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Research Article

Cyclicity of Special Operators on a BK with AK Space

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Let Ω be a complex domain and let F be a reflexive BK space with AK such that $\widehat{F} \subset H(\Omega)$ and the functional of evaluation at λ is bounded for all $\lambda \in \Omega$. We will investigate the cyclicity for the adjoint of a weighted composition operator acting on $\widehat{F}$.

1. Introduction

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ϕ denote the set of all finite sequences. By $e^{(n)}$, we denote the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ whenever $k \neq n$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section. Given any subset F of ω, we write $\widehat{F}$ for the set of all formal power series $\widehat{f}$ with $\widehat{f}(z) = \sum_{k=0}^{\infty} f_k z^k$ where $f = (f_k)_{k=0}^{\infty} \in F$, regardless of whether or not the series converges for any value of z. If $\widehat{F}$ endowed with the norm of F, then F and $\widehat{F}$ are norm isomorphic. Let $\widehat{M}_z : \widehat{F} \to \widehat{\omega}$ be defined by $(\widehat{M}_z \widehat{f}) = \sum_{k=0}^{\infty} f_k z^{k+1}$, so the corresponding shift operator $M : F \to \omega$ is defined by $(Mf)_n = f_{n-1}$ if $n \geq 1$ and 0 else.

A BK space is a Banach sequence space with the property that convergence implies coordinatewise convergence. A BK space F containing ϕ is said to have AK if every sequence $f = (f_k)_{k=0}^{\infty} \in F$ has a unique representation $f = \sum_{k=0}^{\infty} f_k e^{(k)}$; that is, $f = \lim_{n\to\infty} f^{[n]}$; it is said to have AD, if ϕ is dense in F. Given any subset F of ω, the set

$$F^{\beta} = \left\{ a \in \omega : \sum_{k=0}^{\infty} a_k f_k \text{ converges for all } f \in F \right\} \quad (1)$$

is called the β-dual of F.

If λ is a complex number, then $e(\lambda)$ denotes the functional of evaluation at λ, defined on the polynomials p by $e(\lambda)(\widehat{p}) = \widehat{p}(\lambda)$. A point λ is said to be a bounded point evaluation on $\widehat{F}$ if the functional $e(\lambda)$ extends to a continuous linear functional on $\widehat{F}$. Finally, we consider the multiplication of formal power series $\widehat{h} = \widehat{f}\widehat{g}$ given by

$$\widehat{h}(z) = \sum_{k=0}^{\infty} h_k z^k = \left(\sum_{k=0}^{\infty} f_k z^k \right) \left(\sum_{k=0}^{\infty} g_k z^k \right), \quad (2)$$

where $h_k = \sum_{j=0}^{k} f_j g_{k-j}$ for all integers $k \geq 0$. If $\widehat{f} \in \widehat{F}$ and $\widehat{p}$ is a polynomial, then to the vector $\widehat{p}(M_z)\widehat{f}$ the formal power series $\widehat{p}(z)\widehat{f}(z)$ corresponds.

Let Ω be a complex domain and let $\widehat{F}$ be a Banach space of formal power series with coefficients in a reflexive BK space with AK such that $\widehat{F} \subset H(\Omega)$. It is convenient and helpful to introduce the notation $\langle x, x^* \rangle$ to stand for $x^*(x)$, for $x \in \widehat{F}$ and $x^* \in (\widehat{F})^*$. We assume $1 \in \widehat{F}$ and the operators $\widehat{M}_z$ and the functional $e(\lambda)$ of evaluation at λ ($\lambda \in \Omega$) are bounded on $\widehat{F}$.

A complex valued function φ on Ω for which $\varphi \widehat{f} \in \widehat{F}$ for every $\widehat{f} \in \widehat{F}$ is called a multiplier of $\widehat{F}$ and the collection of all of these multipliers is denoted by $\mathscr{M}(\widehat{F})$. If $\widehat{M}_z$ is a bounded operator on $\widehat{F}$, the adjoint $(\widehat{M}_z)^* : (\widehat{F})^* \to (\widehat{F})^*$ satisfies $(\widehat{M}_z)^* e(\lambda) = \lambda e(\lambda)$. In general each multiplier φ of $\widehat{F}$ determines a multiplication operator $\widehat{M}_\varphi$ defined by $\widehat{M}_\varphi \widehat{f} = \varphi \widehat{f}$, $\widehat{f} \in \widehat{F}$. Also $(\widehat{M}_\varphi)^* e(\lambda) = \varphi(\lambda) e(\lambda)$. It is well-known that each multiplier is a bounded analytic function on Ω. Indeed $|\varphi(\lambda)| \leq \|\widehat{M}_\varphi\|$ for each λ in Ω.

Let φ be an analytic self-map of the open unit disk U. A composition operator C_φ maps an analytic function $\widehat{f} \in \widehat{F}$ into $(C_\varphi \widehat{f})(z) = \widehat{f}(\varphi(z))$. If $w \in \mathscr{M}(\widehat{F})$ and C_φ is bounded,

M_wC_φ is called a weighted composition operator on $\mathscr{M}(\widehat{F})$. By φ_n we will mean the nth iterate of φ.

Let X be a Banach space. We denote by $B(X)$ the set of bounded operators on the Banach space X. Let $A \in B(X)$ and $x \in X$. We say that x is a cyclic vector of A if X is equal to the closed linear span of the set $\{A^n x : n = 0, 1, 2, \ldots\}$. An operator $A \in B(X)$ is called cyclic if it has a cyclic vector. In this paper, we investigate the cyclicity of weighted composition operators on some BK spaces with AK.

2. Main Results

The sequence spaces has been the focus of attention for several decades and many properties of operators on these spaces have been studied (e.g., [1]). For $p = (p_k)_k^\infty$, a sequence with $p_k > 0$ for all k, Malkowsky considered the space $bv(p) = \{x \in \omega : \sum_{k=0}^\infty |x_k - x_{k-1}|^{p_k} < \infty\}$ and studied its β-dual and characterized some linear operators on $bv(p)$ [2]. In [3] the reflexivity of Λ-summable sequences from a Banach space is investigated whenever Λ is a Banach perfect sequence space. Some BK spaces including the spaces c_0 and c have been introduced in [4] and also their duals have been computed. In [5], Aydin and Basar have introduced new classes of sequence spaces which include the spaces l_p and l_∞, and the characterization of some other classes of sequence spaces have also been derived. Malafosse has given some properties and applications of Banach algebras of bounded operators $B(X)$, when X is a BK space [6]. In [7], Mursaleen and Noman introduced some spaces of difference sequences which are the BK spaces of nonabsolute type and proved that these spaces are linearly isomorphic to the spaces c_0 and c, respectively. In [8, 9], Mursaleen and Noman established some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain operators on some BK spaces of weighted means have been investigated. Furthermore, by using the Hausdorff measure of noncompactness, they applied their results to characterize some classes of compact operators on those spaces. In [10] some identities or estimates for the operator norms and the Hausdorff measure of noncompactnesss of certain matrix operators on some BK spaces have been established. In [11], Basarir and Kara have characterized some classes of compact operators on special normed Riesz sequence spaces by using the Hausdorff measure of noncompactness. A characterization of compact operators between certain BK spaces has been given by Malkowsky in [12]. Also, Malkowsky gave general bounded linear operators on special BK spaces that are strongly summable to 0, summable and bounded with index equal to or greater than 1 [13]. In [14], Kirisci gave well-known result related to some properties, dual spaces, and matrix transformations of the sequence space bv and introduced the matrix domain of space bv with arbitrary triangle matrix. The reflexivity of multiplication operators on some BK spaces with AK property has been studied in [15]. Cyclicity of the adjoint of weighted composition operators on Hilbert function spaces, Fock spaces, and weighted Hardy spaces has been studied in [16–18]. In this section we want to study the cyclicity of the adjoint of a weighted composition operator acting on a space of formal power series with coefficients in a BK space with AK property. By the notations $\sigma_p(\widehat{M}_z^*)$ and U we mean the point spectrum of $\widehat{M}_z^*$ and the open unit disc, respectively.

Theorem 1. *Let F be a BK space with AK. A complex number λ is a bounded point evaluation on $\widehat{F}$ if and only if $\lambda \in \sigma_p(\widehat{M}_z^*)$ and if and only if $\{\lambda^n\}_{n=0}^\infty \in F^\beta$.*

Proof. If λ is a bounded point evaluation it is clear that $\lambda \in \sigma_p((\widehat{M}_z)^*)$. Conversely, let $\lambda \in \sigma_p(\widehat{M}_z^*)$ and $g^\lambda = \{g_n^\lambda\}_{n=0}^\infty \in F^\beta$ be a corresponding eigenvector in $(\widehat{F})^*$. For $f \in F$, we have

$$\begin{aligned}\left\langle \widehat{M}_z \widehat{f}, \left(g^\lambda\right)^\wedge \right\rangle &= \left\langle \widehat{f}, \left(\widehat{M}_z\right)^* \left(g^\lambda\right)^\wedge \right\rangle = \left\langle \widehat{f}, \lambda \left(g^\lambda\right)^\wedge \right\rangle \\ &= \lambda \left\langle \widehat{f}, \left(g^\lambda\right)^\wedge \right\rangle.\end{aligned} \tag{3}$$

Hence for all n, we get

$$\begin{aligned}\left\langle \left(e^{(n+1)}\right)^\wedge, \left(g^\lambda\right)^\wedge \right\rangle &= \left\langle \widehat{M}_z \left(e^{(n)}\right)^\wedge, \left(g^\lambda\right)^\wedge \right\rangle \\ &= \lambda^{n+1} \left\langle \left(e^{(0)}\right)^\wedge, \left(g^\lambda\right)^\wedge \right\rangle\end{aligned} \tag{4}$$

and so $g_{n+1}^\lambda = \lambda^{n+1} g_0^\lambda$. Since $g^\lambda \neq 0$ we have $g_0^\lambda \neq 0$. Put $E(\lambda) = (g^\lambda)^\wedge / g_0^\lambda$. Then for all n we have $\langle (e^{(n+1)})^\wedge, E(\lambda)\rangle = \lambda^{n+1}$ and this implies that $\langle \widehat{p}, E(\lambda)\rangle = \widehat{p}(\lambda)$ for all polynomials p. Since polynomials are dense in $\widehat{F}$, λ is a bounded point evaluation and intact $E(\lambda) = e(\lambda)$. Now it is clear that λ is a bounded point evaluation if and only if $\{\lambda^n\}_{n=0}^\infty \in F^\beta$. □

Theorem 2. *Let F be a BK space with AK and AD such that each point of U is a bounded point evaluation on $\widehat{F}$. Then a polynomial $\widehat{p}$ is cyclic for $\widehat{M}_z$ if and only if $\widehat{p}$ vanishes at no point in U.*

Proof. Let $\widehat{p}(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ be such that $\lambda_i \notin U$ for $i = 1, \ldots, m$. Fix $k \in \{1, \ldots, m\}$ and consider $M_k \in (\widehat{F})^*$ satisfying $M_k(\widehat{M}_z)^n (z - \lambda_k) = 0$ for all integers $n \geq 0$. Since $F^* = F^\beta$, there exists $h^{(k)} \in F^\beta$ such that $M_k \widehat{f} = \langle \widehat{f}, \widehat{h}(k)\rangle$ for all $f \in F$. Note that

$$\begin{aligned}M_k \left(\widehat{M}_z\right)^n \left(z - \lambda_k\right) &= M_k \left(z^{n+1} - \lambda_k z^n\right) \\ &= h_{n+1}^{(k)} - \lambda_k h_n^{(k)}\end{aligned} \tag{5}$$

for all integers $n \geq 0$. Since $M_k(\widehat{M}_z)^n(z - \lambda_k) = 0$, we get $h_{n+1}^{(k)} = \lambda_k h_n^{(k)}$ and so $h_{n+1}^{(k)} = \lambda_k^{n+1} h_0^{(k)}$ for all $n \geq 0$. But $\{\lambda_k^n\}_n \notin F^\beta$ and $h^{(k)} \in F^\beta$; hence $h_n^{(k)} = 0$ for all n and so $M_k = 0$. Thus by the Hahn Banach Theorem, $z - \lambda_k$ is cyclic for $k = 1, \ldots, m$ and so $\widehat{p}(z)$ is a cyclic vector for $\widehat{M}_z$. The converse case is clear. □

Theorem 3. *Suppose that F is a BK space with AK and AD, $\mathscr{M}(\widehat{F}) = H^\infty$, φ is an analytic function on U satisfying $\|\varphi\|_U < 1$, and $\widehat{w} \in \mathscr{M}(\widehat{F})$. Also, let $\{e(z) : z \in U\}$ be bounded and there exists $z_0 \in U$ satisfying $\widehat{w}(\varphi_k(z_0)) \neq 0$ for all $k \geq 0$, and* *assume that the set* $\{\varphi_k(z_0) : k \geq 0\}$ *has a limit point in* U. *Then* $e(z_0)$ *is a cyclic vector for the operator* $(\widehat{M}_{\widehat{w}}C_\varphi)^*$ *acting on* $\widehat{F}^\beta$.

Proof. Let the map $L : \mathscr{M}(\widehat{F}) \rightarrow B(\widehat{F})$ be given by $L(\widehat{\psi}) = \widehat{M}_{\widehat{\psi}}$. We prove that L is continuous. For this we use the closed graph theorem. Suppose $\widehat{\psi}_n$ converges to $\widehat{\psi}$ in $\mathscr{M}(\widehat{F})$ and $L(\widehat{\psi}_n) = \widehat{M}_{\widehat{\psi}_n}$ converges to A in $B(\widehat{F})$. Then, for each f in F,

$$A\widehat{f} = \lim_n \widehat{M}_{\widehat{\psi}_n}\widehat{f} = \lim_n \widehat{\psi}_n\widehat{f}. \tag{6}$$

Thus $\{\widehat{\psi}_n\widehat{f}\}_n$ is convergent in $\widehat{F}$. Now by the continuity of point evaluations $\widehat{\psi}_n\widehat{f}$ converges pointwise to $\widehat{\psi}\widehat{f}$ on U. So $A\widehat{f}$ is analytic and agrees with $\widehat{\psi}\widehat{f}$ on U. Hence $A\widehat{f} = \widehat{\psi}\widehat{f}$ and $A = \widehat{M}_{\widehat{\psi}}$. Therefore, L is continuouse and there is a constant c such that $\|\widehat{M}_{\widehat{\psi}}\| \leq c\|\widehat{\psi}\|_U$ for all $\widehat{\psi}$ in $\mathscr{M}(\widehat{F})$. But $\|\widehat{\psi}\| \leq \|\widehat{M}_{\widehat{\psi}}\|$ for all $\widehat{\psi}$ in $\mathscr{M}(\widehat{F})$. Thus $\|\widehat{\psi}\| \leq c\|\widehat{\psi}\|_U$ for all $\psi \in \mathscr{M}(\widehat{F})$. Since $\varphi \in H^\infty$ and $\mathscr{M}(\widehat{F}) = H^\infty$ we will use $\widehat{\varphi}$ instead of φ. Let $f \in F$; then $C_{\widehat{\varphi}}\widehat{f} = \widehat{f}o\widehat{\varphi} \in H^\infty$ since $\|\widehat{\varphi}\|_U < 1$. So

$$\left\|\widehat{f}o\widehat{\varphi}\right\| \leq c\left\|\widehat{f}o\widehat{\varphi}\right\|_U \leq c\left\|\widehat{f}\right\|_U, \tag{7}$$

since $\widehat{\varphi}(U) \subseteq U$. On the other hand, note that, for all f in F, $\|\widehat{f}\|_U \leq \gamma\|\widehat{f}\|$, where $\gamma = \sup\{\|e(z)\| : z \in U\}$. Now we get $\|C_{\widehat{\varphi}}\widehat{f}\| \leq c\gamma\|\widehat{f}\|$, which implies that $C_{\widehat{\varphi}}$ and so $\widehat{M}_{\widehat{w}}C_{\widehat{\varphi}}$ are bounded. Now, put $A = \widehat{M}_{\widehat{w}}C_{\widehat{\varphi}}$. To complete the proof we show that if $\langle\widehat{g}, (A^*)^k e(z_0)\rangle = 0$ for all $k \geq 0$ then $\widehat{g}$ should be the zero constant function. For this note that

$$\left\langle \widehat{g}, (A^*)^k e(z_0)\right\rangle = \left(\prod_{i=0}^{k-1}\overline{\widehat{w}(\widehat{\varphi}_i(z_0))}\right)\widehat{g}(\widehat{\varphi}_k(z_0)). \tag{8}$$

By the assumptions, clearly we get $\widehat{g}(\widehat{\varphi}_k(z_0)) = 0$ for all $k \geq 0$. Since $\{\widehat{\varphi}_k(z_0) : k \geq 0\}$ has limit point in U, it should be $\widehat{g} = 0$. Thus, $e(z_0)$ is a cyclic vector for the operator $(\widehat{M}_{\widehat{w}}C_{\widehat{\varphi}})^*$ acting on $\widehat{F}^\beta$. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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