Hindawi Publishing Corporation
Journal of Operators
Volume 2015, Article ID 216060, 8 pages
http://dx.doi.org/10.1155/2015/216060

Research Article

Frame Multiresolution Analysis on Local Fields of Positive Characteristic

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Received 21 August 2014; Accepted 12 January 2015

Academic Editor: Gerd Teschke

We present a notion of frame multiresolution analysis on local fields of positive characteristic based on the theory of shift-invariant spaces. In contrast to the standard setting, the associated subspace V_0 of $L^2(K)$ has a frame, a collection of translates of the scaling function φ of the form $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of nonnegative integers. We investigate certain properties of multiresolution subspaces which provides the quantitative criteria for the construction of frame multiresolution analysis (FMRA) on local fields of positive characteristic. Finally, we provide a characterization of wavelet frames associated with FMRA on local field K of positive characteristic using the shift-invariant space theory.

1. Introduction

Multiresolution analysis is considered as the heart of wavelet theory. The concept of multiresolution analysis provides a natural framework for understanding and constructing discrete wavelet systems. A multiresolution analysis is an increasing family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$, represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of multiresolution analysis to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$ (see [1, 2]).

On the other hand, this elegant tool for the construction of wavelet bases has been extensively studied by several authors on the various spaces, namely, Cantor dyadic groups [3], locally compact Abelian groups [4], p-adic fields [5], zero-dimensional groups [6], and Vilenkin groups [7]. Recently, R. L. Benedetto and J. J. Benedetto [8] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach; their method is based on the theory of wavelet sets. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$ whereas local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. The structures and metrics of the local fields of zero and positive characteristic are similar, but their wavelet and MRA theory are quite different. The concept of multiresolution analysis on a local field K of positive characteristic was introduced by Jiang et al. [9]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [10] using extension principles. As far as the characterization of wavelets on local fields is concerned, Behera and Jahan [11] have given the characterization of all wavelets associated with multiresolution analysis on local field K based on results on affine and quasiaffine frames. Recently, Shah and Abdullah [12] have introduced the notion of nonuniform multiresolution analysis on local field K of positive characteristic and

obtained the necessary and sufficient condition for a function φ to generate a nonuniform multiresolution analysis on local fields. More results in this direction can also be found in [13, 14] and the references therein.

Since the use of multiresolution analysis has proven to be a very efficient tool in wavelet theory mainly because of its simplicity, it is of interest to try to generalize this notion as much as possible while preserving its connection with wavelet analysis. In this connection, Benedetto and Li [15] considered the dyadic *semiorthogonal frame multiresolution analysis* of $L^2(\mathbb{R})$ with a single scaling function and successfully applied the theory in the analysis of narrow band signals. The characterization of the dyadic semiorthogonal frame multiresolution analysis with a single scaling function admitting a single frame wavelet whose dyadic dilations of the integer translates form a frame for $L^2(\mathbb{R})$ was obtained independently by Benedetto and Treiber by a direct method [16] and by Kim and Lim by using the theory of shift-invariant spaces [17]. Later on, Yu [18] extended the results of Benedetto and Li's theory of FMRA to higher dimensions with arbitrary integral expansive matrix dilations and has established the necessary and sufficient conditions to characterize semiorthogonal multiresolution analysis frames for $L^2(\mathbb{R}^n)$.

In this paper, we introduce the notion of *frame multiresolution analysis* (FMRA) on local field K of positive characteristic by extending the above described methods. We first investigate the properties of multiresolution subspaces, which will provide the quantitative criteria for the construction of FMRA on local fields of positive characteristic. We also show that the scaling property of an FMRA also holds for the wavelet subspaces and that the space $L^2(K)$ can be decomposed into the orthogonal sum of these wavelet subspaces. Finally, we study the characterization of wavelet frames associated with FMRA on local field K of positive characteristic using the shift-invariant space theory.

The paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic including the definition of a frame. The notion of frame multiresolution analysis of $L^2(K)$ is introduced in Section 3 and its quantitative criteria are given by means of Theorem 12. In Section 4, we establish a complete characterization of wavelet frames generated by a finite number of mother wavelets on local field K of positive characteristic.

2. Preliminaries on Local Fields

Let K be a field and a topological space. Then, K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence, by a local field, we mean a field K which is locally compact, nondiscrete, and totally disconnected. The p-adic fields are examples of local fields. More details are referred to in [19, 20]. In the rest of this paper, we use $\mathbb{N}, \mathbb{N}_0$, and $\mathbb{Z}$ to denote the sets of natural and nonnegative integers and integers, respectively.

Let K be a fixed local field. Then, there is an integer $q = \mathfrak{p}^r$, where $\mathfrak{p}$ is a fixed prime element of K and r is a positive integer, and a norm $|\cdot|$ on K such that for all $x \in K$ we have $|x| \geq 0$ and for each $x \in K \setminus \{0\}$ we get $|x| = q^k$ for some integer k. This norm is non-Archimedean; that is, $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$ and $|x + y| = \max\{|x|, |y|\}$ whenever $|x| \neq |y|$. Let dx be the Haar measure on the locally compact, topological group $(K, +)$. This measure is normalized so that $\int_{\mathfrak{D}} dx = 1$, where $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$, and hence as a result $\mathfrak{B}$ is both principal and prime. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it is easy to verify that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x'$, $x' \in \mathfrak{D}^*$. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is a compact subgroup of K^+ and is known as the *fractional ideals* of K^+ (see [19]). Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$; then, every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$, implying that if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx)$ for $x \in K$. Suppose that χ_u is any character on K^+; then, clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K^+, then, as it was proved in [20], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\widehat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\widehat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx. \tag{1}$$

It is noted that

$$\widehat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\,\chi(-\xi x)\,dx. \tag{2}$$

Furthermore, the properties of Fourier transform on local field are much similar to those on the real line. In particular, Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n) : n \in \mathbb{N}_0\}$. Since $\mathfrak{D}/\mathfrak{B} \cong GF(q)$, where $GF(q)$ is a c-dimensional vector space over the field $GF(q)$ (see [20]), we choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ such that $0 \leq n < q$, we have

$$n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \leq a_k < p, \quad k = 0, 1, \ldots, c-1. \tag{3}$$

Define

$$u(n) = \left(a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1}\right)\mathfrak{p}^{-1}. \tag{4}$$

For $n \in \mathbb{N}_0$ and $0 \leq b_k < q$, $k = 0, 1, 2, \ldots, s$, we write

$$n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \tag{5}$$

such that

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \quad (6)$$

Also, for $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, we have

$$u\left(rq^k + s\right) = u(r)\mathfrak{p}^{-k} + u(s). \quad (7)$$

Further, it is easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter, we use the notation $\chi_n := \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and let $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi\left(\zeta_\mu \mathfrak{p}^{-j}\right) = \begin{cases} \exp\left(\dfrac{2\pi i}{p}\right), & \mu = 0,\ j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \ne 1. \end{cases} \quad (8)$$

Definition 1. Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k : k \in \mathbb{N}_0\}$ in $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|_2^2 \le \sum_{k\in\mathbb{N}_0} |\langle f, f_k\rangle|^2 \le B\|f\|_2^2, \quad \forall f \in \mathbb{H}. \quad (9)$$

The largest constant A and the smallest constant B satisfying (9) are called the *upper* and the *lower frame bound*, respectively. A frame is said to be *tight* if it is possible to choose $A = B$ and a frame is said to be *exact* if it ceases to be a frame when any one of its elements is removed. An exact frame is also known as a *Riesz basis*.

The following theorem gives us an elementary characterization of frames.

Theorem 2 (see [15]). *A sequence $\{f_k : k \in \mathbb{N}_0\}$ in a Hilbert space $\mathbb{H}$ is a frame for $\mathbb{H}$ if and only if there exists a sequence $a = \{a_k\} \in l^2(\mathbb{N}_0)$ with $\|a\|_{l^2(\mathbb{N}_0)} \le C\|f\|$, $C > 0$, such that*

$$f = \sum_{k\in\mathbb{N}_0} a_k f_k, \quad (10)$$

and $\sum_{k\in\mathbb{N}_0} |\langle f, f_k\rangle|^2 < \infty$, for every $f \in \mathbb{H}$.

For $j \in \mathbb{Z}$ and $y \in K$, we define the dilation operator δ_j and the translation operator τ_y as follows:

$$\delta_j f(x) = q^{j/2} f\left(\mathfrak{p}^{-j}x\right), \qquad \tau_y f(x) = f(x - y), \quad f \in L^2(K). \quad (11)$$

Our study uses the theory of shift-invariant spaces developed in [21, 22] and the references therein. A closed subspace S of $L^2(K)$ is said to be *shift-invariant* if $\tau_k f \in S$ whenever $f \in S$ and $k \in \mathbb{N}_0$. A closed shift-invariant subspace S of $L^2(K)$ is said to be *generated* by $\Phi \subset L^2(K)$ if $S = \overline{\text{span}}\{\tau_k\varphi(x) := \varphi(x - u(k)) : k \in \mathbb{N}_0, \varphi \in \Phi\}$. The cardinality of the smallest generating set Φ for S is called the *length* of S which is denoted by $|S|$. If $|S|$ = finite, then S is called a *finite shift-invariant space* (FSI) and if $|S| = 1$, then S is called a *principal shift-invariant space* (PSI). Moreover, the *spectrum* of a shift-invariant space is defined to be

$$\sigma(S) = \left\{\xi \in \mathfrak{D} : \widehat{S}(\xi) \ne \{0\}\right\}, \quad (12)$$

where $\widehat{S}(\xi) = \{\widehat{f}(\xi + u(k)) \in l^2(\mathbb{N}_0) : f \in S,\ k \in \mathbb{N}_0\}$.

3. Frame Multiresolution Analysis on Local Fields

We first introduce the notion of a *frame multiresolution analysis* (FMRA) of $L^2(K)$.

Definition 3. Let K be a local field of positive characteristic $p > 0$ and let $\mathfrak{p}$ be a prime element of K. A frame multiresolution analysis of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(c) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(d) the function f lying in V_0 implies that the collection $f(\cdot - u(k)) \in V_0$, for all $k \in \mathbb{N}_0$;

(e) the sequence $\{\tau_k\varphi := \varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a frame for the subspace V_0.

The function φ is known as the *scaling function* while the subspaces V_j's are known as *approximation spaces* or *multiresolution subspaces*. A frame multiresolution analysis is said to be *nonexact* and, respectively, *exact* if the frame for the subspace V_0 is *nonexact* and, respectively, *exact*. In multiresolution analysis studied in [9], the frame condition is replaced by that of an orthonormal basis or an exact frame.

Next, we establish several properties of multiresolution subspaces that will help in the construction of frame multiresolution analysis on local field K of positive characteristic. The following proposition shows that, for every $j \in \mathbb{Z}$, the sequence $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$, where

$$\varphi_{j,k}(x) = q^{j/2}\varphi\left(\mathfrak{p}^{-j}x - u(k)\right), \quad (13)$$

is a frame for V_j.

Proposition 4. *Let $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ be a frame for $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$ and*

$$V_j = \left\{f \in L^2(K) : f\left(\mathfrak{p}^j\cdot\right) \in V_0\right\}, \quad j \in \mathbb{Z}. \quad (14)$$

Then, the sequence $\{\varphi_{j,k} : k \in \mathbb{N}_0\}$ defined in (13) is a frame for V_j with the same bounds as those for V_0.

Proof. For any $f \in V_j$, we have

$$\sum_{k\in\mathbb{N}_0} \left|\left\langle \delta^{-j} f, \tau_k\varphi\right\rangle\right|^2 = \sum_{k\in\mathbb{N}_0} \left|\int_K q^{-j/2} f(\mathfrak{p}^j x)\overline{\varphi(x-u(k))}dx\right|^2 \tag{15}$$

$$= \sum_{k\in\mathbb{N}_0} \left|\int_K f(x) q^{j/2}\overline{\varphi(\mathfrak{p}^{-j}x-u(k))}dx\right|^2 \tag{16}$$

$$= \sum_{k\in\mathbb{N}_0} \left|\left\langle f, \varphi_{j,k}\right\rangle\right|^2. \tag{17}$$

Since $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ is a frame for V_0, therefore we have

$$A\|f\|_2^2 = A\left\|\delta^{-j} f\right\|_2^2 \le \sum_{k\in\mathbb{N}_0} \left|\left\langle f, \varphi_{j,k}\right\rangle\right|^2 \le B\left\|\delta^{-j} f\right\|_2^2 = B\|f\|_2^2. \tag{18}$$

This completes the proof of the proposition. □

We now characterize all functions of FSI space by virtue of its Fourier transforms.

Proposition 5. *Let $\{\tau_k\varphi : k \in \mathbb{N}_0, \varphi \in \Phi\}$ be a frame for its closed linear* span *V, where $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\} \subset L^2(K)$. Then, $f \in L^2(K)$ lies in V if and only if there exist integral periodic functions $h_\ell \in L^2(\mathfrak{D})$, $\ell = 1, \ldots, L$, such that*

$$\widehat{f}(\xi) = \sum_{\ell=1}^{L} h_\ell(\xi)\,\widehat{\varphi}_\ell(\xi). \tag{19}$$

Proof. Since the system $\{\tau_k\varphi : k \in \mathbb{N}_0, \varphi \in \Phi\}$ is a frame for V, then, by Theorem 2, there exists a sequence $\{a_k^\ell\} \in l^2(\mathbb{N}_0)$, for $\ell = 1, \ldots, L$, such that

$$f(x) = \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0} a_k^\ell \varphi_\ell(x-u(k)). \tag{20}$$

Taking Fourier transform on both sides of (20), we obtain

$$\widehat{f}(\xi) = \sum_{\ell=1}^{L} h_\ell(\xi)\,\widehat{\varphi}_\ell(\xi), \tag{21}$$

where $h_\ell(\xi) = \sum_{k\in\mathbb{N}_0} a_k^\ell \chi_k(\xi)$ are the integral periodic functions in $L^2(\mathfrak{D})$. The converse is established by taking h_ℓ as above and applying the inverse Fourier transform on both sides of (19). □

We now study some properties of the multiresolution subspaces V_j of the form (14) by means of the Fourier transform.

Proposition 6. *Let $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ be a frame for $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$ and, for $j \in \mathbb{Z}$, define V_j by (14). Then, for any function $\psi \in V_1$, there exists periodic function $G \in L^2(\mathfrak{D})$ such that*

$$\widehat{\psi}\left(\mathfrak{p}^{-1}\xi\right) = q^{1/2} G(\xi)\,\widehat{\varphi}(\xi). \tag{22}$$

Proof. By the definition of V_j, it follows that $\psi(\mathfrak{p}\cdot) \in V_0$. By Proposition 5, there exists a periodic function $G \in L^2(\mathfrak{D})$ such that $(\psi(\mathfrak{p}\cdot))^\wedge = \widehat{\psi}(\mathfrak{p}^{-1}\xi) = q^{1/2}G(\xi)\widehat{\varphi}(\xi)$ lies in $L^2(K)$. □

The following theorem establishes a sufficient condition to ensure that the nesting property holds for the subspaces V_j's.

Theorem 7. *Let $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ be a frame for $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$ and, for $j \in \mathbb{Z}$, define V_j by (14). Assume that there exists a periodic function $H \in L^\infty(\mathfrak{D})$ such that*

$$\widehat{\varphi}(\xi) = q^{1/2} H(\mathfrak{p}\xi)\,\widehat{\varphi}(\mathfrak{p}\xi). \tag{23}$$

Then, $V_j \subseteq V_{j+1}$, for every $j \in \mathbb{Z}$.

Proof. Given any $f \in V_j$, there exists a sequence $\{a_k\}_{k\in\mathbb{N}_0} \in l^2(\mathbb{N}_0)$ such that

$$f(x) = q^{j/2}\sum_{k\in\mathbb{N}_0} a_k \varphi\left(\mathfrak{p}^{-j}x - u(k)\right). \tag{24}$$

Let $m_0(\xi) = \sum_{k\in\mathbb{N}_0} a_k\chi_k(\xi) \in L^2(\mathfrak{D})$ and let $m_1(\mathfrak{p}\xi) = m_0(\xi)H(\mathfrak{p}\xi)$. Then, clearly m_1 lies in $L^2(\mathfrak{D})$ as H lies in $L^\infty(\mathfrak{D})$. Therefore, by Parseval's identity, there exists a sequence $\{b_k\}_{k\in\mathbb{N}_0} \in l^2(\mathbb{N}_0)$ such that $m_1(\xi) = \sum_{k\in\mathbb{N}_0} b_k\chi_k(\xi)$ lies in $L^2(K)$.

Taking Fourier transform of (24) and using assumption (23), we obtain

$$\begin{aligned}\widehat{f}(\xi) &= q^{j/2} m_0\left(\mathfrak{p}^j\xi\right)\widehat{\varphi}\left(\mathfrak{p}^j\xi\right)\\ &= q^{(j+1)/2} m_0\left(\mathfrak{p}^j\xi\right) H\left(\mathfrak{p}^{j+1}\xi\right)\widehat{\varphi}\left(\mathfrak{p}^{j+1}\xi\right)\\ &= q^{(j+1)/2} m_1\left(\mathfrak{p}^{j+1}\xi\right)\widehat{\varphi}\left(\mathfrak{p}^{j+1}\xi\right).\end{aligned} \tag{25}$$

By implementing inverse Fourier transform to (25), we have

$$f(x) = q^{(j+1)/2}\sum_{k\in\mathbb{N}_0} b_k \varphi\left(\mathfrak{p}^{-j-1}x - u(k)\right). \tag{26}$$

Using Proposition 4, we observe that $f \in V_{j+1}$. Moreover, it is easy to verify that the function H in (23) is not unique. □

The following theorem is the converse to Theorem 7.

Theorem 8. *Let $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ be a frame for $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$ and, for $j \in \mathbb{Z}$, define V_j by (14). Assume that $V_0 \subseteq V_1$ and $\Phi(\xi) = \|\widehat{\varphi}(\xi - u(k))\|^2_{l^2(\mathbb{N}_0)}$. Then, there exists periodic function $H \in L^\infty(\mathfrak{D})$ such that (23) holds.*

Proof. Since $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ is a frame for V_0, therefore there exist positive constants A and B such that

$$A \le \Phi(\xi) \le B \quad \text{a.e. on } \sigma(V_0). \tag{27}$$

Since $V_0 \subseteq V_1$, we have $\varphi \in V_1$. By Proposition 6, there exists a periodic function $H_0 \in L^2(\mathfrak{D})$ such that

$$\widehat{\varphi}\left(\mathfrak{p}^{-1}\xi\right) = q^{1/2} H_0(\xi)\,\widehat{\varphi}(\xi). \tag{28}$$

Therefore, we have

$$\left|\widehat{\varphi}(\xi)\right|^2 = q\left|H_0(\mathfrak{p}\xi)\right|^2\left|\widehat{\varphi}(\mathfrak{p}\xi)\right|^2 \quad \text{a.e.} \tag{29}$$

Let $\mathcal{S} = \mathfrak{B} \setminus \sigma(V_0)$ and $H \in L^2(\mathfrak{D})$ be a periodic function such that $H = H_0$, a.e. on $\sigma(V_0)$, and H is bounded on $\mathcal{S}$ by a positive constant C. Then, it follows from the above fact that H is not unique so that (29) also holds for H; that is,

$$\left|\widehat{\varphi}(\xi)\right|^2 = q\left|H(\mathfrak{p}\xi)\right|^2\left|\widehat{\varphi}(\mathfrak{p}\xi)\right|^2 \quad \text{a.e.} \tag{30}$$

Taking $n = kp + r$, where $k \in \mathbb{N}_0$ and $r = 0, 1, \ldots, q-1$, we have

$$\begin{aligned}&\left|\widehat{\varphi}(\xi + u(n))\right|^2\\ &\quad = q\left|H(\mathfrak{p}\xi + \mathfrak{p}u(r))\right|^2\left|\widehat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(r) + u(k)\right|^2 \quad \text{a.e.}\end{aligned} \tag{31}$$

Summing up (31) for all $k \in \mathbb{N}_0$ and $r = 0, 1, \ldots, q-1$, we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}_0}\left|\widehat{\varphi}(\xi + u(n))\right|^2\\ &= q\sum_{r=0}^{q-1}\left|H(\mathfrak{p}\xi + \mathfrak{p}u(r))\right|^2\sum_{k\in\mathbb{N}_0}\left|\widehat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(r) + u(k)\right|^2 \quad \text{a.e.},\end{aligned} \tag{32}$$

which is equivalent to

$$\Phi(\xi) = q\sum_{r=0}^{q-1}\left|H(\mathfrak{p}\xi + \mathfrak{p}u(r))\right|^2\Phi(\mathfrak{p}\xi + \mathfrak{p}u(r)) \quad \text{a.e.} \tag{33}$$

or

$$\Phi\left(\mathfrak{p}^{-1}\xi\right) = q\sum_{r=0}^{q-1}\left|H(\xi + \mathfrak{p}u(r))\right|^2\Phi(\xi + \mathfrak{p}u(r)) \quad \text{a.e.} \tag{34}$$

Note that $\Phi(\mathfrak{p}^{-1}\xi) \le B$ a.e. and, hence, (34) becomes

$$\sum_{r=0}^{q-1}\left|H(\xi + \mathfrak{p}u(r))\right|^2\Phi(\xi + \mathfrak{p}u(r)) \le qB \quad \text{a.e.} \tag{35}$$

This implies that, for almost every $\xi \in \mathfrak{B}^{-1}$ and $r = 0, 1, \ldots, q-1$, we have

$$\left|H(\xi + \mathfrak{p}u(r))\right|^2\Phi(\xi + \mathfrak{p}u(r)) \le qB. \tag{36}$$

Also, if $\Phi(\xi + \mathfrak{p}u(r)) = 0$, then $|H(\xi + \mathfrak{p}u(r))| \le C$ and if $\Phi(\xi+\mathfrak{p}u(r)) > 0$, then we may assume that $A \le \Phi(\xi+\mathfrak{p}u(r)) \le B$. Thus, for almost every $\xi \in \mathfrak{B}^{-1}$ and $r = 0, 1, \ldots, q-1$, we have

$$\left|H(\xi + \mathfrak{p}u(r))\right|^2 \le \max\left\{C^2, qBA^{-1}\right\}. \tag{37}$$

Hence, H is essentially bounded on $\mathfrak{D}$. This proves the theorem completely. □

The following two propositions are proved in [23].

Proposition 9. *Suppose $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$ and, for each $j \in \mathbb{Z}$, define V_j by (14) such that $V_0 \subseteq V_1$. Assume that $|\widehat{\varphi}| > 0$, a.e on a neighborhood of zero. Then, the union $\bigcup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(K)$.*

Proposition 10. *Let $\varphi \in L^2(K)$ and define $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$. For each $j \in \mathbb{Z}$, define V_j by (14). Then, one has $\bigcap_{j\in\mathbb{Z}}V_j = \{0\}$.*

Lemma 11. *Let V_j be the family of subspaces defined by (14) with $V_j \subseteq V_{j+1}$, for each $j \in \mathbb{Z}$. Suppose $\varphi \in L^2(K)$ is a nonzero function with $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$. Then, for every $j \in \mathbb{Z}$, V_j is a proper subspace of V_{j+1}.*

Proof. Suppose that $V_\ell = V_{\ell+1}$ for some $\ell \in \mathbb{Z}$. Let $f \in V_{j+1}$; then, for any given $j \in \mathbb{Z}$, we have $f(\mathfrak{p}^{j+1-\ell-1}x) \in V_{j+1}$. Since $f(\mathfrak{p}^{j-\ell}x) \in V_\ell$, therefore f lies in V_j and $V_j = V_{j+1}$. Hence, $\bigcap_{j\in\mathbb{Z}}V_j = V_0$. By Proposition 10, it follows that $V_j = \{0\}$, which is a contradiction. □

Combining all our results so far, we have the following theorem.

Theorem 12. *Let $\varphi \in L^2(K)$ and define $V_0 = \overline{\text{span}}\{\tau_k\varphi : k \in \mathbb{N}_0\}$. For each $j \in \mathbb{Z}$, define V_j by (14) and $\Phi(\xi) = \|\widehat{\varphi}(\xi - u(k))\|^2_{l^2(\mathbb{N}_0)}$. Suppose that the following hold:*

(i) *$A \le \Phi(\xi) \le B$ a.e. on $\sigma(V_0)$,*

(ii) *there exists a periodic function $H \in L^\infty(\mathfrak{D})$ such that*

$$\widehat{\varphi}(\xi) = q^{1/2}H(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}\xi), \quad \text{a.e.} \tag{38}$$

(iii) *$|\widehat{\varphi}| > 0$, a.e. on a neighborhood of zero.*

Then, $\{V_j : j \in \mathbb{Z}\}$ defines a frame multiresolution analysis of $L^2(K)$.

Proof. Since V_0 is a shift-invariant subspace of $L^2(K)$, therefore the system $\{\tau_k\varphi : k \in \mathbb{N}_0\}$ forms a frame for V_0 with frame bounds A and B. By Theorem 7 and Lemma 11, it follows that $V_j \subset V_{j+1}$, for every $j \in \mathbb{Z}$. Hence, by the definition of V_j, f lies in V_j if and only if $f(\mathfrak{p}^j\cdot)$ lies in V_0, while $f(\mathfrak{p}^{-1}\cdot)$ lies in V_{j+1} if and only if $f(\mathfrak{p}^{j+1}\cdot)$ lies in V_0. Thus, f lies in V_j if and only if $f(\mathfrak{p}^{-1}\cdot)$ lies in V_{j+1}. Moreover, by assumption (iii) and Proposition 10, it follows that $\bigcup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(K)$ and $\bigcap_{j\in\mathbb{Z}}V_j = \{0\}$. Thus, the sequence $\{V_j : j \in \mathbb{Z}\}$ satisfies all the conditions to be a frame multiresolution analysis of $L^2(K)$. □

In order to construct *wavelet frames* associated with frame multiresolution analysis on local fields K of positive characteristic, we introduce the orthogonal complement subspaces $\{W_j : j \in \mathbb{Z}\}$ of V_j in V_{j+1}. It is easy to verify that the sequence of subspaces $\{W_j : j \in \mathbb{Z}\}$ also satisfies the scaling property; that is,

$$W_j = \left\{f \in L^2(K) : f\left(\mathfrak{p}^j\cdot\right) \in W_0\right\}, \quad j \in \mathbb{Z}. \tag{39}$$

Theorem 13. *Let $\{V_j : j \in \mathbb{Z}\}$ be an increasing sequence of closed subspaces of $L^2(K)$ such that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. Let W_j be the orthogonal complement of V_j in V_{j+1}, for each $j \in \mathbb{Z}$. Then, the subspaces W_j are pairwise orthogonal and*

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{40}$$

Proof. Assume that $i < j$; then, $\langle f_i, f_j\rangle = 0$, for any $f_i \in W_j$ as $W_i \subset V_{i+1} \subset V_j$. Let P_j be the orthogonal projection operators from $L^2(K)$ onto V_j; then, $\lim_{j\to\infty} P_j f = f$, $\lim_{j\to-\infty} P_j f = 0$, and $W_j = \{f - P_j f : f \in V_{j+1}\}$. Therefore, for any $f \in L^2(K)$, we have

$$f = \sum_{j\in\mathbb{Z}} \left(P_{j+1} f - P_j f\right). \tag{41}$$

Thus, the result of the direct sum follows since $P_{j+1} - P_j$ is the orthogonal projector from $L^2(K)$ onto W_j. □

4. Characterization of Wavelet Frames on Local Fields

In this section, we give the characterization of wavelet frames associated with frame multiresolution analysis on local fields of positive characteristic. First, we will characterize the existence of a function ψ in W_0, where W_0 is the orthogonal complement of V_0 in V_1, by virtue of the *analysis filters* G and H, defined as in Section 3.

Theorem 14. *Let H be a periodic function associated with the frame multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$ satisfying the condition (23). Define W_0 as the orthogonal complement of V_0 in V_1. Let $\psi \in V_1$ such that*

$$\widehat{\psi}(\xi) = q^{1/2} G(\mathfrak{p}\xi)\, \widehat{\varphi}(\mathfrak{p}\xi), \tag{42}$$

where G is a periodic function in $L^2(\mathfrak{D})$. Then, ψ lies in W_0 if and only if

$$\sum_{r=0}^{q-1} H(\mathfrak{p}\xi + \mathfrak{p}u(r))\, \Phi(\mathfrak{p}\xi + \mathfrak{p}u(r))\, \overline{G(\mathfrak{p}\xi + \mathfrak{p}u(r))} = 0 \quad a.e.\ \xi. \tag{43}$$

Proof. We note that ψ lies in W_0 if and only if

$$\langle \psi, \tau_k \psi\rangle = \langle \psi, \psi(\cdot - u(k))\rangle = 0, \quad \forall k \in \mathbb{N}_0. \tag{44}$$

Define

$$F(\xi) = \sum_{k\in\mathbb{N}_0} \widehat{\varphi}(\xi + u(k))\, \overline{\widehat{\psi}(\xi + u(k))}. \tag{45}$$

Then, it is easy to verify that F lies in $L^1(\mathfrak{D})$ by using Monotonic Convergence Theorem and the Plancherel Theorem as

$$\begin{aligned}
\int_{\mathfrak{D}} |F(\xi)|\, d\xi &\le \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} |\widehat{\varphi}(\xi + u(k))\, \widehat{\psi}(\xi + u(k))|\, d\xi \\
&= \sum_{k\in\mathbb{N}_0} \int_{\mathfrak{D}} |\widehat{\varphi}(\xi + u(k))\, \widehat{\psi}(\xi + u(k))|\, d\xi \\
&= \int_K |\widehat{\varphi}(\xi)\, \widehat{\psi}(\xi)|\, d\xi \\
&\le \|\widehat{\varphi}\|_2 \|\widehat{\psi}\|_2 = \|\varphi\|_2 \|\psi\|_2.
\end{aligned} \tag{46}$$

For a fixed $n \in \mathbb{N}_0$, we define F_M as

$$F_M(\xi) = \sum_{k=0}^{M} \widehat{\varphi}(\xi + u(k))\, \overline{\widehat{\psi}(\xi + u(k))}\, \chi_n(\xi). \tag{47}$$

Then, in view of (23) and (42), we have

$$\begin{aligned}
F_M(\xi) = q \sum_{r=0}^{q-1} \sum_{qk+r\le M} & H(\mathfrak{p}\xi + \mathfrak{p}u(r)) \left|\widehat{\varphi}(\mathfrak{p}\xi + \mathfrak{p}u(r) + u(k))\right|^2 \\
& \cdot \overline{G(\mathfrak{p}\xi + \mathfrak{p}u(r))}\chi_n(\xi).
\end{aligned} \tag{48}$$

Using Monotonic Convergence Theorem and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\|F_M - F\chi_n\|_{L^2(\mathfrak{D})} \\
&\le \int_{\mathfrak{D}} \sum_{k\ge M+1} |\widehat{\varphi}(\xi + u(k))\, \widehat{\psi}(\xi + u(k))|\, d\xi \\
&= \sum_{k\ge M+1} \int_{\mathfrak{D}} |\widehat{\varphi}(\xi + u(k))\, \widehat{\psi}(\xi + u(k))|\, d\xi \\
&= \sum_{k\ge M+1} \int_{x+\mathfrak{D}} |\widehat{\varphi}(\xi)\, \widehat{\psi}(\xi)|\, d\xi \\
&\le \int_{|\xi|>M} |\widehat{\varphi}(\xi)\, \widehat{\psi}(\xi)|\, d\xi \\
&\le \left\{\int_{|\xi|>M} |\widehat{\varphi}(\xi)|^2 d\xi\right\}^{1/2} \left\{\int_{|\xi|>M} |\widehat{\psi}(\xi)|^2 d\xi\right\}^{1/2} \\
&\longrightarrow 0 \quad \text{as } M \longrightarrow \infty.
\end{aligned} \tag{49}$$

Hence,

$$\lim_{M\to\infty} \|F_M - F\chi_n\|_{L^2(\mathfrak{D})} = 0. \tag{50}$$

Therefore, there exists a subsequence $\{F_{M_j}\}$ such that

$$\lim_{j\to\infty} \left\|F_{M_j} - F\chi_n\right\|_{L^2(\mathfrak{D})} = 0, \quad \text{a.e.} \tag{51}$$

Hence,

$$\begin{aligned}
F(\xi) = q \sum_{r=0}^{q-1} & H(\mathfrak{p}\xi + \mathfrak{p}u(r)) \\
& \cdot \Phi(\mathfrak{p}\xi + \mathfrak{p}u(r))\, \overline{G(\mathfrak{p}\xi + \mathfrak{p}u(r))} \quad \text{a.e.}
\end{aligned} \tag{52}$$

Using (50) and the Dominated Convergence Theorem, we have, for all $n \in \mathbb{N}_0$,

$$\begin{aligned}\langle \psi, \tau_{-n}\varphi\rangle &= \int_K \widehat{\psi}(\xi)\,\overline{\widehat{\varphi}(\xi)}\chi_n(\xi)\,d\xi \\ &= \sum_{k\in\mathbb{N}_0}\int_{x+\mathfrak{D}} \widehat{\psi}(\xi)\,\overline{\widehat{\varphi}(\xi)}\chi_n(\xi)\,d\xi \\ &= \lim_{M\to\infty}\sum_{k=0}^{M}\int_{\mathfrak{D}} \widehat{\psi}(\xi+u(k)) \\ &\qquad \cdot \overline{\widehat{\varphi}(\xi+u(k))}\chi_n(\xi)\,\chi_k(\xi)\,d\xi \\ &= \lim_{M\to\infty}\int_{\mathfrak{D}} F_M(\xi)\,d\xi \\ &= \int_{\mathfrak{D}} F(\xi)\,\chi_n(\xi)\,d\xi.\end{aligned} \tag{53}$$

Consequently, $F = 0$, a.e., is the necessary and sufficient condition for (44) to hold for all $n \in \mathbb{N}_0$. □

Lemma 15. *Let $\{W_j : j \in \mathbb{Z}\}$ be a sequence of pairwise orthogonal closed subspaces of $L^2(K)$ such that $L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j$. Then, for every $f \in L^2(K)$, there exist $f_j \in W_j$, $j \in \mathbb{Z}$, such that $f(x) = \sum_{j\in\mathbb{Z}} f_j(x)$. Furthermore,*

$$\|f\|_2^2 = \sum_{j\in\mathbb{Z}} \|f_j\|_2^2. \tag{54}$$

Proof. For any arbitrary function $f \in L^2(K)$, we have

$$\lim_{n\to\infty}\left\| f - \sum_{j=-n}^{n} f_j\right\|_2 = 0, \tag{55}$$

where $f_j \in W_j$, for each $j \in \mathbb{Z}$. Moreover, for a fixed $n \in \mathbb{N}$, we have

$$\left\|\sum_{j=-n}^{n} f_j\right\|_2^2 = \sum_{j=-n}^{n}\|f_j\|_2^2. \tag{56}$$

Since the norm $\|\cdot\|_2$ is continuous, therefore the desired result is obtained by taking $n \to \infty$ on both sides of the above equality. □

Theorem 16. *Let φ be the scaling function for a frame multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$ and suppose that W_j is the orthogonal complement of V_j in V_{j+1}. Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subset W_0$. Then, the collection*

$$\begin{aligned}\mathcal{F}_\Psi = \Big\{&\psi_{j,k}^{\ell}(x) := q^{j/2}\psi^{\ell}\left(\mathfrak{p}^{-j}x - u(k)\right), \\ &j \in \mathbb{Z},\ k \in \mathbb{N}_0,\ \ell = 1,\ldots,L\Big\}\end{aligned} \tag{57}$$

constitutes a wavelet frame for $L^2(K)$ with frame bounds A and B if and only if

$$\left\{\tau_k\psi^{\ell} : k \in \mathbb{N}_0,\ \ell = 1,\ldots,L\right\} \tag{58}$$

forms a frame for W_0 with frame bounds A and B.

Proof. Suppose that the system $\mathcal{F}_\Psi$ given by (57) is a wavelet frame for $L^2(K)$ with bounds A and B. Then, it follows from (39) that the family of functions $\psi_{j,k}^{\ell}$ lies in W_j, for $\ell = 1,\ldots,L$, $j \in \mathbb{Z}$, and $k \in \mathbb{N}_0$.

By applying Theorem 13 to an arbitrary function $f \in W_0$, we have

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f, \psi_{j,k}^{\ell}\right\rangle\right|^2 = \sum_{k\in\mathbb{N}_0}\left|\left\langle f, \tau_k\psi^{\ell}\right\rangle\right|^2. \tag{59}$$

Using the frame property of the system $\mathcal{F}_\Psi$, we have

$$A\|f\|_2^2 \le \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\left|\left\langle f, \tau_k\psi^{\ell}\right\rangle\right|^2 \le B\|f\|_2^2, \tag{60}$$

and it follows that the collection $\{\tau_k\psi^{\ell} : k \in \mathbb{N}_0,\ \ell = 1,\ldots,L\}$ is a frame for W_0.

Conversely, suppose that the collection $\{\tau_k\psi^{\ell} : k \in \mathbb{N}_0,\ \ell = 1,\ldots,L\}$ is a frame for W_0 with bounds A and B. For any fixed $j \in \mathbb{Z}$ and $f \in W_j$, we have from (39) that $f(\mathfrak{p}^j\cdot) \in W_0$. Moreover, by making use of the fact that

$$\begin{aligned}\left\langle f, \psi_{j,k}^{\ell}\right\rangle &= q^{j/2}\int_K f(x)\,\overline{\psi^{\ell}\left(\mathfrak{p}^{-j}x - u(k)\right)}dx, \\ \left\|q^{-j/2}f(\mathfrak{p}^j\cdot)\right\|_2^2 &= q^{-j}\int_K \left|f\left(\mathfrak{p}^j x\right)\right|^2 dx = \|f\|_2^2,\end{aligned} \tag{61}$$

we have

$$A\left\|q^{-j/2}f(\mathfrak{p}^j\cdot)\right\|_2^2 \le \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\left|\left\langle f, \psi_{j,k}^{\ell}\right\rangle\right|^2 \le B\left\|q^{-j/2}f(\mathfrak{p}^j\cdot)\right\|_2^2. \tag{62}$$

Thus, for a given $j \in \mathbb{Z}$, the collection $\{\psi_{j,k}^{\ell} : k \in \mathbb{N}_0,\ \ell = 1,\ldots,L\}$ constitutes a frame for W_j with frame bounds A and B.

Let f be an arbitrary function in $L^2(K)$; then, by Theorem 13 and Lemma 15, there exist $f_j \in W_j$ such that

$$f = \sum_{j\in\mathbb{Z}} f_j, \quad \left\langle f_i, \psi_{j,k}^{\ell}\right\rangle = 0, \quad i \ne j. \tag{63}$$

Therefore, we have

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f, \psi_{j,k}^{\ell}\right\rangle\right|^2 &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\sum_{i\in\mathbb{Z}}\left\langle f_i, \psi_{j,k}^{\ell}\right\rangle\right|^2 \\ &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\left\langle f_j, \psi_{j,k}^{\ell}\right\rangle\right|^2.\end{aligned} \tag{64}$$

Using (62), we obtain

$$A\sum_{j\in\mathbb{Z}}\|f_j\|_2^2 \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f_j, \psi_{j,k}^{\ell}\rangle\right|^2 \le B\sum_{j\in\mathbb{Z}}\|f_j\|_2^2. \tag{65}$$

Combining (64), (65), and Lemma 15, we have

$$A\|f_j\|_2^2 \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f_j, \psi_{j,k}^{\ell}\rangle\right|^2 \le B\|f_j\|_2^2. \tag{66}$$

This completes the proof of the theorem. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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