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Research Article

Maximal Midpoint-Free Subsets of Integers

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A set $S \subset \mathbb{Z}$ is midpoint-free if no ordered triple $(a, b, c) \in S^3$ satisfies $a + c = 2b$ and $a < b < c$. Midpoint-free subsets of $\mathbb{Z}^+$ and $\mathbb{Z}$ are studied, with emphasis on those sets characterized by restrictions on the base m digits of their elements when $3 \le m \le 14$, and with particular attention to maximal midpoint-free subsets with $m \in \{3, 4, 7, 9, \ldots, 13\}$.

1. Introduction

An ordered triple (a, b, c) of integers is a *midpoint triple* of $\mathbb{Z}$ if $a + c = 2b$ and $a < b < c$. The *midpoint* of this triple is b, its *lower endpoint* is a and its *upper endpoint* is c. For any subset $X \subseteq \mathbb{Z}$, let $\Lambda(X)$ denote the set of all midpoint triples $(a, b, c) \in X^3$. When $S \subset X \subseteq \mathbb{Z}$, let

$$\begin{aligned} A(S, X) &:= \{a \in X \mid \exists b, c \in S : (a, b, c) \in \Lambda(X)\}, \\ B(S, X) &:= \{b \in X \mid \exists a, c \in S : (a, b, c) \in \Lambda(X)\}, \qquad (1) \\ C(S, X) &:= \{c \in X \mid \exists a, b \in S : (a, b, c) \in \Lambda(X)\}. \end{aligned}$$

Generically, the members of $A(S, X) \cup B(S, X) \cup C(S, X)$ are the *balance points* for S in X. The balance points comprise the *lower endpoint set* $A(S, X)$, the *midpoint set* $B(S, X)$, and the *upper endpoint set* $C(S, X)$, for S in X. The members of $E(S, X) := X \setminus (S \cup A(S, X) \cup B(S, X) \cup C(S, X))$ are the *eccentric points* for S in X. Attention to these sets appears to be a new focus, suggested by the underlying geometrical viewpoint. There is an extensive literature associated with treating $a + c = 2b$ as specifying an arithmetic progression of length 3. A compact discussion and rich bibliography are given in Guy's survey work [1, Section E10]. For an example of recent work in this field, see Dybizbański [2].

If $\Lambda(S) = \emptyset$ it will be convenient to say that S is *midpoint-free*; moreover, S is a *maximal* midpoint-free subset of X if $\Lambda(S) = \emptyset$ and $\Lambda(T) \neq \emptyset$, whenever $S \subset T \subseteq X$. Hence we have $\Lambda(S \cup \{x\}) \neq \emptyset$ for any $x \in X \setminus S$. This implies the following characterization.

Theorem 1. *If $S \subset X \subseteq \mathbb{Z}$, then S is a maximal midpoint-free subset of X if and only if $A(S, X) \cup B(S, X) \cup C(S, X) = X \setminus S$, or equivalently, if and only if $\Lambda(S) = \emptyset$ and $E(S, X) = \emptyset$.*

Note that if X is infinite, any maximal midpoint-free subset $S \subset X$ must also be infinite: any pair of elements of S has one midpoint and two endpoints, so precludes at most three elements of $X \setminus S$ from membership of $E(S, X)$; thus $E(S, X)$ would be infinite if S were finite, but then $S \cup \{x\}$ would be midpoint-free for any $x \in E(S, X)$, contradicting maximality of S.

In [3] the notions of midpoint triple and maximal midpoint-free subset are studied for several "natural" subsets of the real numbers, but for simplicity in the present note we restrict X to $\mathbb{Z}$ and subsets, especially $\mathbb{Z}^+ := \{x \in \mathbb{Z} \mid x \ge 0\}$ and $\mathbb{Z}^- := \mathbb{Z} \setminus \mathbb{Z}^+ = -\mathbb{Z}^+ \setminus \{0\}$. Here, the main focus will be on $A(S, X)$, $B(S, X)$, and $C(S, X)$ when S is a maximal midpoint-free subset of $X = \mathbb{Z}^+$, $\mathbb{Z}^-$ or $\mathbb{Z}$.

Initially S has been defined to be midpoint-free if S^3 contains no midpoint triple. There are several semantic equivalents for this condition.

Theorem 2. *If $S \subset X \subseteq \mathbb{Z}$, then any one of the sets*

$$A(S, X) \cap S, \qquad B(S, X) \cap S, \qquad C(S, X) \cap S \qquad (2)$$

is empty if and only if all three are empty.

Proof. Consider the contrary. If $b \in B(S, X) \cap S$ there is a triple $(a, b, c) \in \Lambda(S)$; this triple ensures that $a \in A(S, X) \cap S$

and $c \in C(S,X) \cap S$. The other two cases follow in the same way. □

Take $X = \mathbb{Z}$ in Theorem 2. Then $B(S,\mathbb{Z}) \cap S = \emptyset$ recovers the "natural" terminology that S is midpoint-free if and only if *it does not contain the midpoint of any two of its members*. Equally, S is midpoint-free if and only if it is lower endpoint-free, or alternatively, if and only if it is upper endpoint-free.

Corollary 3. *A subset $S \subset X \subseteq \mathbb{Z}$ is midpoint-free if and only if*

$$(A(S,X) \cup B(S,X) \cup C(S,X)) \cap S = \emptyset. \tag{3}$$

This yields another semantic equivalent: S is midpoint-free exactly when it is balance point-free.

For any $x \in X$ define the *multiplicity* of x as a lower endpoint, midpoint, or upper endpoint for S, respectively, as

$$\begin{gathered}\alpha(x,S) := \#\left\{(b,c) \in S^2 \mid (x,b,c) \in \Lambda(X)\right\},\\ \beta(x,S) := \#\left\{(a,c) \in S^2 \mid (a,x,c) \in \Lambda(X)\right\},\\ \gamma(x,S) := \#\left\{(a,b) \in S^2 \mid (a,b,x) \in \Lambda(X)\right\}.\end{gathered} \tag{4}$$

The multiplicities for $x \in X$ in particular cases of $S \subset X \subseteq \mathbb{Z}$ will be of interest.

2. An Explicit Example, Involving Base m Digit Restrictions

Let us begin with an explicit example to illustrate these notions and sample some of the typical features encountered.

For any integer $m \geq 3$, let $\mathbb{Z}_m^+(0,1)$ be the set of integers $x \in \mathbb{Z}^+$ with regular base m representation in which all digits are restricted to the set $\{0,1\}$. When a,b,c are distinct members of $\mathbb{Z}_m^+(0,1)$, all digits of the base m representation of $2b$ lie in the set $\{0,2\}$, but base m computation of $a+c$ involves no carry over, so $a+c$ contains the digit 1 in each place where the base m digits of a and c differ. Hence $a+c=2b$ is impossible. It follows that $\mathbb{Z}_m^+(0,1)$ is midpoint-free.

Let $(x)_m$ denote the regular base m representation of x. An easy base m computation shows that

$$(112)_m = (111)_m + (1)_m = (101)_m + (11)_m. \tag{5}$$

This corresponds to the identities

$$m^2+m+2 = \left(m^2+m+1\right)+1 = \left(m^2+1\right)+(m+1). \tag{6}$$

Note that $m(m+1)+2$ is an even positive integer, so $z := (1/2)(m^2+m+2)$ is a midpoint for $S := \mathbb{Z}_m^+(0,1)$ when this set is treated as a subset of $X := \mathbb{Z}^+$. Digit considerations make it clear that $2z = (112)_m = (a)_m + (c)_m$ has no other solutions with $a,c \in \mathbb{Z}_m^+(0,1)$, so the midpoint multiplicity of z is 2. In summary,

$$\begin{gathered}z := \frac{1}{2}\left(m^2+m+2\right) \in B\left(\mathbb{Z}_m^+(0,1),\mathbb{Z}^+\right),\\ \beta\left(z,\mathbb{Z}_m^+(0,1)\right) = 2.\end{gathered} \tag{7}$$

Again, the general base m computation

$$(112)_m + (110)_m = (222)_m = 2\cdot(111)_m \tag{8}$$

corresponds to the identity

$$\left(m^2+m+2\right)+\left(m^2+m\right) = 2\left(m^2+m+1\right), \tag{9}$$

which implies

$$\begin{gathered}2z = m^2+m+2 \in C\left(\mathbb{Z}_m^+(0,1),\mathbb{Z}^+\right),\\ \gamma\left(2z,\mathbb{Z}_m^+(0,1)\right) = 1,\end{gathered} \tag{10}$$

when $m \geq 4$. Additionally, in the special case $m=3$ we also have

$$\begin{gathered}(112)_3 + (11)_3 = 2\cdot(100)_3;\\ (112)_3 + (101)_3 = 2\cdot(110)_3;\\ (112)_3 + (1111)_3 = 2\cdot(1000)_3.\end{gathered} \tag{11}$$

Hence $(112)_3 = 14 \in A(\mathbb{Z}_3^+(0,1),\mathbb{Z}^+) \cap C(\mathbb{Z}_3^+(0,1),\mathbb{Z}^+)$. There are no other solutions to $(112)_3 + (c)_3 = 2\cdot(b)_3$ with $b,c \in \mathbb{Z}_3^+(0,1)$, so the multiplicities of 14 as a lower and upper endpoint are $\alpha(14,\mathbb{Z}_3^+(0,1)) = 1$, $\gamma(14,\mathbb{Z}_3^+(0,1)) = 3$.

When $m \geq 5$, it can be seen that $(24)_m + (c)_m = 2\cdot(b)_m$ has no solutions with $b,c \in \mathbb{Z}_m^+(0,1)$, so $2m+4$ is not an endpoint for $\mathbb{Z}_m^+(0,1)$, and

$$\alpha\left(2m+4,\mathbb{Z}_m^+(0,1)\right) = \gamma\left(2m+4,\mathbb{Z}_m^+(0,1)\right) = 0. \tag{12}$$

Now seek $a,c \in \mathbb{Z}_m^+(0,1) = \{0,1,m,m+1,m^2,m^2+1,m^2+m,m^2+m+1,m^3,\ldots\}$ such that $a+c = 2(2m+4)$ and $a < 2m+4 < c$. As $m^2 > 2m+4$, it follows that $a \in \{0,1,m,m+1\}$. Then it is routine to verify that

$$c \in \{3m+7,3m+8,4m+7,4m+8\} \cap \mathbb{Z}_m^+(0,1) = \emptyset \tag{13}$$

when $m \geq 5$, but note that digit arguments must be sensitive to the magnitude of m; for instance, $4m+7 = (47)_m$ when $m \geq 8$, while for smaller m we have

$$\begin{gathered}4m+7 = m^2+m+3 = (113)_m \quad \text{when } m=4,\\ 4m+7 = m^2+2 = (102)_m \quad \text{when } m=5,\\ 4m+7 = 5m+1 = (51)_m \quad \text{when } m=6,\\ 4m+7 = 5m = (50)_m \quad \text{when } m=7.\end{gathered} \tag{14}$$

It follows that

$$\beta\left(2m+4,\mathbb{Z}_m^+(0,1)\right) = 0, \tag{15}$$

and $2m+4$ is not a midpoint for $\mathbb{Z}_m^+(0,1)$ when $m \geq 5$. This completes the demonstration that $2m+4$ is an eccentric point, so the midpoint-free subset $\mathbb{Z}_m^+(0,1) \subset \mathbb{Z}^+$ is not maximal if $m \geq 5$.

We will later return to a more systematic study of $\mathbb{Z}_3^+(0,1)$ and $\mathbb{Z}_4^+(0,1)$.

3. New Maximal Midpoint-Free Sets from Old

If $S \subset \mathbb{Z}$ and $c, d \in \mathbb{Z}$ with $c \neq 0$, then $cS + d := \{cs + d \mid s \in S\}$ is an *affine transform* of S. Clearly $(x, y, z) \in \Lambda(\mathbb{Z})$ if and only if $c(x, y, z) + d \in \Lambda(\mathbb{Z})$ when $c > 0$ or $c(z, y, x) + d \in \Lambda(\mathbb{Z})$ when $c < 0$, so any affine transform $cS + d$ is midpoint-free if and only if S is midpoint-free.

For example, let $\mathbb{Z}^+_m(0, 2)$ be the set of all integers $x \in \mathbb{Z}^+$ with regular base $m \geq 3$ representation in which all digits are restricted to the set $\{0, 2\}$. Then $\mathbb{Z}^+_m(0, 2)$ is midpoint-free because $\mathbb{Z}^+_m(0, 1)$ is midpoint-free when $m \geq 3$, and $\mathbb{Z}^+_m(0, 2) = 2\mathbb{Z}^+_m(0, 1)$ is an affine transform of $\mathbb{Z}^+_m(0, 1)$. The last identity shows that $\mathbb{Z}^+_m(0, 2) \subset 2\mathbb{Z}^+$. Similarly, $-(\mathbb{Z}^+_m(0, 2) + 1) = -2\mathbb{Z}^+_m(0, 1) - 1$ is an affine transform of $\mathbb{Z}^+_m(0, 1)$ so is a midpoint-free subset of $2\mathbb{Z}^- + 1$ when $m \geq 3$. Then

$$\mathbb{Z}^+_m(0, 2) \cup -\left(\mathbb{Z}^+_m(0, 2) + 1\right) \tag{16}$$

is a midpoint-free subset of $\mathbb{Z}$, since the positive and negative components of this set are midpoint-free, and $a + c = 2b$ has no solution with $a \in 2\mathbb{Z}^- + 1$, $c \in 2\mathbb{Z}^+$, and $b \in \mathbb{Z}$, because these conditions require $a + c$ to be an odd integer and $2b$ to be an even integer.

Suppose $S \subset \mathbb{Z}^+$ is known to be a maximal midpoint-free subset of $\mathbb{Z}$. It turns out that the principle illustrated by the example in the previous paragraph holds strongly for S.

Theorem 4. *If $S \subset \mathbb{Z}^+$ is a maximal midpoint-free subset of $\mathbb{Z}$, then the set $S^{(2)} := 2S \cup -(2S + 1)$ is a maximal midpoint-free subset of $\mathbb{Z}$.*

Proof. As S is a maximal midpoint-free subset of $\mathbb{Z}$, Theorem 1 implies

$$A(S, \mathbb{Z}) \cup B(S, \mathbb{Z}) \cup C(S, \mathbb{Z}) = \mathbb{Z} \setminus S. \tag{17}$$

But $S \subset \mathbb{Z}^+$, so $B(S, \mathbb{Z}) \subset \mathbb{Z}^+$ and $C(S, \mathbb{Z}) \subset \mathbb{Z}^+$; hence $B(S, \mathbb{Z}) = B(S, \mathbb{Z}^+)$ and $C(S, \mathbb{Z}) = C(S, \mathbb{Z}^+)$. Therefore, $A(S, \mathbb{Z}) = A(S, \mathbb{Z}^+) \cup \mathbb{Z}^-$.

Given $x \in A(S, \mathbb{Z}) \cup C(S, \mathbb{Z})$, there exist $b, y \in S$ such that $x + y = 2b$ and $b \neq y$. Let $\delta \in \{0, 1\}$. Then $(2x + \delta) + (2y + \delta) = 2(2b + \delta)$, so

$$\begin{aligned} x < y &\Longrightarrow 2x + \delta \in A(2S + \delta, \mathbb{Z}); \\ x > y &\Longrightarrow 2x + \delta \in C(2S + \delta, \mathbb{Z}). \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} 2A(S, \mathbb{Z}) + \delta &\subseteq A(2S + \delta, \mathbb{Z}), \\ 2C(S, \mathbb{Z}) + \delta &\subseteq C(2S + \delta, \mathbb{Z}). \end{aligned} \tag{19}$$

Conversely, suppose $u \in A(2S + \delta, \mathbb{Z}) \cup C(2S + \delta, \mathbb{Z})$, so there exist $d, v \in 2S + \delta$ such that $u + v = 2d$ and $d \neq v$. But $d = 2d' + \delta$, $v = 2v' + \delta$ where $d', v' \in S$, so $u + v \equiv 0 \bmod 2$ implies $u \equiv v \equiv \delta \bmod 2$, whence $u = 2u' + \delta$ with $u' \in \mathbb{Z}$. But S is midpoint-free, so $2S + \delta$ is midpoint-free, and therefore, $u \in \mathbb{Z} \setminus (2S + \delta)$. Thus $u' \in \mathbb{Z} \setminus S$. Now $u + v = 2d$ and $d \neq v$ imply that $u' + v' = 2d'$ and $d' \neq v'$, so $u' \in A(S, \mathbb{Z}) \cup C(S, \mathbb{Z})$. Therefore, the reverse containments also hold:

$$\begin{aligned} u < v &\Longrightarrow u \in 2A(S, \mathbb{Z}) + \delta; \\ u > v &\Longrightarrow u \in 2C(S, \mathbb{Z}) + \delta. \end{aligned} \tag{20}$$

Hence

$$\begin{aligned} A(2S + \delta, \mathbb{Z}) &= 2A(S, \mathbb{Z}) + \delta, \\ C(2S + \delta, \mathbb{Z}) &= 2C(S, \mathbb{Z}) + \delta. \end{aligned} \tag{21}$$

For $x \in B(S, \mathbb{Z})$, similar reasoning shows that $2x + \delta \in B(2S + \delta, \mathbb{Z}^+)$, so

$$2B(S, \mathbb{Z}) + \delta \subseteq B(2S + \delta, \mathbb{Z}). \tag{22}$$

However, this containment can in fact be proper. For instance, if $S = \{2^n \mid n \in \mathbb{Z}^+\}$, then $2B(S, \mathbb{Z})$ only contains even integers, whereas $\{2^{n+1} + 1 \mid n \in \mathbb{Z}^+\} \subset B(2S, \mathbb{Z})$.

Since S is a maximal midpoint-free subset of $\mathbb{Z}$, no integers are eccentric for S, so no members of $2\mathbb{Z}^+ + \delta$ are eccentric for $2S + \delta$. Then the positive integers eccentric for $2S + \delta$ satisfy

$$E\left(2S + \delta, \mathbb{Z}^+\right) = \left(2\mathbb{Z}^+ + \varepsilon\right) \setminus B(2S + \delta, \mathbb{Z}) \subseteq 2\mathbb{Z}^+ + \varepsilon, \tag{23}$$

where $\varepsilon := 1 - \delta$. Since $A(S, \mathbb{Z}) = A(S, \mathbb{Z}^+) \cup \mathbb{Z}^-$ it follows that

$$E(2S + \delta, \mathbb{Z}) \subseteq 2\mathbb{Z} + \varepsilon. \tag{24}$$

Specifically, all integers eccentric for $2S$ are odd, and all integers eccentric for $2S + 1$ are even. Therefore, no integer is eccentric for both $2S$ and $-(2S + 1)$, so $E(S^{(2)}, \mathbb{Z}) = \emptyset$. Thus $S^{(2)}$ is a maximal midpoint-free subset of $\mathbb{Z}$. □

Corollary 5. *If $S \subset \mathbb{Z}^+$ is a maximal midpoint-free subset of $\mathbb{Z}$, the balance point sets for S and $2S + \delta$ with $\delta \in \{0, 1\}$ satisfy*

$$\begin{gathered} A(S, \mathbb{Z}) = A\left(S, \mathbb{Z}^+\right) \cup \mathbb{Z}^-, \qquad B(S, \mathbb{Z}) = B\left(S, \mathbb{Z}^+\right), \\ C(S, \mathbb{Z}) = C\left(S, \mathbb{Z}^+\right), \qquad A(2S + \delta, \mathbb{Z}) = 2A(S, \mathbb{Z}) + \delta, \\ C(2S + \delta, \mathbb{Z}) = 2C(S, \mathbb{Z}) + \delta, \\ B(2S + \delta, \mathbb{Z}) \supseteq 2B(S, \mathbb{Z}) + \delta. \end{gathered} \tag{25}$$

It is convenient to refer to the construction in Theorem 4 as "doubling" the given set S. Other constructions involving affine transforms of a set are also of interest. For example, since $\mathbb{Z}^+_m(0, 1)$ is midpoint-free when $m \geq 3$, it follows that the disjoint sets $m\mathbb{Z}^+_m(0, 1)$ and $m\mathbb{Z}^+_m(0, 1) + 1$ are midpoint-free when $m \geq 3$. In fact, their union is midpoint-free. This turns out to be "trivial." Multiplying a member of $\mathbb{Z}^+_m(0, 1)$ by m simply shifts its base m digits one place, and a terminal 0 emerges to occupy the zeroth place; then adding 1 replaces the terminal 0 by 1. Hence $m\mathbb{Z}^+_m(0, 1) \cup (m\mathbb{Z}^+_m(0, 1) + 1) = \mathbb{Z}^+_m(0, 1)$.

When $m \geq 4$, the disjoint midpoint-free sets $3\mathbb{Z}^+_m(0, 1)$ and $3\mathbb{Z}^+_m(0, 1) + 1$ are more interesting. Let us verify that

$$S := 3\mathbb{Z}^+_m(0, 1) \cup \left(3\mathbb{Z}^+_m(0, 1) + 1\right) \tag{26}$$

is also midpoint-free. The two component sets are midpoint-free, so any midpoint triple $(a,b,c) \in \Lambda(S)$ must have at least one member in each set. Thus $\{a,b,c\} \cap (3\mathbb{Z}^+ + \delta) \neq \emptyset$ in each case with $\delta \in \{0,1\}$. If $\{a,c\} \subset 3\mathbb{Z}_m^+(0,1) + \delta$, then $b \in 3\mathbb{Z}_m^+(0,1) + \varepsilon$ for $\varepsilon := 1-\delta \in \{0,1\}$. Then $a+c \in 3\mathbb{Z}^+ + 2\delta$ and $2b \in 3\mathbb{Z}^+ + 2\varepsilon$. But $\{2\delta, 2\varepsilon\} = \{0,2\}$, so $a+c \neq 2b$. If $\{a,b\} \subset 3\mathbb{Z}_m^+(0,1) + \delta$ then $c \in 3\mathbb{Z}_m^+(0,1) + \varepsilon$, while if $\{b,c\} \subset 3\mathbb{Z}_m^+(0,1) + \delta$ then $a \in 3\mathbb{Z}_m^+(0,1) + \varepsilon$. In each case $a+c \in 3\mathbb{Z}^+ + 1$ and $2b \in 3\mathbb{Z}^+ + 2\delta$, so $a + c \neq 2b$ because $2\delta \in \{0,2\}$. Thus $\Lambda(S) = \emptyset$, as claimed.

Generalising the latter example, a "trebling" construction which produces new maximal midpoint-free subsets of $\mathbb{Z}$ will now be studied.

Theorem 6. *If $S \subset \mathbb{Z}^+$ is a maximal midpoint-free subset of $\mathbb{Z}$, and all members of $\mathbb{Z}^+ \setminus S$ are endpoints for S, then $S^{(3)} := 3S \cup (3S+1)$ is a maximal midpoint-free subset of $\mathbb{Z}$, and all members of $\mathbb{Z} \setminus S^{(3)}$ are endpoints of $S^{(3)}$.*

Proof. Because S is midpoint-free, each of the affine transforms $3S$ and $3S+1$ is midpoint-free. Assume $(a,b,c) \in \Lambda(S^{(3)})$. Since $S^{(3)} \subset 3\mathbb{Z}^+ \cup (3\mathbb{Z}^+ + 1)$, there is a $\delta \in \{0,1\}$ such that $b \in 3\mathbb{Z}^+ + \delta$. Then $a + c = 2b \in 3\mathbb{Z}^+ + 2\delta$, so $a,b,c \in 3\mathbb{Z}^+ + \delta$. Thus $(a,b,c) \in \Lambda(3S+\delta) = \emptyset$, so no such triple exists. Hence $S^{(3)} := 3S \cup (3S+1)$ is midpoint-free.

By hypothesis, $A(S,\mathbb{Z}^+) \cup C(S,\mathbb{Z}^+) = \mathbb{Z}^+ \setminus S$. Also $A(S,\mathbb{Z}) = A(S,\mathbb{Z}^+) \cup \mathbb{Z}^-$ by Corollary 5, so every $x \in \mathbb{Z} \setminus S$ is an endpoint for S. Suppose $x + y = 2b$ and $b, y \in S$. Two integers from complementary sets cannot be equal, so x, y, b must be different. Also

$$(3x + r) + (3y + t) = 2(3b + s) \tag{27}$$

holds when $(r,s,t) \in \{(0,0,0),(1,1,1),(2,1,0),(-1,0,1)\}$. It follows that

$$\begin{gathered} x < y \Longrightarrow x \in A(S,\mathbb{Z}), \\ 3x + \{-1,0,1,2\} \subset A\left(S^{(3)},\mathbb{Z}\right), \\ x > y \Longrightarrow x \in C(S,\mathbb{Z}), \\ 3x + \{-1,0,1,2\} \subset C\left(S^{(3)},\mathbb{Z}\right). \end{gathered} \tag{28}$$

If $s < s'$ are consecutive members of S, all members of the interval

$$\left[s+1, s'-1\right] := \left\{z \in \mathbb{Z} \mid s+1 \le z \le s'-1\right\} \tag{29}$$

are endpoints for S. Then $3s + 1 < 3s'$ are consecutive members of $S^{(3)}$ and all members of $[3(s+1)-1, 3(s'-1)+2] = [3s+2, 3s'-1]$ are endpoints for $S^{(3)}$. Also each $x \in \mathbb{Z}^-$ is an endpoint for S, so all members of $[3x-1, 3x+2]$ are endpoints for $S^{(3)}$. Thus $A(S^{(3)},\mathbb{Z}) \cup C(S^{(3)},\mathbb{Z}) = \mathbb{Z} \setminus S^{(3)}$, and $E(S^{(3)},\mathbb{Z}) = \emptyset$. □

Let $U_n := \{x \in \mathbb{Z}_3^+(0,1) \mid 0 \le x < 3^n\}$ for each $n \in \mathbb{Z}^+$, and let $S^{(3,0)} := S$. Iterating the construction in Theorem 6 and combining with Theorem 4 yields the following result.

Corollary 7. *If $S \subset \mathbb{Z}^+$ is a maximal midpoint-free subset of $\mathbb{Z}$, and all members of $\mathbb{Z}^+ \setminus S$ are endpoints for S, then the set*

$$S^{(3,n)} := \bigcup_{x \in U_n} \left(3^n S + x\right) \tag{30}$$

is a maximal midpoint-free subset of $\mathbb{Z}$, for any integer $n \in \mathbb{Z}^+$, and all members of $\mathbb{Z} \setminus S^{(3,n)}$ are endpoints of $S^{(3,n)}$. Moreover, the set

$$S^{(2,3,n)} := 2S^{(3,n)} \cup -\left(2S^{(3,n)} + 1\right) \tag{31}$$

is a maximal midpoint-free subset of $\mathbb{Z}$.

4. Subsets of $\mathbb{Z}^+$ with Base m Digit Restrictions

Fix an integer $m \ge 3$. Let $\{0\} \subseteq D \subset \{x \in \mathbb{Z}^+ \mid 0 \le x < m\} := [0,m)$. Then D is a *digit subset* for base m representations of the integers or, briefly, a *base m digit subset*. Let $\mathbb{Z}_m^+(D)$ be the set of nonnegative integers with base m representation using only digits in D, and let $[\![x]\!]_{m,i}$ denote the digit in position $i \ge 0$ of the regular base m representation of $x \in \mathbb{Z}^+$, so

$$\mathbb{Z}_m^+(D) := \left\{x \in \mathbb{Z}^+ \mid \forall i \in \mathbb{Z}^+ : [\![x]\!]_{m,i} \in D\right\}. \tag{32}$$

Let us say that D is midpoint-free as a base m digit subset if $2 \cdot \max(D) < m$ and there is no ordered triple $(d,e,f) \in D^3$ such that $d \neq f$ and $d + f = 2e$.

Theorem 8. *If D is a midpoint-free base m digit subset with $g := \max D \ge 1$ and $m \ge 2g+1$, then the set $\mathbb{Z}_m^+(D)$ is midpoint-free.*

Proof. Suppose $(a,b,c) \in \Lambda(\mathbb{Z}_m^+(D))$. Then $a + c = 2b$. There is no carry-over in computing this sum in base m arithmetic since all its digits are less than $m/2$, so

$$[\![a]\!]_{m,i} + [\![c]\!]_{m,i} = [\![2b]\!]_{m,i} = 2\,[\![b]\!]_{m,i} \tag{33}$$

for every $i \ge 0$. Since D is midpoint-free, it follows that

$$[\![a]\!]_{m,i} = [\![b]\!]_{m,i} = [\![c]\!]_{m,i} \tag{34}$$

for every $i \ge 0$, so $a = b = c$, contradicting the initial choice of (a,b,c). Thus $\Lambda(\mathbb{Z}_m^+(D)) = \emptyset$, so $\mathbb{Z}_m^+(D)$ is midpoint-free. □

Three early instances of Theorem 8, the first of which was independently demonstrated earlier, are of considerable interest.

Corollary 9. *Each $\mathbb{Z}_m^+(0,1)$ is midpoint-free when $m \ge 3$.*

Corollary 10. *Each $\mathbb{Z}_m^+(0,1,3)$ is midpoint-free when $m \ge 7$.*

Corollary 11. *Each $\mathbb{Z}_m^+(0,1,3,4)$ is midpoint-free when $m \ge 9$.*

If D is a midpoint-free base $m \ge 3$ digit subset, it is of interest to decide whether $\mathbb{Z}^+$ has any members that are eccentric for $\mathbb{Z}_m^+(D)$, since this is equivalent to deciding whether $\mathbb{Z}_m^+(D)$ is a maximal midpoint-free subset of $\mathbb{Z}^+$.

For $x \in \mathbb{Z}^+$, let the *support* for x, as a lower endpoint, midpoint, or upper endpoint for S, be defined by

$$\begin{aligned}
&\operatorname*{supp}_A\left(x, \mathbb{Z}_m^+(D)\right) \\
&\qquad := \bigcup \left\{b, c \in \mathbb{Z}_m^+(D) \mid (x, b, c) \in \Lambda\left(\mathbb{Z}^+\right)\right\}, \\
&\operatorname*{supp}_B\left(x, \mathbb{Z}_m^+(D)\right) \\
&\qquad := \bigcup \left\{a, c \in \mathbb{Z}_m^+(D) \mid (a, x, c) \in \Lambda\left(\mathbb{Z}^+\right)\right\}, \\
&\operatorname*{supp}_C\left(x, \mathbb{Z}_m^+(D)\right) \\
&\qquad := \bigcup \left\{a, b \in \mathbb{Z}_m^+(D) \mid (a, b, x) \in \Lambda\left(\mathbb{Z}^+\right)\right\}.
\end{aligned} \tag{35}$$

The following result is useful for settling whether the eccentric set $E(S, X)$ is empty in specific cases.

Theorem 12. *Suppose $m \geq 3$ and D is a midpoint-free base m digit subset. If $x \in [0, m^k)$ with $k \geq 1$, then*

$$\begin{aligned}
&\operatorname*{supp}_A\left(x, \mathbb{Z}_m^+(D)\right) \subset \left[0, m^{k+1}\right) \cap \mathbb{Z}_m^+(D), \\
&\operatorname*{supp}_B\left(x, \mathbb{Z}_m^+(D)\right) \subset \left[0, 2m^k\right) \cap \mathbb{Z}_m^+(D), \\
&\operatorname*{supp}_C\left(x, \mathbb{Z}_m^+(D)\right) \subset \left[0, m^k\right) \cap \mathbb{Z}_m^+(D).
\end{aligned} \tag{36}$$

Proof. Fix the nonnegative integer $x \in [0, m^k)$ with $k \geq 1$. (For simplicity we do not explicitly require $x \notin \mathbb{Z}_m^+(D)$, although Theorem 8 does imply that members of $\mathbb{Z}_m^+(D)$ have empty support sets.)

Suppose there exist $b, c \in \mathbb{Z}_m^+(D)$ such that $(x, b, c) \in \Lambda(\mathbb{Z}^+)$, so $x + c = 2b$. Base m arithmetic for this sum takes the form

$$[\![x]\!]_{m,i} + [\![c]\!]_{m,i} + \delta_i = [\![2b]\!]_{m,i} + m\delta_{i+1} \tag{37}$$

for every $i \geq 0$, with carry-overs $\delta_i \in \{0, 1\}$ satisfying $\delta_0 = 0$ and

$$\begin{aligned}
&[\![x]\!]_{m,i} + [\![c]\!]_{m,i} + \delta_i < m \Longrightarrow \delta_{i+1} = 0, \\
&[\![x]\!]_{m,i} + [\![c]\!]_{m,i} + \delta_i \geq m \Longrightarrow \delta_{i+1} = 1
\end{aligned} \tag{38}$$

for $i > 0$. But $x < m^k$ so $[\![x]\!]_{m,i} = 0$ for $i \geq k$. In particular,

$$[\![x]\!]_{m,k} + [\![c]\!]_{m,k} + \delta_k < \frac{m}{2} + 1 < m \tag{39}$$

so $\delta_{k+1} = 0$. For all $i \geq k + 1$, it follows that $\delta_i = 0$ and

$$[\![x]\!]_{m,i} + [\![c]\!]_{m,i} + \delta_i = [\![c]\!]_{m,i} = [\![2b]\!]_{m,i} = 2\,[\![b]\!]_{m,i}. \tag{40}$$

But D is midpoint-free, so $[\![b]\!]_{m,i} = [\![c]\!]_{m,i} = 0$ for all $i \geq k + 1$, since otherwise $(0, [\![b]\!]_{m,i}, [\![c]\!]_{m,i}) \in \Lambda(D)$ yields the contradiction $\Lambda(D) \neq \emptyset$. Hence $b < c < m^{k+1}$.

The other two cases follow simply by noting that if $(a, x, c) \in \Lambda(\mathbb{Z}^+)$ then $a < c \leq a + c = 2x < 2m^k$, and if $(a, b, x) \in \Lambda(\mathbb{Z}^+)$ then $a < b < x < m^k$. □

Corollary 13. *If D is a midpoint-free base $m \geq 3$ digit subset, then relative to $\mathbb{Z}_m^+(D)$ the three multiplicities of any $x \in \mathbb{Z}^+$ are finite.*

Corollary 14. *If D is a midpoint-free base $m \geq 3$ digit subset and $x \in [0, m^k)$ with $k \geq 1$, then $M_k := [0, m^{k+1}) \cap \mathbb{Z}_m^+(D)$ contains all three support sets for x relative to $\mathbb{Z}_m^+(D)$. Moreover $\#M_k = d^{k+1}$, where $d := \#D$.*

Corollary 15. *If $m \geq 5$ then $m + 3$ is eccentric for $\mathbb{Z}_m^+(0, 1)$.*

Corollary 16. *If $m \geq 8$ then $2m + 4$ is eccentric for $\mathbb{Z}_m^+(0, 1, 3)$.*

Corollary 17. *If $m \geq 14$ then $4m + 9$ is eccentric for $\mathbb{Z}_m^+(0, 1, 3, 4)$.*

The last three corollaries leave open the possibility that their subject sets $\mathbb{Z}_m^+(D)$ might be maximal midpoint-free subsets of $\mathbb{Z}^+$ when m is small enough. In the next section we shall consider $\mathbb{Z}_3^+(0, 1)$ and pursue other cases later.

5. Greedy Midpoint-Free Subset of $\mathbb{Z}^+$

The *greedy midpoint-free subset* of $\mathbb{Z}^+$ is the set

$$S_0 := \{s_i \in \mathbb{Z}^+ \mid i \in \mathbb{Z}^+\} \tag{41}$$

in which $s_0 = 0$ and each s_i with $i > 0$ is the smallest integer satisfying $s_i > s_{i-1}$ such that $\{s_0, s_1, \ldots, s_i\}$ is midpoint-free. It has long been known [4] that

$$S_0 = \mathbb{Z}_3^+(0, 1) = \{0, 1, 3, 4, 9, 10, 12, 13, 27, \ldots\}, \tag{42}$$

corresponding to sequence A00536 of OEIS [5]. For brevity, let

$$\begin{aligned}
A_0^+ := A\left(S_0, \mathbb{Z}^+\right), \qquad B_0^+ := B\left(S_0, \mathbb{Z}^+\right), \\
C_0^+ := C\left(S_0, \mathbb{Z}^+\right).
\end{aligned} \tag{43}$$

We shall attach the adjective *greedy* when referring to these sets. Before we prove that each of these greedy balance point sets contains all positive integers not in $\mathbb{Z}_3^+(0, 1)$, let us check the example $32 = (1012)_3 \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0, 1)$. The following base 3 computations are transparent:

$$\begin{aligned}
(1012)_3 + (1010)_3 &= 2 \cdot (1011)_3, \\
(1012)_3 + (1111)_3 &= 2 \cdot (1100)_3, \\
2 \cdot (1012)_3 = (2101)_3 &= (1000)_3 + (1101)_3.
\end{aligned} \tag{44}$$

Hence we have the midpoint triples $(30, 31, 32)$, $(32, 36, 40)$, $(27, 32, 37) \in \Lambda(\mathbb{Z}^+)$, showing that $32 \in A_0^+ \cap B_0^+ \cap C_0^+$. Now consider the general case.

Theorem 18. *The greedy midpoint-free subset $\mathbb{Z}_3^+(0, 1) \subset \mathbb{Z}^+$ has greedy balance point sets satisfying*

$$A_0^+ = B_0^+ = C_0^+ = \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0, 1). \tag{45}$$

Proof. Let $x \in \mathbb{Z}^+ \backslash \mathbb{Z}_3^+(0,1)$. First we show that x is a midpoint for $\mathbb{Z}_3^+(0,1)$. Note that there is at least one $j \geq 0$ such that $[\![2x]\!]_{3,j} = 1$. Specify $a, c \in \mathbb{Z}_3^+(0,1)$ by their base 3 digits; thus

$$[\![2x]\!]_{3,i} = 1 \Longrightarrow [\![a]\!]_{3,i} = 0, \qquad [\![c]\!]_{3,i} = 1;$$
$$[\![2x]\!]_{3,i} \neq 1 \Longrightarrow [\![a]\!]_{3,i} = [\![c]\!]_{3,i} = \frac{1}{2}[\![2x]\!]_{3,i}. \tag{46}$$

We have $[\![a]\!]_{3,i} \leq [\![c]\!]_{3,i} \leq [\![2x]\!]_{3,i}$ for all $i \geq 0$. Also $[\![a]\!]_{3,j} = 0 < [\![c]\!]_{3,j} = 1$, because $[\![2x]\!]_{3,j} = 1$. Hence $0 \leq a < c \leq 2x$. It is easily checked that

$$[\![a]\!]_{3,i} + [\![c]\!]_{3,i} = [\![2x]\!]_{3,i} \tag{47}$$

for all $i \geq 0$, so $a + c = 2x$. Hence $x \in B_0^+$.

Next we show that x is an upper endpoint for $\mathbb{Z}_3^+(0,1)$. Let $a, b \in \mathbb{Z}_3^+(0,1)$ be specified by their base 3 digits; thus

$$[\![x]\!]_{3,i} = 2 \Longrightarrow [\![a]\!]_{3,i} = 0, \qquad [\![b]\!]_{3,i} = 1;$$
$$[\![x]\!]_{3,i} \neq 2 \Longrightarrow [\![a]\!]_{3,i} = [\![b]\!]_{3,i} = [\![x]\!]_{3,i}. \tag{48}$$

There is at least one $k \geq 0$ such that $[\![x]\!]_{3,k} = 2$, so $0 \leq a < b < x$. Also

$$[\![a]\!]_{3,i} + [\![x]\!]_{3,i} = 2[\![b]\!]_{3,i} = [\![2b]\!]_{3,i} \tag{49}$$

for all $i \geq 0$, so $a + x = 2b$. Hence $x \in C_0^+$.

Finally we show that x is a lower endpoint for $\mathbb{Z}_3^+(0,1)$. We need to find $b, c \in \mathbb{Z}_3^+(0,1)$ such that $0 < x < b < c$ and $x + c = 2b$. Base 3 arithmetic for this sum takes the form

$$[\![x]\!]_{3,i} + [\![c]\!]_{3,i} + \delta_i = [\![2b]\!]_{3,i} + 3\delta_{i+1} \tag{50}$$

for every $i \geq 0$, with carry-overs $\delta_i \in \{0,1\}$ satisfying $\delta_0 = 0$ and

$$[\![x]\!]_{3,i} + [\![c]\!]_{3,i} + \delta_i \leq 2 \Longrightarrow \delta_{i+1} = 0,$$
$$[\![x]\!]_{3,i} + [\![c]\!]_{3,i} + \delta_i > 2 \Longrightarrow \delta_{i+1} = 1. \tag{51}$$

Specify b, c by their base 3 digits; thus

$$[\![x]\!]_{3,i} + \delta_i \in \{0,3\} \Longrightarrow [\![c]\!]_{3,i} = [\![b]\!]_{3,i} = 0,$$
$$[\![x]\!]_{3,i} + \delta_i = 1 \Longrightarrow [\![c]\!]_{3,i} = [\![b]\!]_{3,i} = 1, \tag{52}$$
$$[\![x]\!]_{3,i} + \delta_i = 2 \Longrightarrow [\![c]\!]_{3,i} = 1, \qquad [\![b]\!]_{3,i} = 0.$$

If $[\![x]\!]_{3,k} = 2$ and $[\![x]\!]_{3,i} < 2$ for $0 \leq i < k$ then $\delta_i = 0$ for $0 \leq i \leq k$. It follows that $[\![c]\!]_{3,k} = 1 > [\![b]\!]_{3,k} = 0$. Since $[\![c]\!]_{3,i} \geq [\![b]\!]_{3,i}$, for every $i \geq 0$, we have $c > b$.

There are integers $h \geq g \geq k$ such that $[\![x]\!]_{3,h} > 0$ and $[\![x]\!]_{3,i} = 0$ for $i > h$ and $[\![x]\!]_{3,g} = 2$ and $[\![x]\!]_{3,i} \leq 1$ for $i > g$. If $\delta_g = 0$ then $[\![c]\!]_{3,g} = 1$, $[\![b]\!]_{3,g} = 0$; if $\delta_g = 1$ then $[\![c]\!]_{3,g} = [\![b]\!]_{3,g} = 0$. In either case it follows that $\delta_{g+1} = 1$. If $\delta_{h+1} = 1$ then $[\![x]\!]_{3,h+1} < [\![c]\!]_{3,h+1} = [\![b]\!]_{3,h+1} = 1$ so $x < b$ since $[\![x]\!]_{3,i} = 0$ for all $i > h$. If $\delta_{h+1} = 0$ then there is an integer f such that $h \geq f > g$ and $\delta_i = 1$ for $f \geq i > g$ while $\delta_{f+1} = 0$. If $[\![x]\!]_{3,i} \leq 1$ and $\delta_i = 0$ then $[\![x]\!]_{3,i} = [\![c]\!]_{3,i} = [\![b]\!]_{3,i} \in \{0,1\}$ so $\delta_{i+1} = 0$. Hence $\delta_i = 0$ for all $i > f$. Also $\delta_f = 1$, $\delta_{f+1} = 0$ and $[\![c]\!]_{3,f} = 1$, so $[\![x]\!]_{3,f} = 0$. Then $[\![x]\!]_{3,f} = 0 < [\![b]\!]_{3,f} = 1$ and $[\![x]\!]_{3,i} = [\![b]\!]_{3,i}$ for all $i > f$, so $x < b$.

Finally, in all three cases $[\![x]\!]_{3,i} + [\![c]\!]_{3,i} + \delta_i = 2[\![b]\!]_{3,i} + 3\delta_{i+1}$ for each $i \geq 0$, so $x + c = 2b$. Hence $x \in A_0^+$. $\square$

Corollary 19. *The greedy midpoint-free subset $\mathbb{Z}_3^+(0,1)$ is maximal in $\mathbb{Z}^+$.*

In [3] it was shown that $S_0 = \mathbb{Z}_3^+(0,1)$ is actually a maximal midpoint-free subset of $\mathbb{Z}$. The proof is not repeated here, but let us note that the single example $-32 \in A(S_0, \mathbb{Z})$ follows from computing $-32 + c = 2b$ with $b, c \in \mathbb{Z}_3^+(0,1)$ in the form $32 + 2b = c$. Base 3 considerations yield

$$(1012)_3 + (22)_3 = (1111)_3, \tag{53}$$

corresponding to $(-32, 4, 40) \in \Lambda(\mathbb{Z})$, so $-32 \in A(S_0, \mathbb{Z})$ as claimed.

For brevity, let $A_0 := A(S_0, \mathbb{Z})$, $B_0 := B(S_0, \mathbb{Z})$, $C_0 := C(S_0, \mathbb{Z})$. With Theorem 18, this yields the following result.

Corollary 20. *The greedy midpoint-free subset $\mathbb{Z}_3^+(0,1) \subset \mathbb{Z}$ has greedy balance point sets satisfying $A_0 = \mathbb{Z} \setminus \mathbb{Z}_3^+(0,1)$, $B_0 = C_0 = \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$.*

Reversing implications, this yields the following result.

Corollary 21. *The greedy midpoint-free subset $\mathbb{Z}_3^+(0,1)$ is maximal in $\mathbb{Z}$.*

Now the multiplicities for $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$ can be examined. First note that the sum of base 3 digits of any even integer is 0 mod 2, so even integers have an even number of base 3 digits equal to 1. If $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, at least one digit in $(2x)_3$ must be 1, so the total number of such digits is $2k$, and $k > 0$.

Theorem 22. *If $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, then the midpoint multiplicity of x is*

$$\beta(x, \mathbb{Z}_3^+(0,1)) = 2^{2k-1}, \tag{54}$$

where $2k$ is the number of digits equal to 1 in $(2x)_3$ and $k > 0$.

Proof. Suppose $a, c \in \mathbb{Z}_3^+(0,1)$ satisfy $a < c$ and $a + c = 2x$. There is no carry-over in the base 3 arithmetic for the sum, so $[\![a]\!]_{3,i} + [\![c]\!]_{3,i} = [\![2x]\!]_{3,i}$, for all $i \geq 0$. If $[\![2x]\!]_{3,i} \in \{0,2\}$ then we must have

$$[\![a]\!]_{3,i} = [\![c]\!]_{3,i} = \frac{1}{2}[\![2x]\!]_{3,i}. \tag{55}$$

There is an integer $j > 0$ such that $[\![2x]\!]_{3,j} = 1$ and $[\![2x]\!]_{3,i} \in \{0,2\}$ for all $i > j$, so $a < c$ forces $[\![a]\!]_{3,j} = 0 < [\![c]\!]_{3,j} = 1$. However, when $j > i \geq 0$ and $[\![2x]\!]_{3,i} = 1$, the requirement $[\![a]\!]_{3,i} + [\![c]\!]_{3,i} = 1$ is satisfied if $\{[\![a]\!]_{3,i}, [\![c]\!]_{3,i}\} = \{0,1\}$. Both possibilities are consistent with $a < c$, so there are 2^{2k-1} solutions in total. $\square$

To illustrate, when $x = 50$ then $2x = (10201)_3$ so $\Lambda(\mathbb{Z}_3^+(0,1),\mathbb{Z}^+)$ has just two triples with 50 as midpoint: $(9,50,91)$, $(10,50,90)$. In contrast, when $x = 20$ then $2x = (1111)_3$ so $\Lambda(\mathbb{Z}_3^+(0,1),\mathbb{Z}^+)$ has eight triples with 20 as midpoint, ranging from $(0,20,40)$ to $(13,20,27)$, namely, $(a,20,40-a)$ for all $a \in \mathbb{Z}_3^+(0,1) \cap [0,13]$.

Fix $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$. The digits of the base 3 representation $(x)_3$ include at least one 2. An ordered pair of integers (j,k) with $0 \le j < k$ is *critical* for x if $[\![x]\!]_{3,j} = 2$ and $[\![x]\!]_{3,k} < 2$, with $[\![x]\!]_{3,i} > 0$ when $j \le i < k$. Any two ordered pairs (j,k) and (j',k') critical for x are *independent* if $[j,k] \cap [j',k'] = \emptyset$. Any set of ordered pairs critical for x is independent if every two members are independent. Let $\mathrm{Crit}(x)$ denote the family of all sets of independent ordered pairs critical for x, including the empty set.

Theorem 23. *If $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, then x has total endpoint multiplicity*

$$\alpha\left(x,\mathbb{Z}_3^+(0,1)\right) + \gamma\left(x,\mathbb{Z}_3^+(0,1)\right) = \#\mathrm{Crit}(x). \tag{56}$$

Proof. Given $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, we seek $y,b \in \mathbb{Z}_3^+(0,1)$ such that $x + y = 2b$. Any solution necessarily satisfies $x \ne y$. Base 3 arithmetic for this sum takes the form

$$[\![x]\!]_{3,i} + [\![y]\!]_{3,i} + \delta_i = 2[\![b]\!]_{3,i} + 3\delta_{i+1} \tag{57}$$

for every $i \ge 0$, with carry-overs $\delta_i \in \{0,1\}$ satisfying $\delta_0 = 0$ and

$$\begin{aligned} [\![x]\!]_{3,i} + [\![y]\!]_{3,i} + \delta_i \le 2 &\Longrightarrow \delta_{i+1} = 0, \\ [\![x]\!]_{3,i} + [\![y]\!]_{3,i} + \delta_i > 2 &\Longrightarrow \delta_{i+1} = 1. \end{aligned} \tag{58}$$

Specifying y,b by their base 3 digits, the equation $x + y = 2b$ requires

$$\begin{aligned} [\![x]\!]_{3,i} + \delta_i \in \{0,3\} &\Longrightarrow [\![y]\!]_{3,i} = [\![b]\!]_{3,i} = 0, \\ [\![x]\!]_{3,i} + \delta_i = 1 &\Longrightarrow [\![y]\!]_{3,i} = [\![b]\!]_{3,i} = 1, \\ [\![x]\!]_{3,i} + \delta_i = 2 &\Longrightarrow [\![y]\!]_{3,i} + [\![b]\!]_{3,i} = 1. \end{aligned} \tag{59}$$

As is easily verified, these digit specifications satisfy the requirements of base 3 arithmetic for $x + y = 2b$.

Any integer $i \ge 0$ is a base 3 *transition point* for the sum $x + y$ if $\delta_{i+1} \ne \delta_i$. Note that the digits $[\![y]\!]_{3,i}$ and $[\![b]\!]_{3,i}$ are forced unless $[\![x]\!]_{3,i} + \delta_i = 2$. In the latter case there are two options:

(1) $[\![y]\!]_{3,i} = 0$, $[\![b]\!]_{3,i} = 1$, $\delta_{i+1} = 0$,

(2) $[\![y]\!]_{3,i} = 1$, $[\![b]\!]_{3,i} = 0$, $\delta_{i+1} = 1$.

If $[\![x]\!]_{3,i} = 2$, $\delta_i = 0$ then choosing option (1) preserves the carry-over state ("off") with $\delta_{i+1} = 0$ whereas choosing option (2) switches the carry-over state (to "on") with $\delta_{i+1} = 1$, so option (2) causes i to be a transition point. If $[\![x]\!]_{3,i} = 1$, $\delta_i = 1$ then choosing option (1) switches the carry-over state (to "off") with $\delta_{i+1} = 0$ and causes i to be a transition point, whereas choosing option (2) preserves the carry-over state ("on") with $\delta_{i+1} = 1$.

Let X be a maximal block of nonzero digits in the base 3 representation $(x)_3$, say $X := ([\![x]\!]_{3,i} \mid g \le i \le h)$, and assume that X contains the digit 2 at least once. There must be at least one such maximal block X in $(x)_3$. Then $\delta_g = 0$, and the "off" carry-over state $\delta_i = 0$ can only switch to $\delta_{j+1} = 1$ for some j such that $g \le j \le h$ when $[\![x]\!]_{3,j} = 2$ and $[\![y]\!]_{3,j} = 1$, corresponding to an option (2) choice in the construction of y. The "on" carry-over state $\delta_i = 1$ can only switch back to $\delta_{k+1} = 0$ for some k such that $j < k \le h$ if $[\![x]\!]_{3,k} = 1$ and $[\![y]\!]_{3,k} = 0$, corresponding to an option (1) choice in constructing y. If no such option is exercised, then maximality of X ensures the carry-over state $\delta_i = 1$ inevitably switches back at $i = h + 1$, for in this case $[\![x]\!]_{3,h} > 0$, $[\![x]\!]_{3,h+1} = 0$, $\delta_h = 1$ and an option (2) choice for $[\![y]\!]_{3,h}$ results in $\delta_{h+1} = 1$, $[\![y]\!]_{3,h+1} = 1$, $[\![b]\!]_{3,h+1} = 1$, $\delta_{h+2} = 0$. Note that the ordered pair (j,k) is critical for x, as is the ordered pair $(j,h+1)$.

Now consider any set $P(X)$ of independent ordered pairs (j,k) critical for x, with $g \le j < k \le h+1$. The set $P(X)$ determines a unique sequence of carry-over digits $\Delta := (\delta_i \mid g \le i \le h+1)$, with $\delta_i = 1$ if $j \le i \le k$ and $(j,k) \in P(X)$, and $\delta_i = 0$ for every other i in the interval $[g,h+1]$. Then X and Δ determine blocks

$$\begin{aligned} Y &:= \left([\![y]\!]_{3,i} \mid g \le i \le h+1\right), \\ B &:= \left([\![b]\!]_{3,i} \mid g \le i \le h+1\right) \end{aligned} \tag{60}$$

such that $X + Y = 2B$. Each member of $\mathrm{Crit}(x)$ is of the form $P(x) := \bigcup_X P(X)$, where X runs through all maximal blocks of nonzero digits containing 2 in $(x)_3$. Any such $P(x)$ uniquely determines a collection of suitable blocks of base 3 digits for y,b while all other digits are forced by base 3 arithmetic for $x + y = 2b$, so the number of solutions for y,b is precisely $\#\mathrm{Crit}(x)$. □

To illustrate, when $x = 50 = (1212)_3$ the possible ordered pairs critical for x are $(0,1)$, $(0,3)$, $(0,4)$, $(2,3)$, $(2,4)$. There are $\#\mathrm{Crit}(50) = 8$ sets of independent critical pairs, namely, $\emptyset$, the five singletons, $\{(0,1),(2,3)\}$ and $\{(0,1),(2,4)\}$. The corresponding triples in $\Lambda(\mathbb{Z}^+)$ with 50 as an endpoint are

$$\begin{aligned} &(4,27,50),(10,30,50),(12,31,50),(28,39,50),(30,40,50), \\ &\text{so } \gamma\left(50,\mathbb{Z}_3^+(0,1)\right) = 5; \\ &(50,81,112),(50,84,118),(50,85,120), \\ &\text{so } \alpha\left(50,\mathbb{Z}_3^+(0,1)\right) = 3. \end{aligned} \tag{61}$$

Given $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, let $X := ([\![x]\!]_{3,i} \mid g \le i \le h)$ be the leading maximal block of nonzero digits containing 2 in $(x)_3$, so $[\![x]\!]_{3,i} < 2$ when $i > h$, and in particular $[\![x]\!]_{3,h+1} = 0$. Then any ordered pair (j,k) critical for x is constrained by $0 \le j < k \le h+1$. In particular, the *leading* ordered pairs critical for x are all those of the form $(j,h+1)$. In this case $g \le j \le h$, and there is at least one such ordered pair since $[\![x]\!]_{3,j} = 2$ for at least one $j \in [g,h]$. Let $\mathrm{LeadCrit}(x)$ denote the family of just those sets of independent ordered pairs critical for x that include a leading pair.

Corollary 24. *If $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, then x has lower endpoint multiplicity*

$$\alpha\left(x, \mathbb{Z}_3^+(0,1)\right) = \#\mathrm{LeadCrit}(x). \tag{62}$$

Proof. Continuing with the definitions and notation in the proof of Theorem 23, if $P(x) \in \mathrm{LeadCrit}(x)$ contains the leading ordered pair $(j, h+1)$ critical for x then the corresponding y, b for $x + y = 2b$ satisfy $[\![x]\!]_{3,j} = 2$, $[\![y]\!]_{3,j} = 1$, $[\![b]\!]_{3,j} = 0$ with $[\![y]\!]_{3,i} = [\![b]\!]_{3,i}$ when $i > j$; moreover, $[\![x]\!]_{3,h+1} = 0$, $[\![y]\!]_{3,h+1} = 1$, and $[\![x]\!]_{3,i} = [\![y]\!]_{3,i}$ when $i \geq h+2$. Thus $x < b < y$ and $b, y \in \mathrm{supp}_A(x, \mathbb{Z}_3^+(0,1))$.

On the other hand, if $P^*(x) \in \mathrm{Crit}(x) \setminus \mathrm{LeadCrit}(x)$ then the corresponding y, b for $x + y = 2b$ satisfy $[\![x]\!]_{3,k} + \delta_k = 2$, $[\![y]\!]_{3,k} = 0$, $[\![b]\!]_{3,k} = 1$ for some $k \in [g, h]$, with $[\![x]\!]_{3,i} = [\![y]\!]_{3,k} = [\![b]\!]_{3,i}$ when $i > k$. Thus $b, y \in \mathrm{supp}_C(x, \mathbb{Z}_3^+(0,1))$. □

The multiplicities in Theorem 23 and Corollary 24 are implicit, so it is of some interest to identify a class of positive integers with endpoint multiplicities that can be specified simply and explicitly. Let $u_{3,n} := (3^n - 1)/2$ be the integer with base 3 representation comprising a block of n digits all equal to 1, so $u_{3,n}$ is a base 3 *rep-unit*, following terminology of Yates [6]. (See A003462 in OEIS [5].)

Corollary 25. *For any integers $n \geq m > 0$ and $s \geq 0$, $t > n + s$, let*

$$x := 3^s\left(u_{3,n} + u_{3,m}\right) + 3^t v + w, \tag{63}$$

where $v \in \mathbb{Z}_3^+(0,1)$ and $w \in \mathbb{Z}_3^+(0,1) \cap [0, 3^s)$. Then

$$\begin{gathered}\alpha\left(x, \mathbb{Z}_3^+(0,1)\right) = m, \\ \gamma\left(x, \mathbb{Z}_3^+(0,1)\right) = m(n-m) + 1.\end{gathered} \tag{64}$$

Proof. The only maximal block of nonzero digits containing 2 in $(x)_3$ is $X = ([\![x]\!]_{3,i} \mid s \leq i < n+s)$, with $[\![x]\!]_{3,i} = 2$ when $s \leq i < m+s$, and $[\![x]\!]_{3,i} = 1$ when $m+s \leq i < n+s$, with $[\![x]\!]_{3,n+s} = 0$. Therefore, the only ordered pairs (j,k) critical for x have $s \leq j < m+s \leq k \leq n+s$. No two of these critical pairs are disjoint, so the only sets in $\mathrm{Crit}(x)$ are $m(n-m+1)$ singletons and the empty set. The sets comprising $\mathrm{LeadCrit}(x)$ are the m singletons $\{(j, n+s)\}$. □

For instance, $x = 862 = (1011221)_3$ arises by taking $m = 2$, $n = 4$, $s = 1$, $t = 6$, and $v = w = 1$ in Corollary 25. The leading critical pairs for x are $(j, 5)$ with $j \in \{1,2\}$, yielding two midpoint triples (x, b, y) with $b, y \in \mathbb{Z}_3^+(0,1)$:

$$b = (11000B1)_3, \qquad y = (1111Y1)_3 \tag{65}$$

for blocks $(B, Y) \in \{(0, 01), (1, 10)\}$, so $(x, b, y) = (862, 973 + r, 1084 + 2r)$ with $r \in \{0, 3\}$. Thus $\alpha(862, \mathbb{Z}_3^+(0,1)) = 2$. The non-leading critical pairs for x are (j,k) with $j \in \{1,2\}$, $k \in \{3,4\}$, yielding five midpoint triples (y, b, x) with $b, y \in \mathbb{Z}_3^+(0,1)$:

$$b = (101B1)_3, \qquad y = (10Y1)_3 \tag{66}$$

for $(B, Y) \in \{(000, 0101), (001, 0110), (100, 1001), (101, 1010), (111, 1100)\}$, so $(y, b, x) = (760 + 2r, 811 + r, 862)$ for $r \in \{0, 3, 27, 30, 39\}$. Thus $\gamma(862, \mathbb{Z}_3^+(0,1)) = 5$.

6. Doubling the Greedy Midpoint-Free Subset of $\mathbb{Z}^+$

Doubling $S_0 := \mathbb{Z}_3^+(0,1)$ as in Theorem 4 shows that $S_0^{(2)} := 2S_0 \cup -(2S_0 + 1)$ is a maximal midpoint-free subset of $\mathbb{Z}$. (This is an alternative demonstration to the proof given in [3].) Here $2S_0 = \mathbb{Z}_3^+(0,2)$ and $2S_0 + 1 = \mathbb{Z}_3^+(0,2;1)$, where $\mathbb{Z}_3^+(0,2;1)$ comprises those positive integers with base 3 representation in which the trailing digit (the last nonzero digit) is 1 and every other digit is in $\{0,2\}$. The relevant balance point sets and multiplicities will now be examined briefly.

For brevity, let us write

$$\begin{gathered}A_{0,2}^+ := A\left(2S_0, \mathbb{Z}^+\right), \qquad B_{0,2}^+ := B\left(2S_0, \mathbb{Z}^+\right), \\ C_{0,2}^+ := C\left(2S_0, \mathbb{Z}^+\right); \\ A_{0,2} := A\left(2S_0, \mathbb{Z}\right), \qquad B_{0,2} := B\left(2S_0, \mathbb{Z}\right), \\ C_{0,2} := C\left(2S_0, \mathbb{Z}\right); \\ A_{0,2}^{(2)} := A\left(S_0^{(2)}, \mathbb{Z}\right), \qquad B_{0,2}^{(2)} := B\left(S_0^{(2)}, \mathbb{Z}\right), \\ C_{0,2}^{(2)} := C\left(S_0^{(2)}, \mathbb{Z}\right).\end{gathered} \tag{67}$$

Note that $A(2S_0+1, \mathbb{Z}^+) = A_{0,2}^+ + 1$, $A(2S_0+1, \mathbb{Z}) = A_{0,2} + 1$. Similar identities hold for other balance point sets of $2S_0 + 1$.

It can be shown that every odd integer $x \in \mathbb{Z}^+$ is the sum of two distinct members of $\mathbb{Z}_3^+(0,1)$. Indeed, there are precisely 4^k such sums, where $2k+1$ is the number of digits equal to 1 in $(x)_3$ for some integer $k \geq 0$. The balance point sets and multiplicities for $2S_0$, $2S_0 + 1$, and $S_0^{(2)}$ can now be specified.

Theorem 26. *The midpoint-free subset $\mathbb{Z}_3^+(0,2) \subset \mathbb{Z}^+$ has balance point sets satisfying $A_{0,2}^+ = C_{0,2}^+ = 2\mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,2)$ and $B_{0,2}^+ = \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,2)$.*

Corollary 27. *If $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,1)$, the endpoint multiplicities satisfy*

$$\begin{gathered}\alpha\left(2x+1, \mathbb{Z}_3^+(0,2;1)\right) = \alpha\left(2x, \mathbb{Z}_3^+(0,2)\right) = \alpha\left(x, \mathbb{Z}_3^+(0,1)\right), \\ \gamma\left(2x+1, \mathbb{Z}_3^+(0,2;1)\right) = \gamma\left(2x, \mathbb{Z}_3^+(0,2)\right) = \gamma\left(x, \mathbb{Z}_3^+(0,1)\right).\end{gathered} \tag{68}$$

Corollary 28. *If $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_3^+(0,2)$, the midpoint multiplicities satisfy*

$$\beta\left(x+1, \mathbb{Z}_3^+(0,2;1)\right) = \beta\left(x, \mathbb{Z}_3^+(0,2)\right) = 2^{k-1}, \tag{69}$$

where $k > 0$ is the number of digits equal to 1 in $(x)_3$.

Corollary 29. *The midpoint-free subset $\mathbb{Z}_3^+(0,2) \subset \mathbb{Z}$ has balance point sets satisfying $A_{0,2} = A_{0,2}^+ \cup 2\mathbb{Z}^-$, $B_{0,2} = B_{0,2}^+$, $C_{0,2} = C_{0,2}^+$.*

Corollary 30. *The midpoint-free subset* $S_0^{(2)} := \mathbb{Z}_3^+(0,2) \cup -\mathbb{Z}_3^+(0,2;1) \subset \mathbb{Z}$ *has balance point sets satisfying*

$$\begin{aligned} A_{0,2}^{(2)} &= A_{0,2} \cup -\left(C_{0,2}+1\right) = \left(\mathbb{Z} \setminus S_0^{(2)}\right) \setminus \left(2\mathbb{Z}^+ + 1\right), \\ C_{0,2}^{(2)} &= C_{0,2} \cup -\left(A_{0,2}+1\right) = \left(\mathbb{Z} \setminus S_0^{(2)}\right) \setminus 2\mathbb{Z}^-, \\ B_{0,2}^{(2)} &= B_{0,2}^+ \cup -B_{0,2,1}^+ = \mathbb{Z} \setminus S_0^{(2)}. \end{aligned} \tag{70}$$

Corollary 31. *The set* $S_0^{(2)}$ *is a maximal midpoint-free subset of* $\mathbb{Z}$*, with balance point sets satisfying* $A_{0,2}^{(2)} \cup C_{0,2}^{(2)} = B_{0,2}^{(2)} = \mathbb{Z} \setminus S_0^{(2)}$.

Note that trebling $S_0 := \mathbb{Z}_3^+(0,1)$ as in Theorem 6 yields $S_0^{(3)} := 3S_0 \cup (3S_0+1)$, and earlier we saw that this is "trivial" in this case, because $S_0^{(3)} = S_0$. In the notation of Corollary 7, for any integer $n \geq 1$ this implies

$$S_0^{(3,n)} = S_0, \qquad S_0^{(2,3,n)} = S_0^{(2)}. \tag{71}$$

If S is any set satisfying the hypotheses of Theorem 6, and $s_0 := \min(S)$, then the *normalized* set $S' = S - s_0$ is an affine transform which satisfies the hypotheses of Theorem 6. Without loss of generality, assume S is any normalized compliant set; then Corollary 7 shows that iterated trebling yields a sequence of maximal midpoint-free subsets of $\mathbb{Z}$ which asymptotically approach the greedy subset S_0, because

$$S^{(3,n)} \cap \left[0, 3^n\right) = S_0 \cap \left[0, 3^n\right). \tag{72}$$

In this sense we may write the asymptotic equivalences

$$S^{(3,n)} \sim S_0, \qquad S^{(2,3,n)} \sim S_0^{(2)}. \tag{73}$$

7. The Midpoint-Free Set $\mathbb{Z}_4^+(0,1)$

Let us now study the set

$$S_1 := \mathbb{Z}_4^+(0,1) = \{0,1,4,5,16,17,20,21,64,\ldots\}, \tag{74}$$

corresponding to sequence A000695 in OEIS [5]. Corollaries 9 and 15 leave open the possibility that S_1 is a maximal midpoint-free subset of $\mathbb{Z}^+$. We now settle that matter. Let $A_1^+ := A(S_1,\mathbb{Z}^+)$, $B_1^+ := B(S_1,\mathbb{Z}^+)$, $C_1^+ := C(S_1,\mathbb{Z}^+)$.

Theorem 32. *The midpoint-free subset* $\mathbb{Z}_4^+(0,1) \subset \mathbb{Z}^+$ *has endpoint sets satisfying* $A_1^+ \cup C_1^+ = \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$.

Proof. Let $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$. We seek $y, b \in \mathbb{Z}_4^+(0,1)$ such that $x + y = 2b$. Clearly any such solution has $x \neq y$. Base 4 computation requires

$$[\![x]\!]_{4,i} + [\![y]\!]_{4,i} + \delta_i = 2[\![b]\!]_{4,i} + 4\delta_{i+1} \tag{75}$$

for every $i \geq 0$, with carry-overs $\delta_i \in \{0,1\}$ satisfying $\delta_0 = 0$ and

$$\begin{aligned} [\![x]\!]_{4,i} + [\![y]\!]_{4,i} + \delta_i \leq 3 &\Longrightarrow \delta_{i+1} = 0, \\ [\![x]\!]_{4,i} + [\![y]\!]_{4,i} + \delta_i > 3 &\Longrightarrow \delta_{i+1} = 1. \end{aligned} \tag{76}$$

Specifying y, b by their base 4 digits, the equation $x + y = 2b$ requires

$$\begin{aligned} [\![x]\!]_{4,i} + \delta_i \in \{0,4\} &\Longrightarrow [\![y]\!]_{4,i} = [\![b]\!]_{4,i} = 0, \\ [\![x]\!]_{4,i} + \delta_i = 1 &\Longrightarrow [\![y]\!]_{4,i} = [\![b]\!]_{4,i} = 1, \\ [\![x]\!]_{4,i} + \delta_i = 2 &\Longrightarrow [\![y]\!]_{4,i} = 0, \qquad [\![b]\!]_{4,i} = 1, \\ [\![x]\!]_{4,i} + \delta_i = 3 &\Longrightarrow [\![y]\!]_{4,i} = 1, \qquad [\![b]\!]_{4,i} = 0. \end{aligned} \tag{77}$$

These digit specifications are easily seen to satisfy the requirements of base 4 arithmetic for $x + y = 2b$. Therefore, every $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$ is in $A_1^+ \cup C_1^+$. Since $\mathbb{Z}_4^+(0,1)$ is midpoint-free, it follows that $A_1^+ \cup C_1^+ = \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$. □

Corollary 33. *The set* $\mathbb{Z}_4^+(0,1)$ *is a maximal midpoint-free subset of* $\mathbb{Z}^+$.

Corollary 34. *For any* $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$ *the total endpoint multiplicity is*

$$\alpha\left(x, \mathbb{Z}_4^+(0,1)\right) + \gamma\left(x, \mathbb{Z}_4^+(0,1)\right) = 1. \tag{78}$$

Any $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$ has at least one base 4 digit in $\{2,3\}$. A *high block* in $(x)_4$ is a maximal block of base 4 digits all in $\{2,3\}$. If $X := ([\![x]\!]_{4,i} \mid j \leq i \leq k)$ is a high block in $(x)_4$, it is *clear* if $[\![x]\!]_{4,k+1} = 0$ and *leading* if $[\![x]\!]_{4,i} \leq 1$ when $i > k$.

Corollary 35. *The lower endpoint set* A_1^+ *for* $\mathbb{Z}_4^+(0,1)$ *in* $\mathbb{Z}^+$ *comprises all positive integers in which the leading base 4 high block is clear and contains 3. The upper endpoint set* C_1^+ *is the complement of* A_1^+ *in* $\mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$.

Proof. With the definitions and notation in the proof of Theorem 32, let $[\![x]\!]_{4,h} > 0$ for some $h > 0$ and $[\![x]\!]_{4,i} = 0$ when $i > h$. Let $X := ([\![x]\!]_{4,i} \mid j \leq i \leq k)$ be the leading base 4 high block for x, so $0 \leq j \leq k \leq h$ and $[\![x]\!]_{4,i} \leq 1$ when $i > k$.

(1) If $[\![x]\!]_{4,h} = 1$ then $\delta_{h+1} = 0$, so $[\![y]\!]_{4,i} = [\![b]\!]_{4,i} = 0$ when $i > h$. Also $[\![x]\!]_{4,k} \geq 2$, while $[\![x]\!]_{4,i} \leq 1$ when $i > k$. If $[\![x]\!]_{4,k} + \delta_k \geq 3$ then $\delta_{k+1} = 1$. Therefore, $[\![x]\!]_{4,k+1} + [\![y]\!]_{4,k+1} = 1$ and $\delta_i = 0$ when $i > k+1$. This case arises just when $[\![x]\!]_{4,t} = 3$ for some t such that $j \leq t \leq k$, and $[\![x]\!]_{4,i} = 2$ if $t < i \leq k$. Also $[\![x]\!]_{4,i} = [\![y]\!]_{4,i}$ when $i > k+1$, so $x > y$ if $[\![x]\!]_{4,k+1} = 1$ and $x < y$ if $[\![x]\!]_{4,k+1} = 0$, the latter condition holding precisely when X is clear. On the other hand, if $[\![x]\!]_{4,k} + \delta_k = 2$ then $[\![x]\!]_{4,k} = 2$, $[\![y]\!]_{4,k} = 0$, while $\delta_i = 0$ and $[\![x]\!]_{4,i} = [\![y]\!]_{4,i}$ when $i > k$, so $x > y$.

(2) If $[\![x]\!]_{4,h} = 2$ and $\delta_h = 0$ then $[\![y]\!]_{4,h} = 0$, so $\delta_i = 0$ and $[\![x]\!]_{4,i} = [\![y]\!]_{4,i} = 0$ when $i > h$. Then $k = h$ but the high block X does not contain 3. In this case $x > y$.

(3) If $[\![x]\!]_{4,h} + \delta_h = 3$ then $[\![y]\!]_{4,h} = 1$, so $\delta_{h+1} = 1$, $[\![x]\!]_{4,h+1} = 0$, $[\![y]\!]_{4,h+1} = 1$, and $\delta_i = [\![x]\!]_{4,i} = [\![y]\!]_{4,i} = 0$ when $i > h+1$. Once again $k = h$, but now the high block X must contain 3. In this case $x < y$.

(4) If $[\![x]\!]_{4,h} = 3$, $\delta_h = 1$ then $\delta_{h+1} = 1$, $[\![x]\!]_{4,h+1} = 0$, $[\![y]\!]_{4,h+1} = 1$ and the same conclusions as in (3) apply. □

Corollary 36. *The midpoint set B_1^+ for $\mathbb{Z}_4^+(0,1)$ in $\mathbb{Z}^+$ comprises those positive integers x with $2x \in \mathbb{Z}_4^+(0,1,2) \setminus \mathbb{Z}_4^+(0,2)$. Any such x has midpoint multiplicity $\beta(x, \mathbb{Z}_4^+(0,1)) = 2^{k-1}$, where $k > 0$ is the number of digits in $(2x)_4$ equal to* 1.

Proof. Let $x \in \mathbb{Z}^+ \setminus \mathbb{Z}_4^+(0,1)$. We seek $a, c \in \mathbb{Z}_4^+(0,1)$ such that $a + c = 2x$. Clearly any such solution has $a \neq x$, so $a \neq c$ holds. Base 4 arithmetic requires

$$[\![a]\!]_{4,i} + [\![c]\!]_{4,i} = [\![2x]\!]_{4,i} \tag{79}$$

for every $i \geq 0$. This forces $[\![2x]\!]_{4,i} \in \{0,1,2\}$. Since $x \notin \mathbb{Z}_4^+(0,1)$ it follows that $[\![2x]\!]_{4,j} = 1$ for some $j \geq 0$, with $[\![2x]\!]_{4,i} \neq 1$ for all $i > j$. Specifying a, c by their base 4 digits, the equation $a + c = 2x$ requires

$$\begin{gathered} [\![2x]\!]_{4,i} \in \{0,2\} \Longrightarrow [\![a]\!]_{4,i} = [\![c]\!]_{4,i} = \frac{1}{2}[\![2x]\!]_{4,i}, \\ [\![2x]\!]_{4,i} = 1 \Longrightarrow [\![a]\!]_{4,i} + [\![c]\!]_{4,i} = 1. \end{gathered} \tag{80}$$

With $[\![a]\!]_{4,j} = 0$, $[\![c]\!]_{4,j} = 1$, clearly all solutions meet base 4 arithmetic requirements for $a + c = 2x$ and $a < c$. □

An alternative characterization of the midpoint set B_1^+ is that it comprises every $x \in \mathbb{Z}^+$ with $[\![x]\!]_{4,i} \in \{2,3\}$ for at least one $i \geq 0$, and in each such instance $[\![x]\!]_{4,i+1} \in \{0,2\}$.

Now consider the midpoint-free set $S_1 := \mathbb{Z}_4^+(0,1)$ as a subset of $\mathbb{Z}$. For brevity, let $A_1 := A(S_1, \mathbb{Z})$, $B_1 := B(S_1, \mathbb{Z})$, $C_1 := C(S_1, \mathbb{Z})$.

Corollary 37. *The midpoint-free subset $\mathbb{Z}_4^+(0,1) \subset \mathbb{Z}$ has balance point sets satisfying $A_1 = A_1^+ \cup \mathbb{Z}^-$, $B_1 = B_1^+$, $C_1 = C_1^+$.*

Proof. Let $-x \in \mathbb{Z}^-$. We seek $b, c \in \mathbb{Z}_4^+(0,1)$ such that $x + 2b = c$. Clearly any such solution has $-x < b < c$. Base 4 computation requires

$$[\![x]\!]_{4,i} + 2[\![b]\!]_{4,i} + \delta_i = [\![c]\!]_{4,i} + 4\delta_{i+1} \tag{81}$$

for every $i \geq 0$, with carry-overs $\delta_i \in \{0,1\}$ satisfying $\delta_0 = 0$ and

$$\begin{gathered} [\![x]\!]_{4,i} + 2[\![b]\!]_{4,i} + \delta_i \leq 3 \Longrightarrow \delta_{i+1} = 0, \\ [\![x]\!]_{4,i} + 2[\![b]\!]_{4,i} + \delta_i > 3 \Longrightarrow \delta_{i+1} = 1. \end{gathered} \tag{82}$$

Specifying b, c by their base 4 digits, we require

$$\begin{gathered} [\![x]\!]_{4,i} + \delta_i \in \{0,4\} \Longrightarrow [\![b]\!]_{4,i} = [\![c]\!]_{4,i} = 0, \\ [\![x]\!]_{4,i} + \delta_i = 1 \Longrightarrow [\![b]\!]_{4,i} = 0, \qquad [\![c]\!]_{4,i} = 1, \\ [\![x]\!]_{4,i} + \delta_i = 2 \Longrightarrow [\![b]\!]_{4,i} = 1, \qquad [\![c]\!]_{4,i} = 0, \\ [\![x]\!]_{4,i} + \delta_i = 3 \Longrightarrow [\![b]\!]_{4,i} = [\![c]\!]_{4,i} = 1. \end{gathered} \tag{83}$$

It is straightforward to verify that b, c are uniquely determined by these specifications, and $(-x, b, c) \in \Lambda(\mathbb{Z})$, so $-x \in A_1$. □

Corollary 38. *The set $\mathbb{Z}_4^+(0,1)$ is a maximal midpoint-free subset of $\mathbb{Z}$.*

Corollary 39. *Any $x \in \mathbb{Z}^-$ has lower endpoint multiplicity $\alpha(x, \mathbb{Z}_4^+(0,1)) = 1$.*

8. Doubling and Trebling the Set $\mathbb{Z}_4^+(0,1)$

By Theorem 32 and Corollary 38, the set $S_1 := \mathbb{Z}_4^+(0,1)$ is compliant with the requirements for the doubling and trebling constructions of Theorems 4 and 6, so both sets

$$S_1^{(2)} := 2S_1 \cup -(2S_1 + 1), \qquad S_1^{(3)} := 3S_1 \cup (3S_1 + 1) \tag{84}$$

are maximal midpoint-free subsets of $\mathbb{Z}$.

Let us now briefly examine the balance point sets associated with $S_1^{(2)}$:

$$\begin{gathered} A_{1,2}^+ := A(2S_1, \mathbb{Z}^+), \qquad B_{1,2}^+ := B(2S_1, \mathbb{Z}^+), \\ C_{1,2}^+ := C(2S_1, \mathbb{Z}^+); \\ A_{1,2} := A(2S_1, \mathbb{Z}), \qquad B_{1,2} := B(2S_1, \mathbb{Z}), \\ C_{1,2} := C(2S_1, \mathbb{Z}); \\ A_{1,2}^{(2)} := A(S_1^{(2)}, \mathbb{Z}), \qquad B_{1,2}^{(2)} := B(S_1^{(2)}, \mathbb{Z}), \\ C_{1,2}^{(2)} := C(S_1^{*}, \mathbb{Z}). \end{gathered} \tag{85}$$

Easy digit and parity considerations in combination with Theorem 32 yield the following corollaries.

Theorem 40. *The midpoint-free subset $2S_1 = \mathbb{Z}_4^+(0,2) \subset \mathbb{Z}^+$ has balance point sets satisfying*

$$\begin{gathered} A_{1,2}^+ = 2A_1^+, \qquad C_{1,2}^+ = 2C_1^+, \\ B_{1,2}^+ = \mathbb{Z}_4^+(0,1,2) \setminus \mathbb{Z}_4^+(0,2). \end{gathered} \tag{86}$$

Corollary 41. *The midpoint-free subset $2S_1 \subset \mathbb{Z}$ has balance point sets satisfying*

$$A_{1,2} = A_{1,2}^+ \cup 2\mathbb{Z}^-, \qquad B_{1,2} = B_{1,2}^+, \qquad C_{1,2} = C_{1,2}^+. \tag{87}$$

Corollary 42. *The subset $S_1^{(2)} := 2S_1 \cup -(2S_1 + 1) \subset \mathbb{Z}$ is midpoint-free and has balance point sets satisfying*

$$\begin{gathered} A_{1,2}^{(2)} = A_{1,2} \cup -(C_{1,2} + 1), \\ C_{1,2}^{(2)} = C_{1,2} \cup -(A_{1,2} + 1), \\ B_{1,2}^{(2)} = B_{1,2}^+ \cup -(B_{1,2}^+ + 1). \end{gathered} \tag{88}$$

Corollary 43. *The sets $2S_1$ and $S_1^{(2)} := 2S_1 \cup -(2S_1 + 1)$ are midpoint-free subsets of $\mathbb{Z}$ with endpoint sets satisfying $A_{1,2} \cup C_{1,2} = 2\mathbb{Z} \setminus 2S_1$ and $A_{1,2}^{(2)} \cup C_{1,2}^{(2)} = \mathbb{Z} \setminus S_1^{(2)}$.*

The balance point sets of $S_1^{(3)} := 3S_1 \cup (3S_1+1)$ can now be considered. Let

$$\begin{gathered} A_{1,3}^{+} := A\left(S_1^{(3)}, \mathbb{Z}^{+}\right), \qquad B_{1,3}^{+} := B\left(S_1^{(3)}, \mathbb{Z}^{+}\right), \\ C_{1,3}^{+} := C\left(S_1^{(3)}, \mathbb{Z}^{+}\right), \\ A_{1,3} := A\left(S_1^{(3)}, \mathbb{Z}\right), \qquad B_{1,3} := B\left(S_1^{(3)}, \mathbb{Z}\right), \\ C_{1,3} := C\left(S_1^{(3)}, \mathbb{Z}\right). \end{gathered} \tag{89}$$

If $a+x=2b$ and $a,b \in S_1 := \mathbb{Z}_4^{+}(0,1)$, the conditions

$$3a+r, 3b+s \in S_1^{(3)} = 3S_1 \cup (3S_1+1) \tag{90}$$

and $(3a+r)+(3x+t) = 2(3b+s)$ are satisfied by the triples $(r,s,t) = (0,0,0), (1,1,1), (1,0,-1), (0,1,2)$. Hence, if $x \in A_1^{+} \cup C_1^{+}$ then $3x + \{-1,0,1,2\} \subset A_{1,3}^{+} \cup C_{1,3}^{+}$. Such observations yield the following results.

Theorem 44. *The subset $S_1^{(3)} := 3S_1 \cup (3S_1+1) \subset \mathbb{Z}^{+}$ is midpoint-free and has balance point sets satisfying*

$$\begin{gathered} A_{1,3}^{+} \cup C_{1,3}^{+} = \mathbb{Z}^{+} \setminus S_1^{(3)}, \\ B_{1,3}^{+} = 3B_1^{+} \cup (3B_1^{+}+1) \cup (3B_1^{\sim}-1), \end{gathered} \tag{91}$$

where $B_1^{\sim} := (B_1^{+} \cup \{1\}) \setminus 2\mathbb{Z}^{+}$.

Corollary 45. *The subset $S_1^{(3)} := 3S_1 \cup (3S_1+1) \subset \mathbb{Z}$ is midpoint-free and has balance point sets satisfying*

$$A_{1,3} = A_{1,3}^{+} \cup \mathbb{Z}^{-}, \qquad B_{1,3} = B_{1,3}^{+}, \qquad C_{1,3} = C_{1,3}^{+}. \tag{92}$$

9. The Midpoint-Free Set $\mathbb{Z}_7^{+}(0,1,3)$

Next consider the set

$$S_2 := \mathbb{Z}_7^{+}(0,1,3) = \{0,1,3,7,8,10,21,22,24,49,\ldots\}. \tag{93}$$

At the time of writing, no corresponding sequence appears in OEIS [5]. However, Corollary 10 asserts that it is midpoint-free, and it will be shown that S_2 is in fact a maximal midpoint-free subset of $\mathbb{Z}$. It will follow from Theorems 4 and 6 that

$$S_2^{(2)} := 2S_1 \cup -(2S_1+1), \qquad S_2^{(3)} := 3S_2 \cup (3S_2+1) \tag{94}$$

are maximal midpoint-free subsets of $\mathbb{Z}$.

The methods used in earlier sections are again applicable, so fewer details are now required. Following earlier practice, let

$$\begin{gathered} A_2^{+} := A(S_2, \mathbb{Z}^{+}), \qquad B_2^{+} := B(S_2, \mathbb{Z}^{+}), \\ C_2^{+} := C(S_2, \mathbb{Z}^{+}), \\ A_2 := A(S_2, \mathbb{Z}), \qquad B_2 := B(S_2, \mathbb{Z}), \\ C_2 := C(S_2, \mathbb{Z}). \end{gathered} \tag{95}$$

Theorem 46. *The midpoint-free subset $S_2 := \mathbb{Z}_7^{+}(0,1,3) \subset \mathbb{Z}^{+}$ has endpoint sets satisfying $A_2^{+} \cup C_2^{+} = \mathbb{Z}^{+} \setminus S_2$.*

Proof. Given $x \in \mathbb{Z}^{+} \setminus S_2$, we seek $y, b \in S_2$ such that $x+y=2b$. Any solution has $x \neq y$. Base 7 computation requires

$$[\![x]\!]_{7,i} + [\![y]\!]_{7,i} + \delta_i = 2[\![b]\!]_{7,i} + 7\delta_{i+1} \tag{96}$$

for $i \geq 0$, with appropriate carry-overs $\delta_i \in \{0,1\}$ beginning with $\delta_0 = 0$. All base 7 digits of y, b are determined by

$$\begin{gathered} [\![x]\!]_{7,i} + \delta_i \in \{0,7\} \Longrightarrow [\![y]\!]_{7,i} = [\![b]\!]_{7,i} = 0, \\ [\![x]\!]_{7,i} + \delta_i = 1 \Longrightarrow [\![y]\!]_{7,i} = [\![b]\!]_{7,i} = 1, \\ [\![x]\!]_{7,i} + \delta_i = 2 \Longrightarrow [\![y]\!]_{7,i} = 0, \qquad [\![b]\!]_{7,i} = 1, \\ [\![x]\!]_{7,i} + \delta_i = 3 \Longrightarrow [\![y]\!]_{7,i} = [\![b]\!]_{7,i} = 3, \\ [\![x]\!]_{7,i} + \delta_i = 4 \Longrightarrow [\![y]\!]_{7,i} = 3, \qquad [\![b]\!]_{7,i} = 0, \\ [\![x]\!]_{7,i} + \delta_i = 5 \Longrightarrow [\![y]\!]_{7,i} = 1, \qquad [\![b]\!]_{7,i} = 3, \\ [\![x]\!]_{7,i} + \delta_i = 6 \Longrightarrow [\![y]\!]_{7,i} = 0, \qquad [\![b]\!]_{7,i} = 3. \end{gathered} \tag{97}$$

It is straightforward to verify that all requirements are satisfied. □

Corollary 47. *The set $\mathbb{Z}_7^{+}(0,1,3)$ is a maximal midpoint-free subset of $\mathbb{Z}^{+}$.*

Corollary 48. *For $S_2 := \mathbb{Z}_7^{+}(0,1,3)$, any $x \in \mathbb{Z}^{+} \setminus S_2$ has endpoint multiplicity*

$$\alpha(x, S_2) + \gamma(x, S_2) = 1. \tag{98}$$

Corollary 49. *The midpoint set B_2^{+} for $S_2 := \mathbb{Z}_7^{+}(0,1,3)$ in $\mathbb{Z}^{+}$ comprises those integers $x \in \mathbb{Z}^{+} \setminus S_2$ with $2x \in \mathbb{Z}_7^{+}([0,6] \setminus \{5\})$. Any such x has midpoint multiplicity $\beta(x, S_2) = 2^{k-1}$, where $k > 0$ is the number of digits in $(2x)_7$ belonging to $\{1,3,4\}$.*

Proof. Given $x \in \mathbb{Z}^{+} \setminus S_2$, we seek $a, c \in S_2$ such that $a + c = 2x$. Necessarily, any solution has $a \neq x$, so $a \neq c$ holds. Base 7 arithmetic requires

$$[\![a]\!]_{7,i} + [\![c]\!]_{7,i} = [\![2x]\!]_{7,i} \tag{99}$$

for every $i \geq 0$. This forces $[\![2x]\!]_{7,i} \in [0,6] \setminus \{5\}$. Since $x \notin S_2$ there is an integer $j \geq 0$ such that $[\![2x]\!]_{7,j} \notin \{0,2,6\}$ but $[\![2x]\!]_{7,i} \in \{0,2,6\}$ for all $i > j$. Specify a, c by

$$\begin{gathered} [\![2x]\!]_{7,i} \in \{0,2,6\} \Longrightarrow [\![a]\!]_{7,i} = [\![c]\!]_{7,i} = \frac{1}{2}[\![2x]\!]_{7,i} \\ [\![2x]\!]_{7,i} = 1 \Longrightarrow \left\{[\![a]\!]_{7,i}, [\![c]\!]_{7,i}\right\} = \{0,1\}, \\ [\![2x]\!]_{7,i} = 3 \Longrightarrow \left\{[\![a]\!]_{7,i}, [\![c]\!]_{7,i}\right\} = \{0,3\}, \\ [\![2x]\!]_{7,i} = 4 \Longrightarrow \left\{[\![a]\!]_{7,i}, [\![c]\!]_{7,i}\right\} = \{1,3\}. \end{gathered} \tag{100}$$

With $[\![a]\!]_{4,j} < [\![c]\!]_{4,j}$, evidently all solutions satisfy $a + c = 2x$ and $a < c$. □

The base 7 representation of the integer $u_{7,n} := (7^n - 1)/6$ is a block of n digits, all equal to 1. Thus $u_{7,n}$ is a base 7 rep-unit [6]. (See A023000 in OEIS [5].) For any $x \in \mathbb{Z}^+$ let $x|_{7,k}$ denote the integer y resulting from $(x)_7$ by deleting all but the last k base 7 digits:

$$\begin{aligned}[\![y]\!]_{7,i} &= 0 \quad \text{for } i \geq k,\\ [\![y]\!]_{7,i} &= [\![x]\!]_{7,i} \quad \text{for } k > i \geq 0.\end{aligned} \tag{101}$$

An alternative characterization of the midpoint set B_2^+ is that it comprises every $x \in \mathbb{Z}^+$ with $[\![x]\!]_{7,i} \in \{2,4,5,6\}$ for at least one $i \geq 0$ and

$$\begin{aligned}[\![x]\!]_{7,k} = 2 &\Longrightarrow x|_{7,k} \leq 3u_{7,k},\\ [\![x]\!]_{7,k} = 6 &\Longrightarrow x|_{7,k} > 3u_{7,k}.\end{aligned} \tag{102}$$

Corollary 50. *The midpoint-free subset $S_2 := \mathbb{Z}_7^+(0,1,3) \subset \mathbb{Z}$ has endpoint sets satisfying $A_2 = A_2^+ \cup \mathbb{Z}^-$, $B_2 = B_2^+$, $C_2 = C_2^+$.*

Proof. Given $-x \in \mathbb{Z}^-$, we seek $b, c \in S_2$ such that $x + 2b = c$. Clearly $-x \neq c$. Base 7 computation requires

$$[\![x]\!]_{7,i} + 2[\![b]\!]_{7,i} + \delta_i = [\![c]\!]_{7,i} + 7\delta_{i+1} \tag{103}$$

for $i \geq 0$, with appropriate carry-overs $\delta_i \in \{0,1\}$ beginning with $\delta_0 = 0$. All base 7 digits of b, c are determined by

$$\begin{gathered}[\![x]\!]_{7,i} + \delta_i \in \{0,7\} \Longrightarrow [\![b]\!]_{7,i} = [\![c]\!]_{7,i} = 0,\\ [\![x]\!]_{7,i} + \delta_i = 1 \Longrightarrow \left([\![b]\!]_{7,i}, [\![c]\!]_{7,i}\right) \in \{(0,1),(1,3),(3,0)\},\\ [\![x]\!]_{7,i} + \delta_i = 2 \Longrightarrow [\![b]\!]_{7,i} = 3, \qquad [\![c]\!]_{7,i} = 1,\\ [\![x]\!]_{7,i} + \delta_i = 3 \Longrightarrow [\![b]\!]_{7,i} = 0, \qquad [\![c]\!]_{7,i} = 3,\\ [\![x]\!]_{7,i} + \delta_i = 4 \Longrightarrow [\![b]\!]_{7,i} = [\![c]\!]_{7,i} = 3,\\ [\![x]\!]_{7,i} + \delta_i = 5 \Longrightarrow [\![b]\!]_{7,i} = 1, \qquad [\![c]\!]_{7,i} = 0,\\ [\![x]\!]_{7,i} + \delta_i = 6 \Longrightarrow [\![b]\!]_{7,i} = [\![c]\!]_{7,i} = 1.\end{gathered} \tag{104}$$

Let x have leading digit in position $h \geq 0$. If $[\![b]\!]_{7,h} < [\![c]\!]_{7,h}$ then $\delta_{h+1} = 0$ so $b < c$, since b, c have leading digits in position h. All requirements are satisfied.

Now suppose that $[\![b]\!]_{7,h} \geq [\![c]\!]_{7,h}$. Then in every case $\delta_{h+1} = 1$, and

$$\left([\![b]\!]_{7,h+1}, [\![c]\!]_{7,h+1}\right) \in \{(0,1),(1,3),(3,0)\}. \tag{105}$$

The first two options here ensure that $\delta_{h+2} = 0$ and b, c have leading digits in position $h+1$, and $b < c$. However, $[\![b]\!]_{7,h+1} > [\![c]\!]_{7,h+1}$ holds if the third option is chosen, and then $\delta_{h+2} = 1$, so

$$\left([\![b]\!]_{7,h+2}, [\![c]\!]_{7,h+2}\right) \in \{(0,1),(1,3),(3,0)\}. \tag{106}$$

This behaviour can be iterated any finite number of times but must terminate at some stage in order to determine integers b, c. We may choose any integer $k > 0$ and assign

$$\left([\![b]\!]_{7,h+k}, [\![c]\!]_{7,h+k}\right) \in \{(0,1),(1,3)\}, \tag{107}$$

with $[\![b]\!]_{7,h+i} = 3$, $[\![c]\!]_{7,h+i} = 0$ if $0 < i < k$. The leading digits of b, c are in position $h + k$, so $b < c$. Once again, all requirements are satisfied. □

Corollary 51. *The set $\mathbb{Z}_7^+(0,1,3)$ is a maximal midpoint-free subset of $\mathbb{Z}$.*

The proof of Corollary 50 implies a surprising result.

Corollary 52. *Let $[\![x]\!]_{7,h} \in \{4,5,6\}$ be the leading base 7 digit of $x \in \mathbb{Z}^+$. Then $-x$ is a lower endpoint for the midpoint-free subset $S_2 := \mathbb{Z}_7^+(0,1,3) \subset \mathbb{Z}$ with multiplicity $\alpha(-x, S_2) = \aleph_0$.*

Proof. The condition $[\![x]\!]_{7,h} \in \{4,5,6\}$ necessitates $[\![b]\!]_{7,h} \geq [\![c]\!]_{7,h}$. Let $b, c \in S_2$ be a solution to $x + 2b = c$ with leading digits $[\![b]\!]_{7,h+1} = 1$, $[\![c]\!]_{7,h+1} = 3$. For $g > h$, replace the leading 1 of $(b)_7$ by a block comprising a leading digit 1 followed by $g - h$ digits all equal to 3. This yields a new solution with the leading 3 of $(c)_7$ replaced by a block comprising a leading digit 3 followed by $g - h$ digits all equal to 0. Using base 7 rep-units, this yields

$$\begin{aligned}&x + 2\left(b - 7^{h+1} + 3\left(u_{7,g+1} - u_{7,h+1}\right) + 7^{g+1}\right)\\ &\quad = c - 3 \cdot 7^{h+1} + 3 \cdot 7^{g+1},\end{aligned} \tag{108}$$

since $6(u_{7,g+1} - u_{7,h+1}) = 7^{g+1} - 7^{h+1}$. Thus,

$$\left(-x, b + 9\left(u_{7,g+1} - u_{7,h+1}\right), c + 18\left(u_{7,g+1} - u_{7,h+1}\right)\right) \in \Lambda(\mathbb{Z}) \tag{109}$$

for every $g > h$, so there are infinitely many triples in $\Lambda(\mathbb{Z})$ having $-x$ as lower endpoint, with midpoint and upper endpoint in S_2. □

10. Closing Remarks

Remark 1. We have seen that the midpoint-free set $S_5 := \mathbb{Z}_5^+(0,1)$ is not maximal in $\mathbb{Z}^+$. The smallest member of $E(S_5, \mathbb{Z}^+)$ is 8, confirming Corollary 15 when $m = 5$. If $S_5 \subset T \subset \mathbb{Z}^+$ and T is a maximal midpoint-free subset of $\mathbb{Z}^+$ then

$$T \setminus S_5 \subset E\left(S_5, \mathbb{Z}^+\right). \tag{110}$$

This raises some intriguing open questions. For which subsets $X \subset E(S_5, \mathbb{Z}^+)$ is $S_5 \cup X$ a maximal midpoint-free subset of $\mathbb{Z}^+$? What is the greedy subset $S_5^* \subset E(S_5, \mathbb{Z}^+)$ which makes $S_5 \cup S_5^*$ a maximal midpoint-free subset of $\mathbb{Z}^+$?

Note that $S_5 \cup \{x\}$ is midpoint-free for any $x \in E(S_5, \mathbb{Z}^+)$, but $S_5 \cup \{x, y\}$ is not always midpoint-free if $x, y \in E(S_5, \mathbb{Z}^+)$. For instance, the midpoint triple $(8, 25, 42)$ comprises $25 \in S_5$ and $8, 42 \in E(S_5, \mathbb{Z}^+)$.

Remark 2. Consider the balance points of $S_5 := \mathbb{Z}_5^+(0,1)$. Using notation defined after the proof of Corollary 49, along with base 5 rep-units $u_{5,k}$ (see A003463 in OEIS [5]), for

$x \in \mathbb{Z}^+ \setminus S_5$ we have $x \in A(S_5, \mathbb{Z}^+) \cup C(S_5, \mathbb{Z}^+)$ precisely when

$$[\![x]\!]_{5,k} = 2 \Longrightarrow x|_{5,k} \le 3u_{5,k},$$
$$[\![x]\!]_{5,k} = 3 \Longrightarrow x|_{5,k} > 3u_{5,k}. \tag{111}$$

Hence, in particular, $\mathbb{Z}_5^+(0,1,4) \setminus S_5 \subset A(S_5, \mathbb{Z}^+) \cup C(S_5, \mathbb{Z}^+)$. Similarly, if $x \in \mathbb{Z}^+ \setminus S_5$ then $x \in B(S_5, \mathbb{Z}^+)$ precisely when $[\![x]\!]_{5,i} \ne 4$ for all $i \ge 0$, and

$$[\![x]\!]_{5,k} = 1 \Longrightarrow x|_{5,k} \le 2u_{5,k},$$
$$[\![x]\!]_{5,k} = 2 \Longrightarrow x|_{5,k} > 2u_{5,k}. \tag{112}$$

In particular, $\mathbb{Z}_5^+(0,3) \setminus \{0\} \subset B(S_5, \mathbb{Z}^+)$.

If $x \in \mathbb{Z}^+ \setminus S_5$ is eccentric for S_5, the digit configuration $(x)_5$ violates each of these conditions. If $x < 5^h$ then $(x + 5^h y)_5$ contains the same digit configuration for every $y \in \mathbb{Z}^+$, so $x + 5^h y \in E(S_5, \mathbb{Z}^+)$. Every eccentric point x is the lower endpoint of infinitely many midpoint triples

$$\left(x, x + 5^h y, x + 2 \cdot 5^h y\right) \in \Lambda\left(E\left(S_5, \mathbb{Z}^+\right)\right), \tag{113}$$

so $E(S_5, \mathbb{Z}^+)$ is densely packed with midpoint triples.

Remark 3. The set $T_5 := \mathbb{Z}_5^+(0,1,3)$ is not midpoint-free, since $(0,3,6) \in \Lambda(T_5)$. The digit 3 is responsible for "most" members of T_5 being midpoints, since it can be shown that $B(T_5, \mathbb{Z}^+) = \mathbb{Z}^+ \setminus S_5$. Moreover, for any $x \in \mathbb{Z}^+ \setminus S_5$ there is at least one triple $(a, x, c) \in \Lambda(\mathbb{Z}^+)$ with $a \in S_5$ and $c \in T_5$.

It can be shown that $A(T_5, \mathbb{Z}^+) \cup C(T_5, \mathbb{Z}^+) = \mathbb{Z}^+$ and $\mathbb{Z}^- \subset A(T_5, \mathbb{Z})$, so no integer is eccentric for T_5. Hence there are maximal midpoint-free subsets $T \subset \mathbb{Z}$ such that $S_5 \subset T \subset T_5$. What is the greedy subset $T_5^* \subset T_5 \setminus S_5$ which makes $S_5 \cup T_5^*$ a maximal midpoint-free subset of $\mathbb{Z}$?

Note also that if $(x, b, c) \in \Lambda(\mathbb{Z})$ and $b, c \in T_5$, then

$$\left(x, b + 3 \cdot 5^h, c + 6 \cdot 5^h\right) \in \Lambda(\mathbb{Z}) \tag{114}$$

and $b + 3 \cdot 5^h, c + 6 \cdot 5^h \in T_5$ for every sufficiently large $h \in \mathbb{Z}^+$. It follows that every $x \in \mathbb{Z}$ has lower endpoint multiplicity $\alpha(x, T_5) = \aleph_0$.

Remark 4. The set $T_6 := \mathbb{Z}_6^+(0,1,3)$ is not midpoint-free. Once again, the digit 3 is responsible for "most" members of T_6 being midpoints, since it can be shown that $T_6 \setminus S_6 \subset B(T_6, \mathbb{Z}^+)$, where $S_6 := \mathbb{Z}_6^+(0,1)$. However, $E(T_6, \mathbb{Z}^+) \ne \emptyset$. For example, it is easily verified that $\{x \in \mathbb{Z}^+ \mid [\![x]\!]_{6,0} = 4, [\![x]\!]_{6,1} = 2\} \subset E(T_6, \mathbb{Z}^+)$. Thus there is no maximal midpoint-free subset $T \subset \mathbb{Z}$ that satisfies $S_6 \subset T \subseteq T_6$. What is the greedy subset $S_6^* \subset E(S_6, \mathbb{Z}^+)$ which makes $S_6 \cup S_6^*$ a maximal midpoint-free subset of $\mathbb{Z}^+$?

Remark 5. The midpoint-free set $S_8 := \mathbb{Z}_8^+(0,1,3)$ is not maximal in $\mathbb{Z}^+$. Indeed, confirming Corollary 16 when $m = 8$, the smallest member of $E(S_8, \mathbb{Z}^+)$ is 20. In fact,

$$\left\{x \in \mathbb{Z}^+ \; [\![x]\!]_{8,0} = 4, [\![x]\!]_{8,1} = 2\right\} \subset E\left(S_8, \mathbb{Z}^+\right). \tag{115}$$

For $x \in \mathbb{Z}^+ \setminus S_8$ it can be shown that $x \in A(S_8, \mathbb{Z}^+) \cup C(S_8, \mathbb{Z}^+)$ precisely when

$$[\![x]\!]_{8,k} = 3 \Longrightarrow x|_{8,k} \le 4u_{8,k},$$
$$[\![x]\!]_{8,k} = 4 \Longrightarrow x|_{8,k} > 4u_{8,k} \tag{116}$$

so, in particular, $\mathbb{Z}_8^+([0,3]) \setminus S_8 \subset A(S_8, \mathbb{Z}^+) \cup C(S_8, \mathbb{Z}^+)$. (For base 8 rep-units $u_{8,k}$, see A023001 in OEIS [5].) Again, if $x \in \mathbb{Z}^+ \setminus S_8$ then $x \in B(S_8, \mathbb{Z}^+)$ precisely when

$$[\![x]\!]_{8,k} \in \{2,3,6,7\} \Longrightarrow x|_{8,k} \le 3u_{8,k} \tag{117}$$

so, in particular, $\mathbb{Z}_8^+([0,3]) \setminus S_8 \subset B(S_8, \mathbb{Z}^+)$. What is the greedy subset $S_8^* \subset E(S_8, \mathbb{Z}^+)$ which makes $S_8 \cup S_8^*$ a maximal midpoint-free subset of $\mathbb{Z}^+$?

Remark 6. The set $S_9 := \mathbb{Z}_9^+(0,1,3,4)$ is a maximal midpoint-free subset of $\mathbb{Z}^+$ since $\mathbb{Z}_9^+(0,1,3,4) = \mathbb{Z}_3^+(0,1)$. This follows immediately from the observation that

$$(3r + s)\, 9^n = r3^{2n+1} + s3^{2n} \quad \text{when } r, s \in \{0,1\},\ n \ge 0. \tag{118}$$

More generally, $\mathbb{Z}_m^+(D) = \mathbb{Z}_3^+(0,1)$ when $m = 3^n$ and $D = S_0 \cap [0, u_{3,n}]$ for any positive integer n.

Remark 7. The midpoint-free sets $S_m := \mathbb{Z}_m^+(0,1,3,4)$ are maximal in $\mathbb{Z}^+$ for each $m \in [9,13]$, and the endpoint sets satisfy $A(S_m, \mathbb{Z}^+) \cup C(S_m, \mathbb{Z}^+) = \mathbb{Z}^+ \setminus S_m$. Thus $S_m^{(2)} := 2S_m \cup -(2S_m + 1)$ and $S_m^{(3)} := 3S_m \cup (3S_m + 1)$ are maximal midpoint-free subsets of $\mathbb{Z}$ when $m \in [9,13]$, by Theorems 4 and 6. However, the midpoint-free set S_{14} is not maximal in $\mathbb{Z}^+$ since, confirming Corollary 17 when $m = 14$, the smallest member of $E(S_{14}, \mathbb{Z}^+)$ is 65. What is the greedy subset $S_{14}^* \subset E(S_{14}, \mathbb{Z}^+)$ which makes $S_{14} \cup S_{14}^*$ a maximal midpoint-free subset of $\mathbb{Z}^+$?

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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