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Research Article

Some General Systems of Rational Difference Equations

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We investigate behavior of solutions of the following systems of rational difference equations: $x_{n+1} = y_{n-(3k-1)}/(\pm 1 \pm y_{n-(3k-1)}x_{n-(2k-1)}y_{n-(k-1)})$, $y_{n+1} = x_{n-(3k-1)}/(\pm 1 \pm x_{n-(3k-1)}y_{n-(2k-1)}x_{n-(k-1)})$, where k is a positive integer and the initial conditions are real numbers. We show that every solution is periodic with $6k$ period, considerably improving the results in the literature.

1. Introduction

Recently, a great effort has been made in studying qualitative properties of the solutions of systems of rational difference equations.

Çinar [1] studied periodicity of the positive solutions of the system of difference equations

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}. \tag{1}$$

Kurbanli et al. [2] studied the behavior of positive solutions of the following system:

$$x_{n+1} = \frac{x_{n-1}}{1 + y_n x_{n-1}}, \qquad y_{n+1} = \frac{y_{n-1}}{1 + x_n y_{n-1}}. \tag{2}$$

Elsayed [3, 4] obtained the solutions of the following systems of difference equations:

$$x_{n+1} = \frac{1}{y_{n-k}}, \qquad y_{n+1} = \frac{y_{n-k}}{x_n y_n}, \tag{3}$$

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + y_n x_{n-1}}, \qquad y_{n+1} = \frac{y_{n-1}}{\pm 1 + x_n y_{n-1}}. \tag{4}$$

Touafek and Elsayed [5] investigated the periodic nature and form of the solutions of the following systems of rational difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}. \tag{5}$$

Mansour et al. [6] investigated form of the solutions and periodic nature of the following systems of rational difference equation:

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm y_{n-2}x_{n-5}}, \qquad y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm x_{n-2}y_{n-5}}. \tag{6}$$

Özkan and Kurbanli [7] gave the solutions of the systems of the difference equations

$$\begin{aligned} x_{n+1} &= \frac{y_{n-2}}{-1 \pm y_{n-2}x_{n-1}y_n}, \\ y_{n+1} &= \frac{x_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}, \\ z_{n+1} &= \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}, \\ & n = 0, 1, \ldots \end{aligned} \tag{7}$$

Similar systems of rational difference equations were investigated (see, e.g., [8–15]).

Motivated by [7], in this paper we investigate periodic solutions of the following systems of rational difference equations:

$$x_{n+1} = \frac{y_{n-(3k-1)}}{-1 + y_{n-(3k-1)}x_{n-(2k-1)}y_{n-(k-1)}}, \quad y_{n+1} = \frac{x_{n-(3k-1)}}{-1 + x_{n-(3k-1)}y_{n-(2k-1)}x_{n-(k-1)}}, \quad n = 0, 1, \ldots, \tag{8}$$

$$x_{n+1} = \frac{y_{n-(3k-1)}}{-1 - y_{n-(3k-1)}x_{n-(2k-1)}y_{n-(k-1)}}, \quad y_{n+1} = \frac{x_{n-(3k-1)}}{-1 - x_{n-(3k-1)}y_{n-(2k-1)}x_{n-(k-1)}}, \quad n = 0, 1, \ldots, \tag{9}$$

$$x_{n+1} = \frac{y_{n-(3k-1)}}{1 + y_{n-(3k-1)}x_{n-(2k-1)}y_{n-(k-1)}}, \quad y_{n+1} = \frac{x_{n-(3k-1)}}{1 - x_{n-(3k-1)}y_{n-(2k-1)}x_{n-(k-1)}}, \quad n = 0, 1, \ldots, \tag{10}$$

$$x_{n+1} = \frac{y_{n-(3k-1)}}{1 - y_{n-(3k-1)}x_{n-(2k-1)}y_{n-(k-1)}}, \quad y_{n+1} = \frac{x_{n-(3k-1)}}{1 + x_{n-(3k-1)}y_{n-(2k-1)}x_{n-(k-1)}}, \quad n = 0, 1, \ldots, \tag{11}$$

where k is a positive integer and the initial conditions are arbitrary real numbers. We show that every solution is periodic with period $6k$. Furthermore, we give the solutions of some systems explicitly.

2. The Case $k = 1$

In this section, we consider the following systems:

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \ldots, \tag{12}$$

$$x_{n+1} = \frac{y_{n-2}}{-1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \ldots, \tag{13}$$

$$x_{n+1} = \frac{y_{n-2}}{1 + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{1 - x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \ldots, \tag{14}$$

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{1 + x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \ldots \tag{15}$$

Özkan and Kurbanli studied periodic solutions of the systems of the difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \quad z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \quad n \in N_0, \tag{16}$$

$$x_{n+1} = \frac{y_{n-2}}{-1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \quad z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \quad n \in N_0, \tag{17}$$

where the initial conditions are arbitrary real numbers. The following two theorems are proved in [7].

Theorem 1. *Let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$, $z_0 = k$, $z_{-1} = p$, and $z_{-2} = q$ be arbitrary real numbers, and let $\{x_n, y_n, z_n\}$ be a solution of the system (16). Also, assume that $fbd \neq 1$ and $cea \neq 1$. Then all six-period solutions of system (16) are as follows:*

$$x_{6n+1} = -\frac{c}{1 - cea}, \quad y_{6n+1} = \frac{f}{fbd - 1}, \quad z_{6n+1} = \frac{f + c}{fbd - 1}$$

$$x_{6n+2} = b(fbd - 1), \quad y_{6n+2} = e(cea - 1),$$

$$z_{6n+2} = (e + b)(cea - 1)$$

$$x_{6n+3} = -\frac{a}{1 - cea}, \quad y_{6n+3} = \frac{d}{fbd - 1}, \quad z_{6n+3} = \frac{d + a}{fbd - 1}$$

$$x_{6n+4} = f, \quad y_{6n+4} = c, \quad z_{6n+4} = \frac{c(fbd - 1) + f(cea - 1)}{fbd - 1}$$

$$x_{6n+5} = e, \quad y_{6n+5} = b, \quad z_{6n+5} = \frac{b(fbd-1)+e(cea-1)}{fbd-1}$$
$$x_{6n+6} = d, \quad y_{6n+6} = a, \quad z_{6n+6} = \frac{a(fbd-1)+d(cea-1)}{fbd-1}, \quad n = 0, 1, \ldots \tag{18}$$

Theorem 2. *Let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$, $z_0 = k$, $z_{-1} = p$, and $z_{-2} = q$ be arbitrary real numbers, and let $\{x_n, y_n, z_n\}$ be a solution of system (17). Also, assume that $fbd \neq -1$ and $cea \neq -1$. Then all six-period solutions of system (17) are as follows:*

$$x_{6n+1} = -\frac{c}{1+cea}, \quad y_{6n+1} = -\frac{f}{fbd+1}, \quad z_{6n+1} = -\frac{f+c}{fbd+1}$$
$$x_{6n+2} = -b(fbd+1), \quad y_{6n+2} = -e(cea+1),$$
$$z_{6n+2} = -(e+b)(cea+1)$$
$$x_{6n+3} = -\frac{a}{1+cea}, \quad y_{6n+3} = -\frac{d}{fbd+1}, \quad z_{6n+3} = -\frac{d+a}{fbd+1}$$
$$x_{6n+4} = f, \quad y_{6n+4} = c, \quad z_{6n+4} = \frac{c(fbd+1)+f(1+cea)}{fbd+1}$$
$$x_{6n+5} = e, \quad y_{6n+5} = b, \quad z_{6n+5} = \frac{b(1+fbd)+e(1+cea)}{fbd+1}$$
$$x_{6n+6} = d, \quad y_{6n+6} = a, \quad z_{6n+6} = \frac{a(fbd+1)+d(1+cea)}{fbd+1}, \quad n = 0, 1, \ldots \tag{19}$$

These theorems give the solutions of systems (12) and (13). So, we do not show the periodic solutions of systems (12) and (13). In this section, we only give the solutions of systems (14) and (15) explicitly.

Theorem 3. *Let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, and $x_{-2} = f$ be arbitrary real numbers, and let $\{x_n, y_n\}$ be a solution of system (14). Also, assume that $fbd \neq 1$ and $cea \neq -1$. Then all six-period solutions of system (14) are as follows:*

$$\begin{aligned}
&x_{6n+1} = \frac{c}{1+cea}, \qquad y_{6n+1} = \frac{f}{1-fbd},\\
&x_{6n+2} = b(1-fbd), \qquad y_{6n+2} = e(1+cea),\\
&x_{6n+3} = \frac{a}{1+cea}, \qquad y_{6n+3} = \frac{d}{1-fbd},\\
&x_{6n+4} = f, \qquad y_{6n+4} = c,\\
&x_{6n+5} = e, \qquad y_{6n+5} = b,\\
&x_{6n+6} = d, \qquad y_{6n+6} = a,
\end{aligned} \tag{20}$$

for $n = 0, 1, \ldots$.

Proof. We prove the theorem by induction. From system (14), we obtain, immediately by iteration, $x_1 = c/(1+cea)$, $y_1 = f/(1-fbd)$, $x_2 = b(1-fbd)$, $y_2 = e(1+cea)$, $x_3 = a/(1+cea)$, $y_3 = d/(1-fbd)$, $x_4 = f$, $y_4 = c$, $x_5 = e$, $y_5 = b$, $x_6 = d$, and $y_6 = a$. Now, assume that (20) holds for a positive integer N; that is,

$$\begin{aligned}
&x_{6N+1} = \frac{c}{1+cea}, \qquad y_{6N+1} = \frac{f}{1-fbd},\\
&x_{6N+2} = b(1-fbd), \qquad y_{6N+2} = e(1+cea),\\
&x_{6N+3} = \frac{a}{1+cea}, \qquad y_{6N+3} = \frac{d}{1-fbd},\\
&x_{6N+4} = f, \qquad y_{6N+4} = c,\\
&x_{6N+5} = e, \qquad y_{6N+5} = b,\\
&x_{6N+6} = d, \qquad y_{6N+6} = a.
\end{aligned} \tag{21}$$

Then, it follows that

$$\begin{aligned}
&x_{6N+7} = \frac{c}{1+cea}, \qquad y_{6N+7} = \frac{f}{1-fbd},\\
&x_{6N+8} = b(1-fbd), \qquad y_{6N+8} = e(1+cea),\\
&x_{6N+9} = \frac{a}{1+cea}, \qquad y_{6N+9} = \frac{d}{1-fbd},\\
&x_{6N+10} = f, \qquad y_{6N+10} = c,\\
&x_{6N+11} = e, \qquad y_{6N+11} = b,\\
&x_{6N+12} = d, \qquad y_{6N+12} = a.
\end{aligned} \tag{22}$$

So, the proof is finished. □

The following theorem can be proved by induction similarly. So, it will be omitted.

Theorem 4. *Let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, and $x_{-2} = f$ be arbitrary real numbers, and let $\{x_n, y_n\}$ be a solution of system (15). Also, assume that $fbd \neq -1$ and $cea \neq 1$. Then all six-period solutions of system (15) are as follows:*

$$\begin{aligned}
&x_{6n+1} = \frac{c}{1-cea}, \qquad y_{6n+1} = \frac{f}{1+fbd},\\
&x_{6n+2} = b(1+fbd), \qquad y_{6n+2} = e(1-cea),\\
&x_{6n+3} = \frac{a}{1-cea}, \qquad y_{6n+3} = \frac{d}{1+fbd},\\
&x_{6n+4} = f, \qquad y_{6n+4} = c,\\
&x_{6n+5} = e, \qquad y_{6n+5} = b,\\
&x_{6n+6} = d, \qquad y_{6n+6} = a,
\end{aligned} \tag{23}$$

for $n = 0, 1, \ldots$.

3. The Case $k = 2$

In this section, we consider the following systems:

$$x_{n+1} = \frac{y_{n-5}}{-1 + y_{n-5}x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{x_{n-5}}{-1 + x_{n-5}y_{n-3}x_{n-1}}, \quad n = 0, 1, \ldots, \tag{24}$$

$$x_{n+1} = \frac{y_{n-5}}{-1 - y_{n-5}x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{x_{n-5}}{-1 - x_{n-5}y_{n-3}x_{n-1}}, \quad n = 0, 1, \ldots, \tag{25}$$

$$x_{n+1} = \frac{y_{n-5}}{1 + y_{n-5}x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{x_{n-5}}{1 - x_{n-5}y_{n-3}x_{n-1}}, \quad n = 0, 1, \ldots, \tag{26}$$

$$x_{n+1} = \frac{y_{n-5}}{1 - y_{n-5}x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{x_{n-5}}{1 + x_{n-5}y_{n-3}x_{n-1}}, \quad n = 0, 1, \ldots \tag{27}$$

We will consider well-defined solutions. So, we will assume that $y_{-5}x_{-3}y_{-1} \neq 1$, $x_{-5}y_{-3}x_{-1} \neq 1$, $y_{-4}x_{-2}y_0 \neq 1$, and $x_{-4}y_{-2}x_0 \neq 1$ for system (24), $y_{-5}x_{-3}y_{-1} \neq -1$, $x_{-5}y_{-3}x_{-1} \neq -1$, $y_{-4}x_{-2}y_0 \neq -1$, and $x_{-4}y_{-2}x_0 \neq -1$ for system (25), $y_{-5}x_{-3}y_{-1} \neq -1$, $x_{-5}y_{-3}x_{-1} \neq 1$, $y_{-4}x_{-2}y_0 \neq -1$, and $x_{-4}y_{-2}x_0 \neq 1$ for system (26), and $y_{-5}x_{-3}y_{-1} \neq 1$, $x_{-5}y_{-3}x_{-1} \neq -1$, $y_{-4}x_{-2}y_0 \neq 1$, and $x_{-4}y_{-2}x_0 \neq -1$ for system (27). We give solutions of these systems in the following theorem explicitly.

Theorem 5. *Let $\{x_n, y_n\}$ be a solution of system (24), or (25), or (26), or (27). Then, $\{x_n, y_n\}$ is periodic with period 12. Morever, the following are true for $x_{-5} = a$, $x_{-4} = b$, $x_{-3} = c$, $x_{-2} = d$, $x_{-1} = e$, $x_0 = f$, $y_{-5} = k$, $y_{-4} = l$, $y_{-3} = m$, $y_{-2} = n$, $y_{-1} = p$, and $y_0 = r$.*

(i) *If $\{x_n, y_n\}$ is a solution of system (24), then all solutions are as follows:*

$$\begin{aligned}
&x_{12n+1} = \frac{k}{kcp - 1}, \qquad y_{12n+1} = \frac{a}{ame - 1},\\
&x_{12n+2} = \frac{l}{ldr - 1}, \qquad y_{12n+1} = \frac{b}{bnf - 1},\\
&x_{12n+3} = m(ame - 1), \qquad y_{12n+3} = c(kcp - 1),\\
&x_{12n+4} = n(bnf - 1), \qquad y_{12n+4} = d(ldr - 1),\\
&x_{12n+5} = \frac{p}{kcp - 1}, \qquad y_{12n+5} = \frac{e}{ame - 1},\\
&x_{12n+6} = \frac{r}{ldr - 1}, \qquad y_{12n+6} = \frac{f}{bnf - 1},\\
&x_{12n+7} = a, \qquad y_{12n+7} = k,\\
&x_{12n+8} = b, \qquad y_{12n+8} = l,\\
&x_{12n+9} = c, \qquad y_{12n+9} = m,\\
&x_{12n+10} = d, \qquad y_{12n+10} = n,\\
&x_{12n+11} = e, \qquad y_{12n+11} = p,\\
&x_{12n+12} = f, \qquad y_{12n+12} = r,
\end{aligned} \tag{28}$$

for $n \geq 0$.

(ii) *If $\{x_n, y_n\}$ is a solution of system (25), then all solutions are as follows:*

$$\begin{aligned}
&x_{12n+1} = \frac{k}{-1 - kcp}, \qquad y_{12n+1} = \frac{a}{-1 - ame},\\
&x_{12n+2} = \frac{l}{-1 - ldr}, \qquad y_{12n+2} = \frac{b}{-1 - bnf},\\
&x_{12n+3} = m(-1 - ame), \qquad y_{12n+3} = c(-1 - kcp),\\
&x_{12n+4} = n(-1 - bnf), \qquad y_{12n+4} = d(-1 - ldr),\\
&x_{12n+5} = \frac{p}{-1 - kcp}, \qquad y_{12n+5} = \frac{e}{-1 - ame},\\
&x_{12n+6} = \frac{r}{-1 - ldr}, \qquad y_{12n+6} = \frac{f}{-1 - bnf},\\
&x_{12n+7} = a, \qquad y_{12n+7} = k,\\
&x_{12n+8} = b, \qquad y_{12n+8} = l,\\
&x_{12n+9} = c, \qquad y_{12n+9} = m,\\
&x_{12n+10} = d, \qquad y_{12n+10} = n,\\
&x_{12n+11} = e, \qquad y_{12n+11} = p,\\
&x_{12n+12} = f, \qquad y_{12n+12} = r,
\end{aligned} \tag{29}$$

for $n \geq 0$.

(iii) *If $\{x_n, y_n\}$ is a solution of system (26), then all solutions are as follows:*

$$\begin{aligned}
&x_{12n+1} = \frac{k}{1 + kcp}, \qquad y_{12n+1} = \frac{a}{1 - ame},\\
&x_{12n+2} = \frac{l}{1 + ldr}, \qquad y_{12n+1} = \frac{b}{1 - bnf},
\end{aligned}$$

$$
\begin{aligned}
&x_{12n+3} = m(1-ame), \qquad y_{12n+3} = c(1+kcp),\\
&x_{12n+4} = n(1-bnf), \qquad y_{12n+4} = d(1+ldr),\\
&x_{12n+5} = \frac{p}{1+kcp}, \qquad y_{12n+5} = \frac{e}{1-ame},\\
&x_{12n+6} = \frac{r}{1+ldr}, \qquad y_{12n+6} = \frac{f}{1-bnf},\\
&x_{12n+7} = a, \qquad y_{12n+7} = k,\\
&x_{12n+8} = b, \qquad y_{12n+8} = l,\\
&x_{12n+9} = c, \qquad y_{12n+9} = m,\\
&x_{12n+10} = d, \qquad y_{12n+10} = n,\\
&x_{12n+11} = e, \qquad y_{12n+11} = p,\\
&x_{12n+12} = f, \qquad y_{12n+12} = r,
\end{aligned}
\tag{30}
$$

for $n \geq 0$.

(iv) *If* $\{x_n, y_n\}$ *is a solution of system (27), then all solutions are as follows:*

$$
\begin{aligned}
&x_{12n+1} = \frac{k}{1-kcp}, \qquad y_{12n+1} = \frac{a}{1+ame},\\
&x_{12n+2} = \frac{l}{1-ldr}, \qquad y_{12n+1} = \frac{b}{1+bnf},\\
&x_{12n+3} = m(1+ame), \qquad y_{12n+3} = c(1-kcp),\\
&x_{12n+4} = n(1+bnf), \qquad y_{12n+4} = d(1-ldr),\\
&x_{12n+5} = \frac{p}{1-kcp}, \qquad y_{12n+5} = \frac{e}{1+ame},\\
&x_{12n+6} = \frac{r}{1-ldr}, \qquad y_{12n+6} = \frac{f}{1+bnf},\\
&x_{12n+7} = a, \qquad y_{12n+7} = k,\\
&x_{12n+8} = b, \qquad y_{12n+8} = l,\\
&x_{12n+9} = c, \qquad y_{12n+9} = m,\\
&x_{12n+10} = d, \qquad y_{12n+10} = n,\\
&x_{12n+11} = e, \qquad y_{12n+11} = p,\\
&x_{12n+12} = f, \qquad y_{12n+12} = r,
\end{aligned}
\tag{31}
$$

for $n \geq 0$.

Proof. (i) Consider system (14). By iteration we obtain immediately $x_1 = k/(kcp-1)$, $y_1 = a/(ame-1)$, $x_2 = l/(ldr-1)$, $y_2 = b/(bnf-1)$, $x_3 = m(ame-1)$, $y_3 = c(kcp-1)$, $x_4 = n(bnf-1)$, $y_4 = d(ldr-1)$, $x_5 = p/(kcp-1)$, $y_5 = e/(ame-1)$, $x_6 = r/(ldr-1)$, $y_6 = f/(bnf-1)$, $x_7 = a$, $y_7 = k$, $x_8 = b$, $y_8 = l$, $x_9 = c$, $y_9 = m$, $x_{10} = d$, $y_{10} = n$, $x_{11} = e$, $y_{11} = p$, $x_{12} = f$, and $y_{12} = r$. Now, assume that (28) holds for a positive integer N; that is,

$$
\begin{aligned}
&x_{12N+1} = \frac{k}{kcp-1}, \qquad y_{12N+1} = \frac{a}{ame-1},\\
&x_{12N+2} = \frac{l}{ldr-1}, \qquad y_{12N+2} = \frac{b}{bnf-1},\\
&x_{12N+3} = m(ame-1), \qquad y_{12N+3} = c(kcp-1),\\
&x_{12N+4} = n(bnf-1), \qquad y_{12N+4} = d(ldr-1),\\
&x_{12N+5} = \frac{p}{kcp-1}, \qquad y_{12N+5} = \frac{e}{ame-1},\\
&x_{12N+6} = \frac{r}{ldr-1}, \qquad y_{12N+6} = \frac{f}{bnf-1},\\
&x_{12N+7} = a, \qquad y_{12N+7} = k,\\
&x_{12N+8} = b, \qquad y_{12N+8} = l,\\
&x_{12N+9} = c, \qquad y_{12N+9} = m,\\
&x_{12N+10} = d, \qquad y_{12N+10} = n,\\
&x_{12N+11} = e, \qquad y_{12N+11} = p,\\
&x_{12N+12} = f, \qquad y_{12N+12} = r.
\end{aligned}
\tag{32}
$$

Then, it follows that

$$
\begin{aligned}
&x_{12N+13} = \frac{k}{kcp-1}, \qquad y_{12N+13} = \frac{a}{ame-1},\\
&x_{12N+14} = \frac{l}{ldr-1}, \qquad y_{12N+14} = \frac{b}{bnf-1},\\
&x_{12N+15} = m(ame-1), \qquad y_{12N+15} = c(kcp-1),\\
&x_{12N+16} = n(bnf-1), \qquad y_{12N+16} = d(ldr-1),\\
&x_{12N+17} = \frac{p}{kcp-1}, \qquad y_{12N+17} = \frac{e}{ame-1},\\
&x_{12N+18} = \frac{r}{ldr-1}, \qquad y_{12N+18} = \frac{f}{bnf-1},\\
&x_{12N+19} = a, \qquad y_{12N+19} = k,\\
&x_{12N+20} = b, \qquad y_{12N+20} = l,\\
&x_{12N+21} = c, \qquad y_{12N+21} = m,\\
&x_{12N+22} = d, \qquad y_{12N+22} = n,\\
&x_{12N+23} = e, \qquad y_{12N+23} = p,\\
&x_{12N+24} = f, \qquad y_{12N+24} = r.
\end{aligned}
\tag{33}
$$

So, the proof of i is finished. Because of the rest of the proof is similar, it is omitted. □

4. The General Case

In this section, we investigate systems (8), (9), (10), and (11).

We will consider well-defined solutions. So, we will assume that $y_{-3k+j}x_{-2k+j}y_{-k+j} \neq 1$ and $x_{-3k+j}y_{-2k+j}x_{-k+j} \neq 1$, $j = 1, 2, \ldots, k$, for system (8), $y_{-3k+j}x_{-2k+j}y_{-k+j} \neq -1$ and $x_{-3k+j}y_{-2k+j}x_{-k+j} \neq -1$, $j = 1, 2, \ldots, k$, for system (9), $y_{-3k+j}x_{-2k+j}y_{-k+j} \neq -1$ and $x_{-3k+j}y_{-2k+j}x_{-k+j} \neq 1$, $j = 1, 2, \ldots, k$, for system (10), and $y_{-3k+j}x_{-2k+j}y_{-k+j} \neq 1$ and $x_{-3k+j}y_{-2k+j}x_{-k+j} \neq -1$, $j = 1, 2, \ldots, k$, for system (11).

We give the following lemma which is useful to prove the following theorem.

Lemma 6. *Let $\{x_n, y_n\}$ be a solution of system (8), or (9), or (10), or (11) where the initial conditions are real numbers and satisfy well-defined solutions. The followings are true for $n \geq 0$:*

(i) $x_{n+k+1}y_{n+2k+1} = y_{n-(2k-1)}x_{n-(k-1)}$;

(ii) $y_{n+k+1}x_{n+2k+1} = x_{n-(2k-1)}y_{n-(k-1)}$.

Proof. (i) Firstly, we prove (i) for system (8). From system (8), we obtain

$$\begin{aligned}
&x_{n+k+1}y_{n+2k+1} \\
&= \frac{y_{n-(2k-1)}}{-1 + y_{n-(2k-1)}x_{n-(k-1)}y_{n+1}} \cdot \frac{x_{n-(k-1)}}{-1 + x_{n-(k-1)}y_{n+1}x_{n-(-k-1)}} \\
&= y_{n-(2k-1)}x_{n-(k-1)} \cdot \left(\left[-1 + y_{n-(2k-1)}x_{n-(k-1)}y_{n+1}\right] \cdot \left[-1 + x_{n-(k-1)}y_{n+1} \cdot \left(\frac{y_{n-(2k-1)}}{-1 + y_{n-(2k-1)}x_{n-(k-1)}y_{n+1}} \right)\right]\right)^{-1} \\
&= y_{n-(2k-1)}x_{n-(k-1)} \cdot \left(\left[-1 + y_{n-(2k-1)}x_{n-(k-1)}y_{n+1}\right] \cdot \left[\frac{1}{-1 + y_{n-(2k-1)}x_{n-(k-1)}y_{n+1}} \right]\right)^{-1} \\
&= y_{n-(2k-1)}x_{n-(k-1)}
\end{aligned} \tag{34}$$

for $n \geq 0$. Considering other systems (9), (10), and (11) individually, the proof can be obtained similarly.

(ii) The proof of (ii) can be obtained similarly. So, it is omitted. □

Theorem 7. *Let $\{x_n, y_n\}$ be a solution of system (8), or (9), or (10), or (11) which hold Lemma 6. Then, $\{x_n, y_n\}$ is periodic with period $6k$.*

Proof. Suppose that $\{x_n, y_n\}$ holds conditions in Theorem 7. We must show $x_{n+6k} = x_n$ and $y_{n+6k} = y_n$, for $n \geq 0$. Firstly considering system (8), we show $x_{n+6k} = x_n$. From system (8) and Lemma 6(i), we obtain

$$\begin{aligned}
x_{n+6k} &= \frac{y_{n+3k}}{-1 + y_{n+3k}x_{n+4k}y_{n+5k}} \\
&= \frac{x_n}{\left(-1 + y_{n+3k}x_{n+4k}y_{n+5k}\right)\left(-1 + x_n y_{n+k} x_{n+2k}\right)} \\
&= x_n \left(1 - y_{n+3k}x_{n+4k}y_{n+5k} - x_n y_{n+k}x_{n+2k} + y_{n+3k}x_{n+4k}y_{n+5k}x_n y_{n+k}x_{n+2k}\right)^{-1} \\
&= x_n \left(1 - y_{n+3k}x_{n+4k}y_{n+5k} - x_n y_{n+k}x_{n+2k} + y_{n+3k}x_{n+4k}y_{n+5k}\left(\frac{x_n}{y_{n+3k}} + 1\right)\right)^{-1} \\
&= \frac{x_n}{1 + x_{n+4k}y_{n+5k}x_n - x_n y_{n+k}x_{n+2k}} \\
&= \frac{x_n}{1 + x_n\left(x_{n+4k}y_{n+5k} - y_{n+k}x_{n+2k}\right)} \\
&= x_n
\end{aligned} \tag{35}$$

for $n \geq 0$. From system (8) and Lemma 6(ii),

$$\begin{aligned}
y_{n+6k} &= \frac{x_{n+3k}}{-1 + x_{n+3k}y_{n+4k}x_{n+5k}} \\
&= \frac{y_n}{\left(-1 + x_{n+3k}y_{n+4k}x_{n+5k}\right)\left(-1 + y_n x_{n+k} y_{n+2k}\right)} \\
&= y_n \left(1 - y_n x_{n+k}y_{n+2k} - x_{n+3k}y_{n+4k}x_{n+5k} + x_{n+3k}y_{n+4k}x_{n+5k}y_n x_{n+k}y_{n+2k}\right)^{-1} \\
&= y_n \left(1 - y_n x_{n+k}y_{n+2k} - x_{n+3k}y_{n+4k}x_{n+5k} + x_{n+3k}y_{n+4k}x_{n+5k}\left(\frac{y_n}{x_{n+3k}} + 1\right)\right)^{-1} \\
&= \frac{y_n}{1 - y_n x_{n+k}y_{n+2k} + y_n y_{n+4k}x_{n+5k}} \\
&= \frac{y_n}{1 + y_n\left(y_{n+4k}x_{n+5k} - x_{n+k}y_{n+2k}\right)} \\
&= y_n.
\end{aligned} \tag{36}$$

So, we have obtained $y_{n+6k} = y_n$ for $n \geq 0$. Considering the other systems individually, the proof can be obtained similarly. So, the proof is finished. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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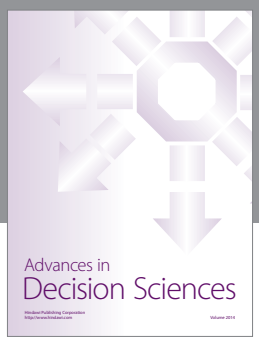

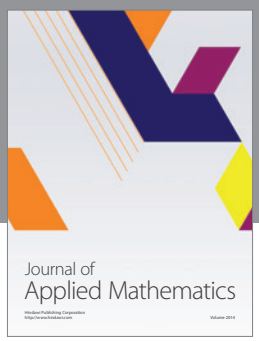

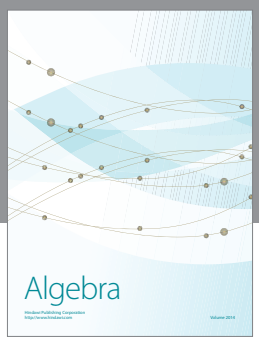

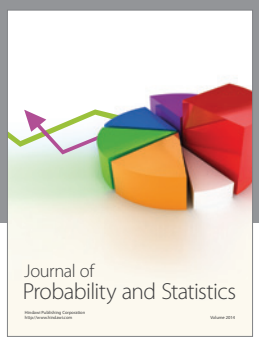

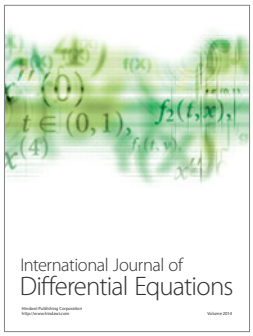

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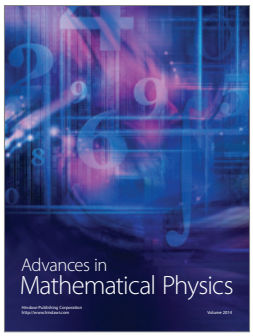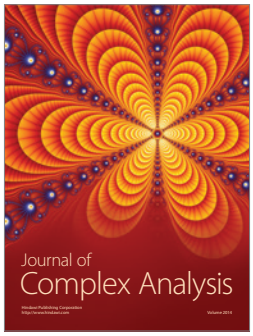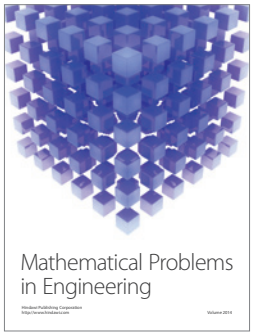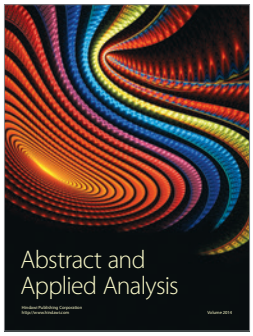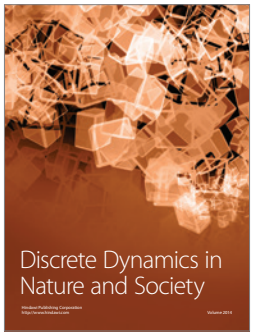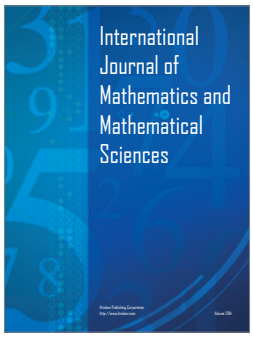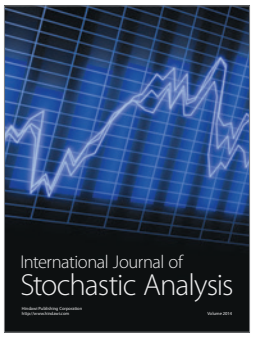
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