tu, C\right),\right. \\ &\left. b\delta^p\left(Sx_{2n}, Bu, C\right) + c\delta^p\left(Tu, Ax_{2n}, C\right)\right\}. \end{aligned} \quad (65)$$

Letting limit $n \to \infty$ in the above equation, we have

$$\lim_{n\to\infty} M\left(x_{2n}, u, C\right) \leq (1-a)\max\left\{\delta^p\left(Bu, z, C\right), b\delta^p\left(z, Bu, C\right)\right\}. \quad (66)$$

Letting limit $n \to \infty$ in (64), we get

$$\begin{aligned} &\psi\left(\delta^p\left(z, Bu, C\right)\right) \\ &\leq \psi\left((1-a)\max\left\{\delta^p\left(Bu, z, C\right), b\delta^p\left(z, Bu, C\right)\right\}\right) \\ &\quad - \phi\left((1-a)\max\left\{\delta^p\left(Bu, z, C\right), b\delta^p\left(z, Bu, C\right)\right\}\right). \end{aligned} \quad (67)$$

This implies that

$$\psi\left(\delta^p\left(z, Bu, C\right)\right) \leq \psi\left((1-a)\delta^p\left(Bu, z, C\right)\right) - \phi\left((1-a)\delta^p\left(Bu, z, C\right)\right). \quad (68)$$

Since ψ is monotone nondecreasing function, we can write

$$\psi\left(\delta^p\left(z, Bu, C\right)\right) \leq \psi\left(\delta^p\left(Bu, z, C\right)\right) - \phi\left((1-a)\delta^p\left(Bu, z, C\right)\right), \quad (69)$$

a contradiction. Hence, u is the coincidence point of B and T; that is, $\{z\} = Bu = \{Tu\}$. Since $\bigcup B(X) \subset S(X)$, for some $v \in X$, we have $\{Sv\} = Bu = \{Tu\}$. If $Av \neq Bu$, then, from (11) and (12), putting $x = v$ and $y = u$, we get

$$\psi\left(\delta^p\left(Av, Bu, C\right)\right) \leq \psi\left(\delta^p\left(v, u, C\right)\right) - \phi\left((1-a)\delta^p\left(v, u, C\right)\right), \quad (70)$$

where

$$\begin{aligned} M\left(v, u, C\right) &= a\delta^p\left(Sv, Tu, C\right) + (1-a) \\ &\cdot \max\left\{\delta^p\left(Av, Sv, C\right), \delta^p\left(Bu, Tu, C\right),\right. \\ &\left. bD^p\left(Sv, Bu, C\right) + cD^p\left(Tu, Av, C\right)\right\}, \\ M\left(v, u, C\right) &= (1-a)\max\left\{\delta^p\left(Av, Sv, C\right),\right. \\ &\left. c\delta^p\left(Tu, Av, C\right)\right\}. \end{aligned} \quad (71)$$

Since $0 \leq b + c \leq 1/2$, $0 < a < 1$, and $b, c \geq 0$, we have

$$\psi\left(\delta^p\left(Av, Bu, C\right)\right) \leq \psi\left((1-a)\delta^p\left(Av, Sv, C\right)\right) - \phi\left((1-a)\delta^p\left(Av, Sv, C\right)\right). \quad (72)$$

Since ψ is monotone nondecreasing function, we can write

$$\psi\left(\delta^p\left(Av, Bu, C\right)\right) \leq \psi\left(\delta^p\left(Av, Sv, C\right)\right) - \phi\left((1-a)\delta^p\left(Av, Sv, C\right)\right), \quad (73)$$

a contradiction. Hence, $Av = Bu = \{Sv\} = \{Tu\} = z$.

Since (A, S) are R-weakly commuting maps, then

$$\begin{aligned} &\delta\left(ASv, SAu, C\right) \\ &\quad \leq R \cdot \max\left\{\delta\left(Av, Sv, C\right), \delta\left(SAv, SAv, C\right)\right\}, \end{aligned} \quad (74)$$

which implies that

$$ASv = SAv \Longrightarrow Az = \{Sz\}. \quad (75)$$

Again, using (11), putting $x = z$ and $y = u$, we have

$$\begin{aligned} &\psi\left(\delta^p\left(Az, z, C\right)\right) = \psi\left(\delta^p\left(Az, Bu, C\right)\right) \leq \psi\left(a\delta^p\left(Sz,\right.\right. \\ &\quad Tu, C) + (1-a)\max\left\{\delta^p\left(Az, Sz, C\right),\right. \\ &\quad \delta^p\left(Bu, Tu, C\right), bD^p\left(Sz, Bu, C\right) \\ &\quad \left.\left. + cD^p\left(Tu, Az, C\right)\right\}\right) - \phi\left(a\delta^p\left(Sz, Tu, C\right) + (1\right. \\ &\quad - a)\max\left\{\delta^p\left(Az, Sz, C\right), \delta^p\left(Bu, Tu, C\right),\right. \\ &\quad \left.\left. bD^p\left(Sz, Bu, C\right) + cD^p\left(Tu, Az, C\right)\right\}\right) \\ &\quad \leq \psi\left(a\delta^p\left(Az, z, C\right) + (1-a)\max\left\{0, 0,\right.\right. \\ &\quad \left.\left. b\delta^p\left(Az, z, C\right) + c\delta^p\left(Az, z, C\right)\right\}\right) - \phi\left(a\delta^p\left(Az, z,\right.\right. \\ &\quad C) + (1-a)\max\left\{0, 0, b\delta^p\left(Az, z, C\right)\right. \\ &\quad \left.\left. + c\delta^p\left(Az, z, C\right)\right\}\right), \\ &\psi\left(\delta^p\left(Az, z, C\right)\right) \leq \psi\left(a + (1-a)(b+c)\delta^p\left(Az, z,\right.\right. \\ &\quad \left. C)\right) - \phi\left(a + (1-a)(b+c)\delta^p\left(Az, z, C\right)\right). \end{aligned} \quad (76)$$

Since ψ is nondecreasing function, we can write

$$\begin{aligned}&(a+(1-a)(b+c))\,\delta^p(Az,z,C)\le\delta^p(Az,z,C)\\&\Longrightarrow\psi((a+(1-a)(b+c))\,\delta^p(Az,z,C))\\&\le\psi(\delta^p(Az,z,C)).\end{aligned}\tag{77}$$

This implies that

$$\begin{aligned}&\psi(\delta^p(Az,z,C))\\&\le\psi(\delta^p(Az,z,C))\\&\quad-\phi(a+(1-a)(b+c)\,\delta^p(Az,z,C)),\end{aligned}\tag{78}$$

a contradiction. Hence, we get $Az=\{Sz\}=\{z\}$ and z is a common fixed point of A and S.

Similarly, we can show that $\{z\}$ is common fixed point of B and T by assuming $\{B,T\}$ is a pair of R-weakly commuting maps. Hence, $Az=Bz=\{Sz\}=\{Tz\}=\{z\}$.

For the uniqueness of common fixed point z, let z^* be another fixed point of A, B, S, and T. By using (11), we have

$$\begin{aligned}&\psi(\delta^p(Az,Bz^*,C))=\psi(\delta^p(z,z^*,C))\\&\le\psi(a\delta^p(Sz,Tz^*,C)+(1-a)\\&\cdot\max\{\delta^p(Az,Sz,C),\delta^p(Bz^*,Tz^*,C),\\&bD^p(Sz,Bz^*,C)+cD^p(Tz^*,Az,C)\})\\&-\phi(a\delta^p(Sz,Tz^*,C)+(1-a)\\&\cdot\max\{\delta^p(Az,Sz,C),\delta^p(Bz^*,Tz^*,C),\\&bD^p(Sz,Bz^*,C)+cD^p(Tz^*,Az,C)\})\\&\le\psi(a\delta^p(Sz,Tz^*,C)+(1-a)\\&\cdot\max\{\delta^p(Az,Sz,C),\delta^p(Bz^*,Tz^*,C),\\&b\delta^p(Sz,Bz^*,C)+c\delta^p(Tz^*,Az,C)\})\\&-\phi(a\delta^p(Sz,Tz^*,C)+(1-a)\\&\cdot\max\{\delta^p(Az,Sz,C),\delta^p(Bz^*,Tz^*,C),\\&b\delta^p(Sz,Bz^*,C)+c\delta^p(Tz^*,Az,C)\}),\\&\psi(\delta^p(z,z^*,C))\le\psi(a\delta^p(z,z^*,C)+(1-a)\\&\cdot\max\{0,0,b\delta^p(z,z^*,C)+c\delta^p(z^*,z,C)\})\\&-\phi(a\delta^p(z,z^*,C)+(1-a)\max\{0,0,\\&b\delta^p(z,z^*,C)+c\delta^p(z^*,z,C)\}).\end{aligned}\tag{79}$$

Since ψ is nondecreasing function, we can write

$$\begin{aligned}&(a+(1-a)(b+c))\,\delta^p(z,z^*,C)\le\delta^p(z,z^*,C)\\&\Longrightarrow\psi((a+(1-a)(b+c))\,\delta^p(z,z^*,C))\\&\le\psi(\delta^p(z,z^*,C)).\end{aligned}\tag{80}$$

This implies that

$$\begin{aligned}&\psi(\delta^p(z,z^*,C))\\&\le\psi(\delta^p(z,z^*,C))\\&\quad-\phi(a+(1-a)(b+c)\,\delta^p(z,z^*,C)),\end{aligned}\tag{81}$$

a contradiction. Hence, we have $z=z^*$; that is, that z is unique common fixed point of A,B,S, and T in X. □

As an immediate consequence of the above theorem, we have the following corollaries.

Corollary 12. *Let S and T be mapping of 2-metric space (X,d) into itself and $A,B:X\to B(X)$ are two set valued mappings satisfying conditions (a) and (b) of Theorem 11 and the following conditions:*

$$\begin{aligned}&\bigcup A(X)\subset T(X),\\&\bigcup B(X)\subset S(X),\end{aligned}\tag{82}$$

for every $x,y\in X$, $C\in B(X)$, and $p>0$,

$$\delta^p(Ax,By,C)\le M(x,y,C)-\phi(M(x,y,z)),\tag{83}$$

where

$$\begin{aligned}M(x,y,z)&=a\delta^p(Sx,Ty,C)+(1-a)\\&\cdot\max\{\delta^p(Ax,Sx,C),\delta^p(By,Ty,C),\\&bD^p(Sx,By,C)+cD^p(Ty,Ax,C)\},\end{aligned}\tag{84}$$

$a\in(0,1)$, $0\le b+c\le 1/2$, $c\ge 0$, and $\phi:[0,\infty)\to[0,\infty)$ is lower semicontinuous, monotone decreasing function with $\phi(t)=0$ if and only if $t=0$ and $\phi(t)>0$ for all $t\in(0,\infty)$.

Then A,B,S, and T have unique common fixed point in X.

Proof. The proof follows from Theorem 11 by taking $\psi(t)=t$. □

Corollary 13. *Let S and T be mapping of 2-metric space (X,d) into itself and $A,B:X\to B(X)$ are two set valued mappings satisfying conditions (a) and (b) of Theorem 11 and the following conditions:*

$$\begin{aligned}&\bigcup A(X)\subset T(X),\\&\bigcup B(X)\subset S(X),\end{aligned}\tag{85}$$

for every $x,y\in X$, $C\in B(X)$, and $p>0$,

$$\delta^p(Ax,By,C)\le k(M(x,y,z)),\tag{86}$$

where

$$\begin{aligned}M(x,y,z)&=a\delta^p(Sx,Ty,C)+(1-a)\\&\cdot\max\{\delta^p(Ax,Sx,C),\delta^p(By,Ty,C),\\&bD^p(Sx,By,C)+cD^p(Ty,Ax,C)\},\end{aligned}\tag{87}$$

$a\in(0,1)$, $0\le b+c\le 1/2$, $c\ge 0$, and $k\in(0,1)$. Then, A,B,S, and T have unique common fixed point in X.

Proof. The proof follows from Corollary 12 by taking $\phi(t) = (1-k)t$, where $k \in (0,1)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in *New Results in Operator Theory and Its Applications*, I. Gohberg and Y. Lyubich, Eds., vol. 98 of *Operator Theory: Advances and Applications*, pp. 7–22, Birkhäuser, Basel, Switzerland, 1997.

[2] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.

[3] P. N. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 406368, 8 pages, 2008.

[4] Q. Zhang and Y. Song, "Common point theory for generalized φ-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 75–78, 2009.

[5] D. Đorić, "Common fixed point for generalized (ψ, φ)-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1896–1900, 2009.

[6] S. Gahler, "2-Metrische Räume und ihre topologische structure," *Mathematische Nachrichten*, vol. 26, pp. 115–148, 1963.

[7] S. Gahler, "Uber die unformesior bakat 2-metriche Reume," *Mathematische Nachrichten*, vol. 28, pp. 235–244, 1965.

[8] S. Gähler, "Zur geometrie 2-metrischer Räume," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 11, pp. 665–667, 1966.

[9] B. C. Dhage, *A study of some fixed points theorems [Ph.D. thesis]*, Marathawada University, Aurangabad, India, 1984.

[10] Z. Mustafa and B. Sims, "Some remarks concerning D-metric spaces," in *Proceedings of the International Conference on Fixed Point Theory and Applications*, pp. 189–198, Valencla, Spain, July 2003.

[11] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[12] S. V. R. Naidu and J. TR. Prasad, "Fixed point theoems in 2-metric spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 17, no. 8, pp. 974–993, 1988.

[13] K. Iseki, "Fixed point theorems in 2-metric spaces," *Mathematics Seminar Notes*, vol. 3, pp. 133–136, 1975.

[14] M. R. Singh, L. S. Singh, and P. P. Murthy, "Common fixed points of set valued mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 25, no. 6, pp. 411–415, 2001.

[15] M. E. Abd EL-Monsef, H. M. Abu-Donia, and K. Abd-Rabou, "New types of common fixed point theorems in 2-metric spaces," *Chaos, Solitons and Fractals*, vol. 41, no. 3, pp. 1435–1441, 2009.

[16] P. P. Murthy and K. Tas, "New common fixed point theorems of Greguš type for R-weakly commuting mappings in 2-metric spaces," *Hacettepe Journel of Mathematics and Statistics*, vol. 38, no. 3, pp. 285–291, 2009.

[17] M. Greguš, "A fixed point theorem in Banach Spaces," *Bollettino dell'Unione Matematica Italiana A*, vol. 17, pp. 193–198, 1980.

[18] M. L. Diviccaro, B. Fisher, and S. Sessa, "A common fixed point theorem of Greguš type," *Publicationes Mathematicae Debrecen*, vol. 34, pp. 83–89, 1987.

[19] B. Fisher and S. Sessa, "On a fixed point theorem of Greguš," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 1, pp. 23–28, 1986.

[20] R. N. Mukherjee and V. Verma, "A note on a fixed point theorem of Greguš," *Mathematica Japonica*, vol. 33, no. 5, pp. 745–749, 1988.

[21] P. P. Murthy, Y. J. Cho, and B. Fisher, "Common fixed points of Greguš type mappings," *Glasnik Matematicki*, vol. 30, no. 50, pp. 335–341, 1995.

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