Hindawi Publishing Corporation
Journal of Difference Equations
Volume 2015, Article ID 486985, 12 pages
http://dx.doi.org/10.1155/2015/486985

*Research Article*

# Stability of Nonhyperbolic Equilibrium Solution of Second Order Nonlinear Rational Difference Equation

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Received 4 January 2015; Accepted 17 February 2015

Academic Editor: Mustafa R. S. Kulenović



This is a continuation part of our investigation in which the second order nonlinear rational difference equation $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$, $n = 0, 1, 2, \ldots$, where the parameters $A \geq 0$ and $B, C, \alpha, \beta, \gamma$ are positive real numbers and the initial conditions $x_{-1}, x_0$ are nonnegative real numbers such that $A + Bx_0 + Cx_{-1} > 0$, is considered. The first part handled the global asymptotic stability of the hyperbolic equilibrium solution of the equation. Our concentration in this part is on the global asymptotic stability of the nonhyperbolic equilibrium solution of the equation.

*Dedicated to Gerry Ladas on the occasion of his retirement*

## 1. Introduction

In part 1 of this investigation [1], we have established the global stability of the hyperbolic equilibrium solution of the second order rational difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where the parameters $A \geq 0$ and $B, C, \alpha, \beta, \gamma$ are positive real numbers and the initial conditions $x_{-1}, x_0$ are nonnegative real numbers such that $A + Bx_0 + Cx_{-1} > 0$. Our aim in this part is on the global attractivity of the nonhyperbolic equilibrium solution of (1).

The periodic character of positive solutions of (1) has been investigated by the authors in [2]. They showed that the period-two solution is locally asymptotically stable if it exists.

Many rational difference equations were studied extensively in [3]. A systematic study of the second order rational difference equation of form (1), where the parameters $\alpha, \beta, \gamma, A, B, C$ and the initial conditions $x_{-1}, x_0$ are nonnegative real numbers, was considered in the monograph of Kulenovic and Ladas [4]. They presented the known results up to 2002. Next, Kulenovic and Ladas [4] derived several ones on the boundedness, the global stability, and the periodicity of solutions of all rational difference equations of form (1). Furthermore, they posed several open problems and conjectures related to this equation and its functional generalization.

Even after a sustained effort by many researchers such as [5–9], there were some difference equations of form (1) that have not been investigated till 2007.

Amleh et al. in [10, 11] give an up-to-date account on recent developments related to (1) up to 2007. Furthermore, they reposed several open problems and conjectures related to this equation.

Camouzis and Ladas in [12] summarize the progress up to 2008. Recently, the work done by many researchers such as [13–21] have solved many open problems and conjectures proposed in [4, 10–12] related to (1) and have led to the development of some general theory about difference equation. However, as confirmed by Professor Kulenovic (personal communication, August, 24, 2014), the case $A = 0$ remains open.

Our approach handles the aforementioned case as well as other cases. Furthermore, the results in this paper, together

with the established results in [1, 2, 4], give a complete picture of the nature of solutions of the second order rational difference equation of form (1).

It is worth mentioning that there are very few results in the literature regarding the stability of nonhyperbolic equilibrium solution of a general difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{2}$$

We believe that our result is an important stepping stone in understanding the behavior of solutions of rational difference equation of form (1) which provides prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of form (2).

The transformation

$$x_n = \frac{\gamma}{C} y_n \tag{3}$$

reduces (1) to the following equation:

$$y_{n+1} = \frac{r + py_n + y_{n-1}}{z + qy_n + y_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where

$$\begin{aligned} r &= \frac{\alpha C}{\gamma^2}, \\ p &= \frac{\beta}{\gamma}, \\ z &= \frac{A}{\gamma}, \\ q &= \frac{B}{C} \end{aligned} \tag{5}$$

are positive real numbers and the initial conditions $y_{-1}, y_0$ are nonnegative real numbers.

That being said, the remainder of this paper is organized as follows. In the next section, a brief description of some definitions and results from the literature that are needed to prove the main results in this paper is given. Section 3 gives necessary and sufficient conditions for (4) to have nonhyperbolic solution. Next, Section 4 examines the existence of intervals which attract all solutions of (4) and shows that the nonhyperbolic equilibrium solution of (4) is globally asymptotically stable. In Section 5 we consider several numerical examples generated by MATLAB to illustrate the results of the previous sections and to support our theoretical discussion. Finally, we conclude in Section 6 with suggestions for future research.

## 2. Preliminaries

For the sake of self-containment and convenience, we recall the following definitions and results from [4].

Let $I$ be a nondegenerate interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_0, x_{-1} \in I$, the difference equation of form (2) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

A constant sequence, $x_n = \overline{x}$ for all $n$ where $\overline{x} \in I$, is called an equilibrium solution of (2) if

$$\overline{x} = f(\overline{x}, \overline{x}). \tag{6}$$

*Definition 1.* Let $\overline{x}$ be an equilibrium solution of (2).

(i) $\overline{x}$ is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x_0, x_{-1} \in I$, with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \delta$, we have

$$|x_n - \overline{x}| < \epsilon \quad \forall n \geq -1. \tag{7}$$

(ii) $\overline{x}$ is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$, such that, for all $x_0, x_{-1} \in I$, with $|x_0 - \overline{x}| + |x_{-1} - \overline{x}| < \gamma$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{8}$$

(iii) $\overline{x}$ is called a global attractor if for every $x_0, x_{-1} \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{9}$$

(iv) $\overline{x}$ is called globally asymptotically stable if it is locally stable and a global attractor.

(v) $\overline{x}$ is called *unstable* if it is not stable.

(vi) $\overline{x}$ is called a *source*, or a *repeller*, if there exists $r > 0$ such that, for all $x_0, x_{-1} \in I$, with $0 < |x_0 - \overline{x}| + |x_{-1} - \overline{x}| < r$, there exists $N \geq 1$ such that

$$|x_N - \overline{x}| \geq r. \tag{10}$$

Clearly a source is an unstable equilibrium.

*Definition 2.* Let

$$\begin{aligned} a &= \frac{\partial f}{\partial u}(\overline{x}, \overline{x}), \\ b &= \frac{\partial f}{\partial v}(\overline{x}, \overline{x}) \end{aligned} \tag{11}$$

denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\overline{x}$ of (2). Then the equation

$$y_{n+1} = ay_n + by_{n-1}, \quad n = 0, 1, \ldots \tag{12}$$

is called the linearized equation associated with (2) about the equilibrium solution $\overline{x}$.

**Theorem 3** (linearized stability). *(a) If both roots of the quadratic equation*

$$\lambda^2 - a\lambda - b = 0 \tag{13}$$

*lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of (2) is locally asymptotically stable.*

*(b) If at least one of the roots of (13) has absolute value greater than one, then the equilibrium $\overline{x}$ of (2) is unstable.*

*(c) A necessary and sufficient condition for both roots of (13) to lie in the open unit disk $|\lambda| < 1$ is*

$$|a| < 1 - b < 2. \tag{14}$$

*In this case the locally asymptotically stable equilibrium $\overline{x}$ is also called a sink.*

*(d) A necessary and sufficient condition for both roots of (13) to have absolute value greater than one is*

$$\begin{gathered} |b| > 1, \\ |a| < |1 - b|. \end{gathered} \tag{15}$$

*In this case $\overline{x}$ is a repeller.*

*(e) A necessary and sufficient condition for one root of (13) to have absolute value greater than one and for the other to have absolute value less than one is*

$$\begin{gathered} a^2 + 4b > 0, \\ |a| > |1 - b|. \end{gathered} \tag{16}$$

*In this case the unstable equilibrium $\overline{x}$ is called a saddle point.*

*(f) A necessary and sufficient condition for a root of (13) to have absolute value equal to one is*

$$|a| = |1 - b| \tag{17}$$

*or*

$$\begin{gathered} b = -1, \\ |a| \le 2. \end{gathered} \tag{18}$$

*In this case the equilibrium $\overline{x}$ is called a nonhyperbolic point.*

**Theorem 4.** *Consider the difference equation (2). Let $I = [a, b]$ be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \tag{19}$$

*is a continuous function satisfying the following properties:*

(a) *$f(x, y)$ is nonincreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is nondecreasing in $y \in [a, b]$ for each $x \in [a, b]$;*

(b) *the difference equation (2) has no solutions of prime period two in $[a, b]$;*

*then (2) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of (2) converges to $\overline{x}$.*

The following result from [12] will become handy in the sequel.

**Theorem 5.** *Let $I$ be a set and let*

$$f : I \times I \longrightarrow I \tag{20}$$

*be a function $f(u, v)$ which decreases in $u$ and increases in $v$. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{21}$$

*the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

(i) *they are both monotonically increasing;*

(ii) *they are both monotonically decreasing;*

(iii) *eventually, one of them is monotonically increasing and the other is monotonically decreasing.*

The following result was established in [2] and will prove to be useful in our investigation.

**Theorem 6.** *(a) When*

$$p + z \ge 1, \tag{22}$$

*(4) has no nonnegative prime period-two solution.*

*(b) When*

$$p + z < 1, \tag{23}$$

*(4) has prime period-two solution,*

$$\ldots, \phi, \psi, \phi, \psi, \ldots; \tag{24}$$

*if and only if condition*

$$r < \frac{(1 - p - z)[q(1 - p - z) - (1 + 3p - z)]}{4}, \tag{25}$$

*where $\phi$ and $\psi$ are the positive and distinct solutions of the quadratic equation*

$$t^2 - (1 - z - p)t + \frac{p(1 - z - p) + r}{q - 1} = 0, \quad q > 1. \tag{26}$$

The following two results were established in part 1 of this investigation [1] and will prove to be useful in our investigation.

**Theorem 7.** *Assume that $p < q$ and $r \le z$; then one has two cases to be considered.*

(1) *If $pz - qr \le 0$, then $[p/q, 1]$ is invariant.*

(2) *If $pz - qr > 0$, then we have two subcases to be considered:*

(a) *if $(pz - qr)/(q - p) \le p/q$, then every positive solution of (4) eventually enters and remains in the interval $[p/q, 1]$;*

(b) *if $(pz - qr)/(q - p) > p/q$, then every positive solution of (4) eventually enters and remains in the interval $(0, p/q)$.*

**Theorem 8.** *Assume that $p < q$ and $r > z$; then one has two cases to be considered.*

(1) *If $(r - z)/(q - p) \le 1$, then every positive solution of (4) eventually enters and remains in the interval $[(r - z)/(q - p), 1]$.*

(2) *If $(r - z)/(q - p) > 1$, then every positive solution of (4) eventually enters and remains in the interval $[1, (r - z)/(q - p)]$.*

## 3. Existence of Nonhyperbolic Equilibrium Solution

In this section, we give explicit conditions on the parameter values of (4) for the equilibrium $\overline{y}$ to be nonhyperbolic.

Equation (4) has a unique positive equilibrium given by

$$\overline{y} = \frac{1+p-z+\sqrt{(1+p-z)^2+4r(q+1)}}{2(q+1)}. \tag{27}$$

The linearized equation associated with (4) about the equilibrium solution is given by

$$z_{n+1} = \frac{p-q\overline{y}}{z+(q+1)\overline{y}} z_n + \frac{1-\overline{y}}{z+(q+1)\overline{y}} z_{n-1}. \tag{28}$$

Therefore, its characteristic equation is

$$\lambda^2 - \frac{p-q\overline{y}}{z+(q+1)\overline{y}}\lambda - \frac{1-\overline{y}}{z+(q+1)\overline{y}} = 0. \tag{29}$$

By applying Theorem 3(f) we have the following result.

**Theorem 9.** *Assume that*

$$p+z<1, \tag{30}$$

*then the positive equilibrium of (4) is nonhyperbolic if and only if*

$$r = \frac{(1-p-z)\left[q(1-p-z)-(1+3p-z)\right]}{4}. \tag{31}$$

*Proof.* By employing Theorem 3(f), conditions (17) and (18) are equivalent to the following two inequalities:

$$\left|\frac{p-q\overline{y}}{z+(q+1)\overline{y}}\right| = \left|1-\frac{1-\overline{y}}{z+(q+1)\overline{y}}\right|, \tag{32}$$

$$\begin{gathered} \underbrace{\frac{1-\overline{y}}{z+(q+1)\overline{y}}}_{(1)} = -1, \\ \left|\frac{p-q\overline{y}}{z+(q+1)\overline{y}}\right| \le 2, \end{gathered} \tag{33}$$

respectively. Notice that Part (1) of (33) implies $-(1+z) = q\overline{y}$, which is impossible to be satisfied since $z, q, \overline{y} > 0$, while (32) is equivalent to the following two inequalities:

$$\frac{p-q\overline{y}}{z+(q+1)\overline{y}} = 1-\frac{1-\overline{y}}{z+(q+1)\overline{y}} \tag{34}$$

or

$$\frac{p-q\overline{y}}{z+(q+1)\overline{y}} = \frac{1-\overline{y}}{z+(q+1)\overline{y}} - 1. \tag{35}$$

Equation (34) implies $1+p-z = 2(q+1)\overline{y}$, which contradicts (27), while (35) is equivalent to

$$\frac{p-q\overline{y}}{z+(q+1)\overline{y}} - \frac{1-\overline{y}}{z+(q+1)\overline{y}} = -1. \tag{36}$$

From which we have

$$1-p-z = 2\overline{y}. \tag{37}$$

Clearly the equilibrium $\overline{y}$ is the positive solution of the quadratic equation

$$(q+1)\overline{y}^2 + (z-p-1)\overline{y} - r = 0. \tag{38}$$

Now set

$$F(u) = (q+1)u^2 + (z-p-1)u - r, \tag{39}$$

and (37) holds if and only if

$$\begin{gathered} p+z<1, \\ F\left(\frac{1-p-z}{2}\right) = 0. \end{gathered} \tag{40}$$

That is,

$$\begin{aligned} &(q+1)\left(\frac{1-p-z}{2}\right)^2 + (z-p-1)\left(\frac{1-p-z}{2}\right) - r \\ &= 0, \\ &\Longleftrightarrow (q+1)(1-p-z)^2 + 2(z-p-1)(1-p-z) \\ &\quad - 4r = 0, \\ &\Longleftrightarrow (1-p-z)\left[(q+1)(1-p-z) + 2(z-p-1)\right] \\ &\quad - 4r = 0, \\ &\Longleftrightarrow (1-p-z)\left[q(1-p-z) - (1+3p-z)\right] - 4r \\ &= 0, \end{aligned} \tag{41}$$

from which (31) follows.

The proof is complete. □

## 4. Global Stability Analysis

In this section, we give necessary and sufficient conditions for the nonhyperbolic solution of (4) to be globally attractive.

The characteristic polynomial associated with (4) about the positive equilibrium is given by

$$\begin{aligned} f(\lambda) &= \lambda^2 - \frac{p-q\overline{y}}{z+(q+1)\overline{y}}\lambda - \frac{1-\overline{y}}{z+(q+1)\overline{y}} = 0 \\ &\Longrightarrow (\lambda+1)\left(\lambda - \frac{1-\overline{y}}{z+(q+1)\overline{y}}\right) = 0 \\ &\Longrightarrow \lambda = -1 \\ &\text{or } \lambda = \frac{1-\overline{y}}{z+(q+1)\overline{y}} < 1. \end{aligned} \tag{42}$$

By the Stable Manifold Theorem, there is a manifold of solutions that converge to the equilibrium solution.

Table 1: Signs of $\partial f/\partial y_n$ and $\partial f/\partial y_{n-1}$ of (3) when $p < q$.

| Case | $pz-qr$ | $r-z$ | $\dfrac{\partial f(y_n, y_{n-1})}{\partial y_n} = \dfrac{q-p}{(z+qy_n+y_{n-1})^2}\left[\dfrac{pz-qr}{q-p} - y_{n-1}\right]$ | $\dfrac{\partial f(y_n, y_{n-1})}{\partial y_{n-1}} = \dfrac{q-p}{(z+qy_n+y_{n-1})^2}\left[y_n - \dfrac{r-z}{q-p}\right]$ |
|---|---|---|---|---|
| 1 | + | − | − if $y_{n-1} > (pz-qr)/(q-p)$ and + if $y_{n-1} < (pz-qr)/(q-p)$ | + |
| 2 | 0 | − | − | + |
| 3 | − | − | − | + |
| 4 | − | 0 | − | + |
| 5 | − | + | − | + if $y_n > (r-z)/(q-p)$ and − if $y_n < (r-z)/(q-p)$ |

Now, since $r > 0$, condition (31) implies

$$q > \frac{1+3p-z}{1-p-z} > 1. \tag{43}$$

Indeed, since $p + z < 1$ we have

$$p < q. \tag{44}$$

With that in mind we examine the existence of intervals which attract all solutions of (1) in the next section.

*4.1. Invariant Intervals.* Table 1 gives the signs of $\partial f(y_n, y_{n-1})/\partial y_n$ and $\partial f(y_n, y_{n-1})/\partial y_{n-1}$ of (4) in all possible nondegenerate cases when $p < q$.

The following two lemmas will be useful in investigating the attracting intervals of solutions of (4).

**Lemma 10.** *Assume that condition (31) holds.*

(a) *If $r \le z$ and $pz - qr > 0$, then $(pz-qr)/(q-p) < p/q$.*

(b) *If $r > z$, then $(r-z)/(q-p) < 1$.*

*Proof.* (a) Consider $r \le z$ and $pz - qr > 0$.

Assume for the sake of contradiction that

$$\frac{pz-qr}{q-p} \ge \frac{p}{q}. \tag{45}$$

Then

$$r \le \frac{pzq - pq + p^2}{q^2}. \tag{46}$$

By condition (31), inequality (46) is equivalent to

$$\begin{aligned} &q^3(1-p-z)^2 - q^2(1-p-z)(1+3p-z) \\ &\quad \le 4\left(pzq - pq + p^2\right). \end{aligned} \tag{47}$$

Since $p + z < 1$ and $q > 1$, the right-hand side of inequality (47) is equivalent to

$$\begin{aligned} 4\left(pq(z-1) + p^2\right) &\le 4\left(pq(-p) + p^2\right) \\ &= -4p^2(q-1) < 0. \end{aligned} \tag{48}$$

Thus, inequality (47) implies

$$\begin{aligned} &\Longrightarrow q^3(1-p-z)^2 - q^2(1-p-z)(1+3p-z) \\ &\le 0, \\ &\Longrightarrow q^3(1-p-z)^2 \le q^2(1-p-z)(1+3p-z), \\ &\Longrightarrow q \le \frac{1+3p-z}{1-p-z}, \end{aligned} \tag{49}$$

which contradicts condition (43).

(b) Consider $r > z$.

Assume for the sake of contradiction that

$$\frac{r-z}{q-p} \ge 1. \tag{50}$$

Then

$$r \ge q - p + z. \tag{51}$$

By condition (31), inequality (51) is equivalent to

$$\begin{aligned} &(1-p-z)\left[q(1-p-z) - (1+3p-z)\right] \\ &\quad \ge 4q + 4(z-p). \end{aligned} \tag{52}$$

Since $p + z < 1$, inequality (52) implies

$$\begin{aligned} &q(1-p-z)^2 - (1-p-z)(1+3p-z) \\ &\quad \ge 4q + 4(z-(1-z)) > 4q - 4, \\ &\Longrightarrow q\left[4 - (1-p-z)^2\right] \\ &\quad < 4 - (1-p-z)(1+3p-z), \\ &\Longrightarrow q < \frac{4-(1-p-z)(1+3p-z)}{4-(1-p-z)^2}, \\ &\Longrightarrow q < \frac{-(1-p-z)(1+3p-z)}{-(1-p-z)^2}, \\ &\Longrightarrow q < \frac{1+3p-z}{1-p-z}, \end{aligned} \tag{53}$$

which contradicts condition (43).

The proof is complete. □

**Lemma 11.** *Assume that condition (31) holds.*

(a) *If $r \le z$, then $p/q \le \overline{y} < 1$.*

(b) *If $r > z$, then $(r-z)/(q-p) \le \overline{y} < 1$.*

*Proof.* Condition (37) implies

$$\overline{y} = \frac{1-p-z}{2}. \tag{54}$$

Since $p + z < 1$, it is clear that $\overline{y} < 1$. To complete the proof we have two cases to be considered.

(a) Consider $r \le z$.

To show that $\overline{y} = (1-p-z)/2 > p/q$, it is enough to show $q > 2p/(1-p-z)$ which is clearly satisfied since condition (43) holds.

(b) Consider $r > z$.

Our interest is to show that

$$\overline{y} \ge \frac{r-z}{q-p}. \tag{55}$$

Assume for the sake of contradiction that

$$\begin{aligned} &\overline{y} < \frac{r-z}{q-p} \\ &\Longrightarrow \frac{1-p-z}{2} < \frac{r-z}{q-p} \\ &\Longrightarrow (q-p)(1-p-z) < 2(r-z) \\ &\Longrightarrow 2r > 2z + (q-p)(1-p-z). \end{aligned} \tag{56}$$

By condition (31), the last inequality is equivalent to

$$\begin{aligned} &\frac{(1-p-z)[q(1-p-z)-(1+3p-z)]}{2} > 2z \\ &\quad + (q-p)(1-p-z) \\ &\Longrightarrow (1-p-z)[q(1-p-z)-(1+3p-z)] > 4z \\ &\quad + 2(q-p)(1-p-z) \\ &\Longrightarrow (1-p-z) \\ &\quad \cdot [q(1-p-z)-(1+3p-z)-2(q-p)] > 4z \\ &\Longrightarrow (1-p-z)[-q-q(p+z)-1-p+z] > 4z \\ &\Longrightarrow (1-p-z)\underbrace{[-q(1+p+z)-(1+p-z)]}_{-\text{ve}} > 4z, \end{aligned} \tag{57}$$

which is impossible.

The proof is complete. □

By Theorem 7, Lemmas 10 and 11, we obtain the following key result.

**Theorem 12.** *Assume that condition (31) holds.*

(a) *If $r \le z$, then every positive solution of (4) eventually enters and remains in the interval $[p/q, 1]$.*

(b) *If $r > z$, then every positive solution of (4) eventually enters and remains in the interval $[(r-z)/(q-p), 1]$.*

*Proof.* We have two cases to be considered.

(a) Consider $r \le z$.

Since $r \le z$ and condition (44) holds, then we have two subcases to be considered:

(1) if $pz - qr \le 0$, Theorem 7 part 1 implies that the interval $[p/q, 1]$ is invariant;

(2) If $pz - qr > 0$, Lemma 10 implies that $(pz - qr)/(q-p) < p/q$, and by Theorem 7 part 2(a) every positive solution of (4) eventually enters and remains in the interval $[p/q, 1]$.

(b) Consider $r > z$.

Since $r > z$ and condition (44) holds, Lemma 10 implies that $(r-z)/(q-p) < 1$. By Theorem 8 part 1. every positive solution of (4) eventually enters and remains in the interval $[(r-z)/(q-p), 1]$.

The proof is complete. □

*4.2. Global Stability of the Nonhyperbolic Equilibrium Solution.* The following lemma will be useful in investigating the global stability of the nonhyperbolic equilibrium solution of (4).

**Lemma 13.** *Assume $r > z$. Then $y_{n+1} \le y_{n-1}$.*

*Proof.* By (4), we have

$$\begin{aligned} &y_{n+1} - y_{n-1} \\ &\quad = \frac{r + py_n + y_{n-1}}{z + qy_n + y_{n-1}} - y_{n-1} \\ &\quad = \frac{r + py_n + y_{n-1} - zy_{n-1} - qy_ny_{n-1} - y_{n-1}^2}{z + qy_n + y_{n-1}} \\ &\quad = \frac{r + py_n + (1-z)y_{n-1} - qy_ny_{n-1} - y_{n-1}^2}{z + qy_n + y_{n-1}}. \end{aligned} \tag{58}$$

Let

$$\begin{aligned} h(y_n, y_{n-1}) = r + py_n + (1-z)y_{n-1} - qy_ny_{n-1} \\ - y_{n-1}^2. \end{aligned} \tag{59}$$

Our interest now is to show that $h(y_n, y_{n-1}) \le 0$ for all $y_n, y_{n-1} \in [(r-z)/(q-p), 1]$.

Observe that $h(y_n, y_{n-1})$ is continuously differentiable map defined on a compact interval. As such, either the maximum is attained at an interior stationary point or on the boundary. Figure 1 depicts the region where $h(y_n, y_{n-1})$ is defined.

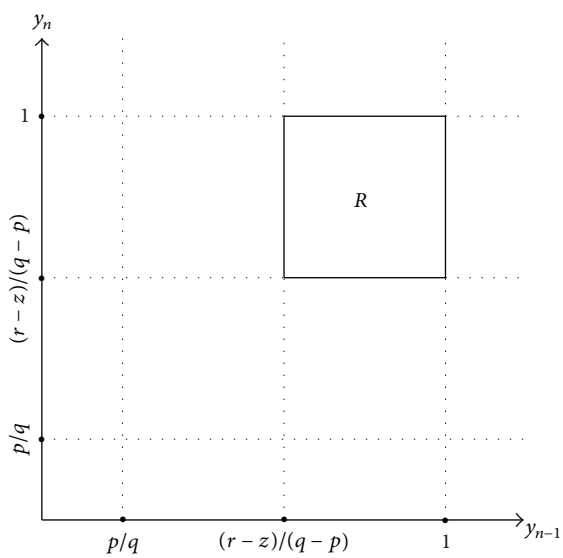


FIGURE 1: The region where $h(y_n, y_{n-1})$ is defined.

Furthermore,

$$\frac{\partial h(y_n, y_{n-1})}{\partial y_n} = p - qy_{n-1},$$
$$\frac{\partial h(y_n, y_{n-1})}{\partial y_{n-1}} = 1 - z - qy_n - 2y_{n-1}. \tag{60}$$

As such, there is one interior stationary point, namely, $((1 - z - 2(p/q))/q, p/q)$.

Recall that we only want stationary point in the region $R$, so $((1 - z - 2(p/q))/q, p/q)$ will be ignored, because $p/q < (r - z)/(q - p)$.

Now, we need to find the absolute maximum of the function $h(y_n, y_{n-1})$ along the boundary of the rectangle $R$.

The boundary of this rectangle is given by the following.

(1) Upper side: $y_n = 1$, $(r - z)/(q - p) \le y_{n-1} \le 1$.

Define

$$g(y_{n-1}) = h(1, y_{n-1}) = (r + p) + (1 - z - q)\, y_{n-1} - y_{n-1}^2. \tag{61}$$

Now, finding the absolute maximum of the function $h(y_n, y_{n-1})$ a long the right side of the rectangle $R$ will be equivalent to finding the absolute maximum of the function $g(y_{n-1})$ in the range $(r - z)/(q - p) \le y_{n-1} \le 1$.

Hence,

$$g' = (1 - z - q) - 2y_{n-1} \Longrightarrow y_{n-1} = \frac{1 - z - q}{2} < 0. \tag{62}$$

This is not in the range $(r - z)/(q - p) \le y_{n-1} \le 1$, so we will ignore it.

The value of this function at the end points is

$$\begin{aligned} g\left(\frac{r-z}{q-p}\right) &= h\left(1, \frac{r-z}{q-p}\right) \\ &= (r+p) + (1-z-q)\left(\frac{r-z}{q-p}\right) \\ &\quad - \left(\frac{r-z}{q-p}\right)^2, \\ g(1) &= h(1,1) = (r+p) + (1-z-q) - 1 \\ &= (r-z) - (q-p). \end{aligned} \tag{63}$$

Since $(r - z)/(q - p) < 1$ by Lemma 10(b), then

$$\begin{aligned} h\left(1, \frac{r-z}{q-p}\right) &< (r+p) + (1-z-q)(1) - (1) \\ &= (r-z) - (q-p) < 0, \\ h(1,1) &= (r-z) - (q-p) < 0. \end{aligned} \tag{64}$$

(2) Lower side: $y_n = (r - z)/(q - p)$, $(r - z)/(q - p) \le y_{n-1} \le 1$.

Define

$$\begin{aligned} g(y_{n-1}) &= h\left(\frac{r-z}{q-p}, y_{n-1}\right) \\ &= r + p\left(\frac{r-z}{q-p}\right) \\ &\quad + \left(1 - z - q\left(\frac{r-z}{q-p}\right)\right) y_{n-1} - y_{n-1}^2, \\ g' &= \left(1 - z - q\left(\frac{r-z}{q-p}\right)\right) - 2y_{n-1} \Longrightarrow y_{n-1} \\ &= \frac{1 - z - q((r-z)/(q-p))}{2} < 0. \end{aligned} \tag{65}$$

This is not in the range $(r - z)/(q - p) \le y_{n-1} \le 1$, so we will ignore it.

The value of this function at the end points is

$$\begin{aligned} g\left(\frac{r-z}{q-p}\right) &= h\left(\frac{r-z}{q-p}, \frac{r-z}{q-p}\right) \\ &= r + (p + 1 - z)\left(\frac{r-z}{q-p}\right) \\ &\quad - (q+1)\left(\frac{r-z}{q-p}\right)^2, \\ g(1) &= h\left(\frac{r-z}{q-p}, 1\right) \\ &= (r-z) - (q-p)\left(\frac{r-z}{q-p}\right). \end{aligned} \tag{66}$$

Since $(r-z)/(q-p) < 1$ by Lemma 10(b), then

$$h\left(\frac{r-z}{q-p}, \frac{r-z}{q-p}\right) < r+p+1-z-q-1 = (r-z)-(q-p) < 0. \tag{67}$$

Furthermore,

$$h\left(\frac{r-z}{q-p}, 1\right) = 0. \tag{68}$$

(3) Right side: $y_{n-1} = 1$, $(r-z)/(q-p) \le y_n \le 1$.
Define

$$f(y_n) = h(y_n, 1) = (r-z) + (p-q)\, y_n, \qquad f' = p-q < 0. \tag{69}$$

Then $f$ decreases and its maximum occurs at $y_n = (r-z)/(q-p)$. Furthermore,

$$f\left(\frac{r-z}{q-p}\right) = h\left(\frac{r-z}{q-p}, 1\right) = 0. \tag{70}$$

(4) Left side: $y_{n-1} = (r-z)/(q-p)$, $(r-z)/(q-p) \le y_n \le 1$.
Define

$$\begin{aligned} f(y_n) &= h\left(y_n, \frac{r-z}{q-p}\right) \\ &= r + py_n + (1-z)\left(\frac{r-z}{q-p}\right) - q\left(\frac{r-z}{q-p}\right) y_n \\ &\quad - \left(\frac{r-z}{q-p}\right)^2, \\ f' &= p - q\left(\frac{r-z}{q-p}\right) < 0. \end{aligned} \tag{71}$$

Then $f$ decreases and the maximum occurs at $y_n = (r-z)/(q-p)$. Furthermore,

$$f\left(\frac{r-z}{q-p}\right) = h\left(\frac{r-z}{q-p}, \frac{r-z}{q-p}\right) < 0 \tag{72}$$

Now, collect up all the function values for $h(y_n, y_{n-1})$:

$$\begin{aligned} h\left(\frac{r-z}{q-p}, \frac{r-z}{q-p}\right) &< 0, \\ h\left(1, \frac{r-z}{q-p}\right) &< 0, \\ h\left(\frac{r-z}{q-p}, 1\right) &= 0, \\ h(1,1) &< 0. \end{aligned} \tag{73}$$

Clearly the maximum value of $h$ is 0 and it occurs at $((r-z)/(q-p), 1)$.

The proof is complete. □

The result about the global stability of the positive nonhyperbolic equilibrium solution of (4) is given in the following theorem.

**Theorem 14.** *The positive nonhyperbolic equilibrium solution of (4) is globally asymptotically stable.*

*Proof.* Let

$$f(y_n, y_{n-1}) = \frac{r + py_n + y_{n-1}}{z + qy_n + y_{n-1}}. \tag{74}$$

We have two cases to be considered.

(a) Consider $r \le z$.

Here we distinguish between two subcases.

(1) Consider $pz - qr \le 0$.
Let $\{y_n\}_{n=0}^{\infty}$ be a positive solution of (4). It follows from Theorem 12(a) that every positive solution of (4) eventually enters and remains in the interval $[p/q, 1]$. Furthermore, $p/q \le \overline{y} < 1$ by Lemma 11. Indeed, function (74) is decreasing in $y_n$ and increasing in $y_{n-1}$, in the interval $[p/q, 1]$. By applying Theorem 5 the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of the solution converge monotonically to finite limits $\ell_1$ and $\ell_2$. Set

$$\ell_1 = \lim_{n\to\infty} y_{2n}, \qquad \ell_2 = \lim_{n\to\infty} y_{2n+1}. \tag{75}$$

By (4) we have

$$y_{2n+2} = \frac{r + py_{2n+1} + y_{2n}}{z + qy_{2n+1} + y_{2n}}, \qquad y_{2n+1} = \frac{r + py_{2n} + y_{2n-1}}{z + qy_{2n} + y_{2n-1}}. \tag{76}$$

Hence,

$$\ell_1 = \frac{r + p\ell_2 + \ell_1}{z + q\ell_2 + \ell_1}, \qquad \ell_2 = \frac{r + p\ell_1 + \ell_2}{z + q\ell_1 + \ell_2}. \tag{77}$$

Thus,

$$z\ell_1 + q\ell_1\ell_2 + \ell_1^2 = r + p\ell_2 + \ell_1, \qquad z\ell_2 + q\ell_1\ell_2 + \ell_2^2 = r + p\ell_1 + \ell_2 \tag{78}$$

and by subtracting we have

$$(\ell_1 - \ell_2)\left[z + \ell_1 + \ell_2 + p - 1\right] = 0. \tag{79}$$

This is true if and only if

$$\ell_1 = \ell_2 \quad \text{or } \ell_1 + \ell_2 = 1 - p - z. \tag{80}$$

However, if $\ell_1 = \ell_2 = \ell$ then by (77) we have

$$\begin{aligned}
\ell &= \frac{r + p\ell + \ell}{z + q\ell + \ell} \\
&\Longrightarrow \ell = \frac{r/\ell + p + 1}{z/\ell + q + 1} \\
&\Longrightarrow z + (q+1)\ell = \frac{r}{\ell} + p + 1 \\
&\Longrightarrow (q+1)\ell - \frac{r}{\ell} = 1 + p - z \\
&\Longrightarrow (q+1)\ell^2 - r = (1 + p - z)\ell \\
&\Longrightarrow (q+1)\ell^2 - (1 + p - z)\ell - r = 0 \\
&\Longrightarrow \ell = \frac{1 + p - z + \sqrt{(1+p-z)^2 + 4r(q+1)}}{2(q+1)} \\
&\Longrightarrow \ell = \overline{y},
\end{aligned} \tag{81}$$

whereas $\ell_1 + \ell_2 = 1 - p - z = 2\overline{y}$ is impossible in this case since by Theorem 6, (4) does not possess a period-two solution.

(2) Consider $pz - qr > 0$.

Let $\{y_n\}_{n=0}^{\infty}$ be a positive solution of (4). It follows from Lemma 10(a) that $(pz - qr)/(q - p) < p/q$. Furthermore, Theorem 12(a) implies that every positive solution of (4) eventually enters and remains in the interval $[p/q, 1]$. Indeed, $p/q \leq \overline{y} < 1$ by Lemma 11. Figure 2 depicts the region where $f(y_n, y_{n-1})$ is defined.

With the understanding that function (74) is decreasing in $y_n$ and increasing in $y_{n-1}$ in the interval $[p/q, 1]$ by Table 1 case 1, and since condition (31) holds, (4) does not possess a period-two solution by Theorem 6. Thus all conditions of Theorem 4 are satisfied and we conclude that $\overline{y}$ is globally asymptotically stable.

(b) Consider $r > z$.

Let $\{y_n\}_{n=0}^{\infty}$ be a positive solution of (4). It follows from Theorem 12(b) that every positive solution of (4) eventually enters and remains in the interval $[(r - z)/(p - q), 1]$. Furthermore, $(r - z)/(q - p) \leq \overline{y} < 1$ by Lemma 11.

Lemma 13 shows that $y_{n+1} \leq y_{n-1}$. Replacing $n$ by $n - 1$ in the previous inequality then $y_n \leq y_{n-2}$. As such, the odd and even terms of any solution $y_n$ of (4) form two monotonic nondecreasing subsequences. Furthermore, both of these subsequences are bounded because $y_n$ is bounded. Hence by Monotone Convergence Theorem the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of the solution converge to finite limits $\jmath_1$ and $\jmath_2$. Set

$$\begin{aligned}
\jmath_1 &= \lim_{n\to\infty} y_{2n}, \\
\jmath_2 &= \lim_{n\to\infty} y_{2n+1}.
\end{aligned} \tag{82}$$

Using technique similar to the one used in proofing Case (a)(1), we have

$$\begin{aligned}
y_{2n+2} &= \frac{r + py_{2n+1} + y_{2n}}{z + qy_{2n+1} + y_{2n}}, \\
y_{2n+1} &= \frac{r + py_{2n} + y_{2n-1}}{z + qy_{2n} + y_{2n-1}}.
\end{aligned} \tag{83}$$

Hence,

$$\begin{aligned}
\jmath_1 &= \frac{r + p\jmath_2 + \jmath_1}{z + q\jmath_2 + \jmath_1}, \\
\jmath_2 &= \frac{r + p\jmath_1 + \jmath_2}{z + q\jmath_1 + \jmath_2}.
\end{aligned} \tag{84}$$

Thus,

$$\begin{aligned}
z\jmath_1 + q\jmath_1\jmath_2 + \jmath_1^2 &= r + p\jmath_2 + \jmath_1, \\
z\jmath_2 + q\jmath_1\jmath_2 + \jmath_2^2 &= r + p\jmath_1 + \jmath_2
\end{aligned} \tag{85}$$

and by subtracting we have

$$(\jmath_1 - \jmath_2)[z + \jmath_1 + \jmath_2 + p - 1] = 0. \tag{86}$$

This is true if and only if

$$\begin{aligned}
&\jmath_1 = \jmath_2 \\
&\text{or } \jmath_1 + \jmath_2 = 1 - p - z.
\end{aligned} \tag{87}$$

However, if $\jmath_1 = \jmath_2 = \jmath$ then by (84) we have

$$\begin{aligned}
\jmath &= \frac{r + p\jmath + \jmath}{z + q\jmath + \jmath} \\
&\Longrightarrow \jmath = \frac{r/\jmath + p + 1}{z/\jmath + q + 1} \\
&\Longrightarrow z + (q+1)\jmath = \frac{r}{\jmath} + p + 1 \\
&\Longrightarrow (q+1)\jmath - \frac{r}{\jmath} = 1 + p - z \\
&\Longrightarrow (q+1)\jmath^2 - r = (1 + p - z)\jmath \\
&\Longrightarrow (q+1)\jmath^2 - (1 + p - z)\jmath - r = 0 \\
&\Longrightarrow \jmath = \frac{1 + p - z + \sqrt{(1+p-z)^2 + 4r(q+1)}}{2(q+1)} \\
&\Longrightarrow \jmath = \overline{y},
\end{aligned} \tag{88}$$

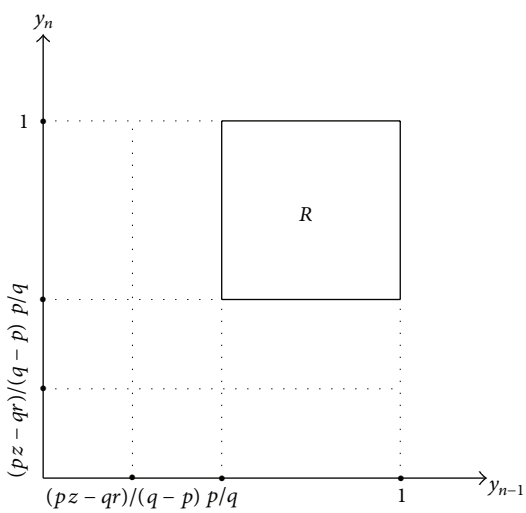


FIGURE 2: The region where $f(y_n, y_{n-1})$ is defined.

whereas, $\jmath_1 + \jmath_2 = 1 - p - z = 2\overline{y}$ is impossible in this case since by Theorem 6, (4) does not possess a period-two solution.

The proof is complete.

□

*Remark 15.* The papers [22–24] give the proof of the existence of both stable and unstable manifolds for second order difference equations decreasing in first and increasing in second argument in nonhyperbolic case of stable type, that is, of the type when second characteristic value is in $(-1, 1)$. In such a way one can avoid the use of center manifold for such equations.

## 5. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussion, we consider several numerical examples generated by MATLAB.

*Example 1.* Consider the following equation:

$$y_{n+1} = \frac{0.01 + 0.6y_n + y_{n-1}}{0.2 + 14y_n + y_{n-1}}. \tag{89}$$

Since $r = 0.01$ satisfies condition (31), by Theorem 9, the equilibrium is nonhyperbolic. Indeed, $r \le z$ and $pz - qr \le 0$, and Theorem 12 implies that every positive solution of (89) eventually enters and remains in the interval $[p/q, 1]$. Furthermore, the equilibrium $\overline{y}$ is globally asymptotically stable by Theorem 14. The dynamics of (89) is shown in Figure 3.

*Example 2.* Consider the following equation:

$$y_{n+1} = \frac{0.0025 + 0.3y_n + y_{n-1}}{0.6 + 14y_n + y_{n-1}}. \tag{90}$$

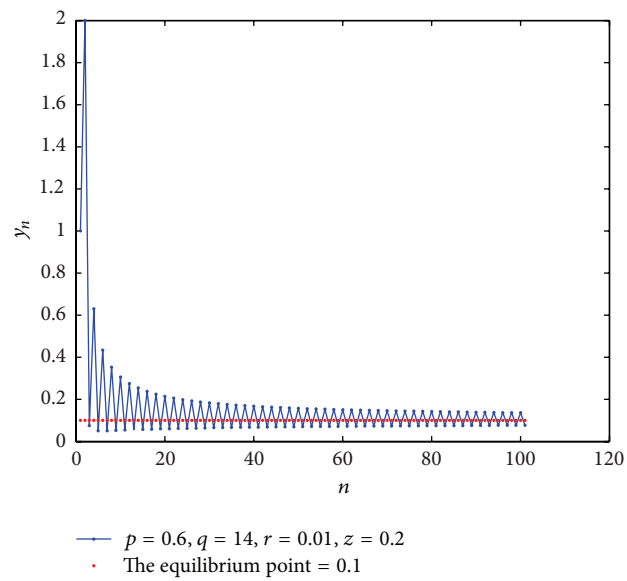


FIGURE 3: Dynamics of $y_{n+1} = (0.01+0.6y_n+y_{n-1})/(0.2+14y_n+y_{n-1})$.

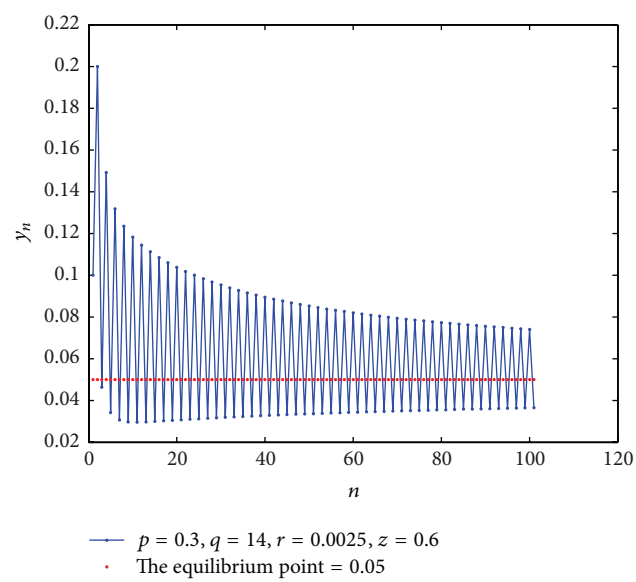


FIGURE 4: Dynamics of $y_{n+1} = (0.0025 + 0.3y_n + y_{n-1})/(0.6 + 14y_n + y_{n-1})$.

Since $r = 0.0025$ satisfies condition (31), by Theorem 9, the equilibrium is nonhyperbolic. Indeed, $r < z$ and $pz - qr > 0$, and Theorem 12 implies that every positive solution of (90) eventually enters and remains in the interval $[p/q, 1]$. Furthermore, the equilibrium $\overline{y}$ is globally asymptotically stable by Theorem 14. The dynamics of (90) is shown in Figure 4.

*Example 3.* Consider the following example:

$$y_{n+1} = \frac{0.031 + 0.3y_n + y_{n-1}}{0.01 + 3y_n + y_{n-1}}. \tag{91}$$

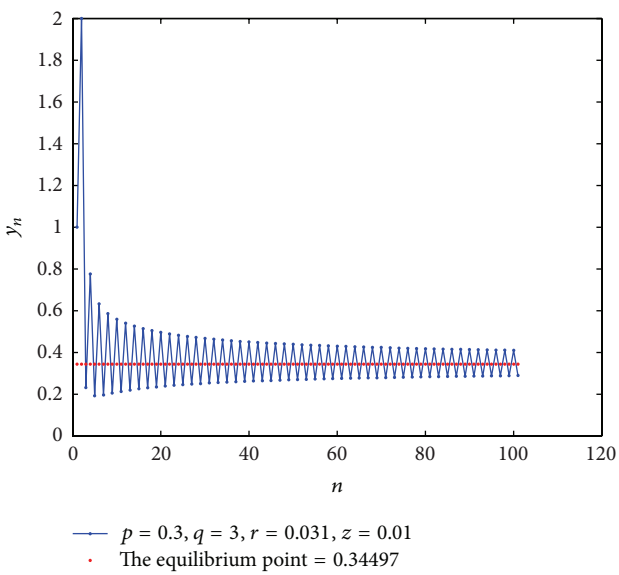


FIGURE 5: Dynamics of $y_{n+1} = (0.031 + 0.3y_n + y_{n-1})/(0.01 + 3y_n + y_{n-1})$.

Since $r = 0.0025$ satisfies condition (31), by Theorem 9, the equilibrium is nonhyperbolic. Indeed, $r > z$, and Theorem 12 implies that every positive solution of (91) eventually enters and remains in the interval $[(r - z)/(q - p), 1]$. Furthermore, the equilibrium $\overline{y}$ is globally asymptotically stable by Theorem 14. The dynamics of (91) is shown in Figure 5.

## 6. Conclusion

In this paper, we have established the global stability of the nonhyperbolic equilibrium solutions of the second order rational difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{92}$$

where the parameters $\alpha$, $\beta$, $\gamma$, $A$, $B$, $C$ are positive real numbers and the initial conditions $x_{-1}$, $x_0$ are nonnegative real numbers.

We believe that our result is an important stepping stone in understanding the behavior of solutions of rational difference equations which provides prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of higher order.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

The authors are very grateful to the anonymous referees for carefully reading the paper and for their comments and valuable suggestions that lead to an improvement in the paper.

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