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Research Article

On the Rank of Elliptic Curves in Elementary Cubic Extensions

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We give a method for explicitly constructing an elementary cubic extension L over which an elliptic curve $E_D : y^2 + Dy = x^3$ ($D \in \mathbb{Q}^*$) has Mordell-Weil rank of at least a given positive integer by finding a close connection between a 3-isogeny of E_D and a generic polynomial for cyclic cubic extensions. In our method, the extension degree $[L : \mathbb{Q}]$ often becomes small.

1. Introduction

Let E be an elliptic curve defined over a number field F. It is well-known that the Mordell-Weil group $E(F)$ of F-rational points on E forms a finitely generated abelian group, and its *rank* $\operatorname{rank} E(F) = \dim_{\mathbb{Q}} E(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is of great interest in arithmetic geometry. The present paper is motivated by the following general problem:

> *To understand the behavior of ranks of elliptic curves in towers of finite extensions over F.*

This problem dates back at least to [1], which first introduced an Iwasawa theory for elliptic curves. In the context of Iwasawa theory, Kurcanov [2] showed that if an elliptic curve E over $\mathbb{Q}$ without complex multiplication has good reduction at a prime number $p > 3$ satisfying a mild condition then E has infinite rank in a certain $\mathbb{Z}_p$-extension, and Harris [3] showed that if E is a modular elliptic curve over F having good ordinary reduction at p and a specific F-rational point arising from the modular curve then there is a p-adic Lie extension of F in which the rank of E grows infinitely. On the other hand, Kida [4] constructed a tower of elementary 2-extensions in which the rank of $y^2 = x^3 - n^2 x$ becomes arbitrarily large by using a result in theory of congruent numbers. Also, Dokchitser [5] proved that for any elliptic curve E over F there are infinitely many cubic extensions K/F so that the rank of E increases in K/F. In the present paper, we consider the specific curve

$$E_D : y^2 + Dy = x^3, \quad D \in \mathbb{Q}^*, \tag{1}$$

and give an explicit construction of an elementary cubic extension over which the elliptic curve E_D is of rank of at least a given positive integer l by finding a close connection between a 3-isogeny of E_D and a generic polynomial for cyclic cubic extensions. We shall call an extension K/F an *elementary cubic extension* if K is a finite compositum of cubic extensions over F. Compared with related results as above, in our method, the extension degree $[L : \mathbb{Q}]$ often becomes small. The question of finding a small elementary cubic extension with rank $\geq l$ seems to be of independent interest.

2. Main Results

Let

$$S_D = \left\{ b \in \mathbb{Z} \mid D = a^2 + 3ab + 9b^2 \ \ (a, b \in \mathbb{Z},\ b > 0), \right. \\ \left. \text{where } a \text{ is relatively prime to } b \right\}. \tag{2}$$

We denote by $\overline{F}$ a fixed algebraic closure of F. Let $F(\sqrt[3]{S_D})$ denote the field obtained by adjoining a real number $\sqrt[3]{b}$ for each $b \in S_D$ to F. Since the set S_D is finite, the extension $F(\sqrt[3]{S_D})/F$ is also finite. The main result of the paper is the following theorem.

Theorem 1. *Let D be any square-free product of distinct primes $p_1, \ldots, p_k \in \mathbb{Z}$ satisfying $p_i \equiv 1 \pmod 3$. Then, for any positive*

Table 1: rank $E_D(L) \geq k$.

D	L	rank $E_D(L)$
$13 \cdot 421$	$\mathbb{Q}$	≥ 2
$19 \cdot 277$	$\mathbb{Q}$	≥ 2
$31 \cdot 373$	$\mathbb{Q}$	≥ 2
$109 \cdot 1153$	$\mathbb{Q}$	≥ 2
$7 \cdot 19 \cdot 37$	$\mathbb{Q}(\sqrt[3]{17})$	≥ 3
$7 \cdot 19 \cdot 229$	$\mathbb{Q}(\sqrt[3]{3})$	≥ 3
$7 \cdot 43 \cdot 127$	$\mathbb{Q}(\sqrt[3]{61})$	≥ 3
$7 \cdot 43 \cdot 127$	$\mathbb{Q}(\sqrt[3]{73})$	≥ 3
$109 \cdot 349 \cdot 457$	$\mathbb{Q}(\sqrt[3]{307})$	≥ 3

integer l ($\leq k$), we can construct a subfield L of $\mathbb{Q}(\sqrt[3]{S_D})$ such that

$$\operatorname{rank} E_D(L) \geq l, \quad [L : \mathbb{Q}] \leq 3^l. \tag{3}$$

As we will see in Section 4, the field L in Theorem 1 is effectively computable from the decomposition of the primes $p_1, \ldots, p_k$ in $\mathbb{Z}[\zeta_3]$, where ζ_3 denotes a primitive cube root of unity. See Remark 4 for this computation. It will also turn out in Section 4 that the field L is constructed using certain products of prime elements in $\mathbb{Z}[\zeta_3]$ above each prime p_i, and in particular L depends only on the primes $p_1, \ldots, p_k$ ($\equiv 1 \pmod 3$) and a positive integer l ($\leq k$). One question arises here: how small can $[L : \mathbb{Q}]$ be when $p_1, \ldots, p_k$ vary through the set of all primes $p \equiv 1 \pmod 3$ for a fixed integer l ($\leq k$)? We give some examples in the case $l = k$ (Table 1).

Corollary 2. *Let D be any square-free product of distinct primes $p_1, \ldots, p_k \in \mathbb{Z}$ satisfying $p_i \equiv 1 \pmod 3$. Then, for any positive integer l, we can construct a subfield L' of $\mathbb{Q}(\sqrt[3]{S_{dD}}, \sqrt[3]{d})$ with some integer d such that*

$$\operatorname{rank} E_D\left(L'\right) \geq l, \quad \left[L' : \mathbb{Q}\right] \leq 3^{l+1}. \tag{4}$$

Note that our construction in Theorem 1 (resp., Corollary 2) differs from Dokchitser's [5], and the extension degree $[L : \mathbb{Q}]$ (resp., $[L' : \mathbb{Q}]$) is often smaller than or equal to 3^{l-1} (resp., 3^l).

3. A Connection between the Elliptic Curve E_D and a Generic Polynomial for Cyclic Cubic Extensions

In this section, let F be a perfect field with characteristic not equal to 3, and let D be any element in F^*. The elliptic curve E_D has a rational 3-torsion point $(0,0)$ and admits a 3-isogeny $\phi : E_D \to \widehat{E_D} = E_D/\langle(0,0)\rangle$ over F. The 3-isogeny ϕ can be explicitly written as

$$\begin{aligned} \phi : E_D &\longrightarrow \widehat{E_D} \\ (x, y) &\longmapsto \left(\frac{x^3 + D^2}{Dx^2}, \frac{-(-y/D)^3 + 3(-y/D) - 1}{(-y/D)(-y/D - 1)}\right). \end{aligned} \tag{5}$$

The isogenous elliptic curve is given by the form

$$\widehat{E_D} : t^2 + 3t + 9 = Ds^3. \tag{6}$$

Taking Galois cohomology of the short exact sequence $0 \to E_D[\phi] \to E_D \xrightarrow{\phi} \widehat{E_D} \to 0$ yields a Kummer map

$$\begin{aligned} \frac{\widehat{E_D}(F)}{\phi(E_D(F))} &\hookrightarrow H^1(F, E_D[\phi]) \\ P &\longmapsto \left\{\sigma \longmapsto \phi^{-1}(P)^{\sigma-1}\right\}. \end{aligned} \tag{7}$$

Combining this map with the map

$$\begin{aligned} H^1(F, E_D[\phi]) &= \operatorname{Hom}\left(\operatorname{Gal}\left(\frac{\overline{F}}{F}\right), \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \\ &\twoheadrightarrow \mathfrak{Gal}\left(\frac{C_3}{F}\right) \\ \xi &\longmapsto \overline{F}^{\operatorname{Ker}\xi}, \end{aligned} \tag{8}$$

where $\mathfrak{Gal}(C_3/F)$ denotes the set of all cyclic cubic extensions over F together with F,

$$\begin{aligned} &\mathfrak{Gal}\left(\frac{C_3}{F}\right) \\ &:= \left\{\frac{K}{F} : \text{a Galois extension in } \overline{F} \,\middle|\, \operatorname{Gal}\left(\frac{K}{F}\right) \hookrightarrow C_3 \simeq \frac{\mathbb{Z}}{3\mathbb{Z}}\right\}, \end{aligned} \tag{9}$$

we have a map

$$\begin{aligned} \delta_F : \frac{\widehat{E_D}(F)}{\phi(E_D(F))} &\longrightarrow \mathfrak{Gal}\left(\frac{C_3}{F}\right) \\ P &\longmapsto F\left(\phi^{-1}(P)\right). \end{aligned} \tag{10}$$

It is well-known that the set $\mathfrak{Gal}(C_3/F)$ is parametrized by a generic polynomial for C_3-extensions [6]

$$\mathfrak{g}(x; t) = x^3 + tx^2 - (t+3)x + 1 \quad \text{with } t \in F, \tag{11}$$

which has discriminant $(t^2+3t+9)^2$. We shall denote by K_t the minimal splitting field of $\mathfrak{g}(x;t)$ over $F_0(t)$, where F_0 stands for the prime field in F. By the property of $\mathfrak{g}(x;t)$, for any $K/F \in \mathfrak{Gal}(C_3/F)$ there is some $t \in F$ so as to be $K = K_t F$ in $\overline{F}$.

The following proposition is the key observation in the present paper.

Proposition 3. *The connecting homomorphism δ_F is given by $\delta_F(s,t) = K_t F$ for any $(s,t) \in \widehat{E_D}(F)$. Furthermore, in the case where F is a number field, for any prime ideal $\mathfrak{p}$ of F not dividing 3, the $\mathfrak{p}$-component of the conductor of $K_t F/F$ is*

$$\begin{aligned} &\mathfrak{p} \quad \text{if } \nu_{\mathfrak{p}}\left(Ds^3\right) > 0,\ \nu_{\mathfrak{p}}(D) \not\equiv 0 \pmod 3 \\ &1 \quad \text{otherwise.} \end{aligned} \tag{12}$$

Here $\nu_{\mathfrak{p}} : F_{\mathfrak{p}}^* \to \mathbb{Z}$ denotes the normalized valuation of the $\mathfrak{p}$-adic completion $F_{\mathfrak{p}}$.

Proof. For any $(s,t) \in \widehat{E_D}(F)$, take a point (x,y) on E_D satisfying $\phi(x,y) = (s,t)$. By the form of the y-coordinate of (5), the splitting field K_t coincides with $F_0(t,y)$. Here $F(x,y) = F(x) = F(y)$. The latter part of the proposition follows from the equation $t^2 + 3t + 9 = Ds^3$ by using Proposition 8.1 in [7]. □

4. A Method for Constructing an Elementary Cubic Extension

From now on, let D be a square-free product of distinct primes $p_1, \ldots, p_k \in \mathbb{Z}$ satisfying $p_i \equiv 1 \pmod 3$, and let l be a positive integer $\leq k$.

Recall a result of Komatsu [8] that the set of all cyclic cubic fields (i.e., cyclic cubic extensions over $\mathbb{Q}$) of conductor $D = p_1 \cdots p_k$ has a one-to-one correspondence to the subset of the algebraic torus $T(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ consisting of 2^{k-1} elements of the form

$$c_{p_1} \pm_T c_{p_2} \pm_T \cdots \pm_T c_{p_{k-1}} \pm_T c_{p_k}, \tag{13}$$

where $+_T$ is the composition of $T(\mathbb{Q})$ given by $a/b +_T c/d = (a/b \cdot c/d - 1)/(a/b + c/d + 1)$, $-_T$ denotes the inverse of $+_T$ given by $-_T a/b = -a/b - 1$, and $c_{p_i} = e_{p_i}/f_{p_i} \in T(\mathbb{Q})$ is a rational number so that the pair (e_{p_i}, f_{p_i}) of integers is unique in the sense of Lemma 2.1 in [8] (especially $f_{p_i} \equiv 0 \pmod 3$, which will be used in the following) and satisfies

$$e_{p_i}^2 + e_{p_i} f_{p_i} + f_{p_i}^2 = \left(e_{p_i} - f_{p_i}\zeta_3\right)\left(e_{p_i} - f_{p_i}\zeta_3^2\right) = p_i. \tag{14}$$

There is a group isomorphism $\varphi : T \to \mathbb{G}_m$ mapping t to $(t - \zeta_3)/(t - \zeta_3^{-1})$, where $\mathbb{G}_m$ denotes the multiplicative group. Then the composition $+_T$ is written as $a/b +_T c/d = \varphi^{-1}(\varphi(a/b)\varphi(c/d))$.

Now, let $\pi_i := e_{p_i} - f_{p_i}\zeta_3$, which is a prime element in $\mathbb{Z}[\zeta_3]$ dividing p_i, and let $\overline{\alpha}$ denote the complex conjugation of α. For each i $(1 \leq i \leq l)$, let

$$t_i := \begin{cases} 3\left(c_{p_1} +_T \cdots +_T c_{p_k}\right) & \text{if } i = 1 \\ 3\left(c_{p_1} +_T \cdots +_T c_{p_{i-1}} -_T c_{p_i} +_T c_{p_{i+1}} +_T \cdots +_T c_{p_k}\right) & \text{if } i \neq 1. \end{cases} \tag{15}$$

Since $f_{p_i} \equiv 0 \pmod 3$ for each i, there exist unique integers a_i, b_i so that

$$a_i - 3b_i\zeta_3 = \begin{cases} \pi_1 \cdots \pi_k & \text{if } i = 1 \\ \pi_1 \cdots \pi_{i-1} \cdot \overline{\pi_i} \cdot \pi_{i+1} \cdots \pi_k & \text{if } i \neq 1, \end{cases} \tag{16}$$

where a_i is relatively prime to $3b_i$. Then $\varphi(t_i/3) = (a_i - 3b_i\zeta_3)/\overline{(a_i - 3b_i\zeta_3)}$, and thus $t_i = 3 \cdot \varphi^{-1}(\varphi(t_i/3)) = a_i/b_i$.

Let $L := \mathbb{Q}(\{\sqrt[3]{b_i} \mid 1 \leq i \leq l\})$.

Remark 4. In order to calculate $t_i = a_i/b_i$, we have only to know the values c_{p_i} by the definition of t_i. For example, one can use Table 5.1 in [8], in which the values c_p for $p < 1000$ are listed. From the construction of L, the field L depends only on $D = p_1 \cdots p_k$ and a positive integer l $(\leq k)$.

Lemma 5. *Every prime number p_i $(1 \leq i \leq l)$ is unramified in L.*

Proof. Since p_i does not divide $3b_1 \cdots b_l$ and is unramified in $\mathbb{Q}(\zeta_3)$, it turns out that p_i is unramified in $\mathbb{Q}(\zeta_3, \{\sqrt[3]{b_i} \mid 1 \leq i \leq l\})$ by Kummer theory. Hence, p_i is also unramified in $L = \mathbb{Q}(\{\sqrt[3]{b_i} \mid 1 \leq i \leq l\})$. □

Proposition 6. *$K_{t_i}L \in \operatorname{Im}\delta_L - \{L\}$ for any i $(1 \leq i \leq l)$.*

Proof. Since $(a_i - 3b_i\zeta_3)\overline{(a_i - 3b_i\zeta_3)} = D$, we have

$$t_i^2 + 3t_i + 9 = D\left(\sqrt[3]{b_i}^{-2}\right)^3. \tag{17}$$

Thus $(\sqrt[3]{b_i}^{-2}, t_i) \in \widehat{E_D}(L)$. It follows from Proposition 3 that $\delta_L(\sqrt[3]{b_i}^{-2}, t_i) = K_{t_i}L$, and hence $K_{t_i}L \in \operatorname{Im}\delta_L$. Combining Proposition 3 with Lemma 5 yields $K_{t_i}L \neq L$. □

Proposition 7. $\dim_{\mathbb{F}_3} \widehat{E_D}(L)/\phi(E_D(L)) \geq l$.

Proof. One can easily verify that $(0, 3\zeta_3)$ is a 3-torsion point in $\widehat{E_D}(L(\zeta_3))$, and the rational function $f := (t - 3\zeta_3)/D^2$ on $\widehat{E_D}$ has a zero of order 3 at $(0, 3\zeta_3)$ and a pole of order 3 at infinity and satisfies

$$f \circ \phi = \left(\frac{y - D\zeta_3}{Dx}\right)^3. \tag{18}$$

Then there exists an injective homomorphism (see [9] for details):

$$\begin{aligned} \frac{\widehat{E_D}(L(\zeta_3))}{\phi(E_D(L(\zeta_3)))} &\longrightarrow \frac{L(\zeta_3)^*}{L(\zeta_3)^{*3}} \\ (s,t) &\longmapsto D(t - 3\zeta_3)L(\zeta_3)^{*3} \\ (0, 3\zeta_3) &\longmapsto D^2 L(\zeta_3)^{*3}. \end{aligned} \tag{19}$$

Combining this with the natural map

$$\frac{\widehat{E_D}(L)}{\phi(E_D(L))} \longrightarrow \frac{\widehat{E_D}(L(\zeta_3))}{\phi(E_D(L(\zeta_3)))}, \tag{20}$$

we have the homomorphism

$$\begin{aligned} \frac{\widehat{E_D}(L)}{\phi(E_D(L))} &\longrightarrow \frac{L(\zeta_3)^*}{L(\zeta_3)^{*3}} \\ (s,t) &\longmapsto D(t - 3\zeta_3)L(\zeta_3)^{*3}. \end{aligned} \tag{21}$$

By Proposition 6, the point $(\sqrt[3]{b_i}^{-2}, t_i) \in \widehat{E_D}(L)$ corresponds to $D(t_i - 3\zeta_3)L(\zeta_3)^{*3} = D(a_i - 3b_i\zeta_3)L(\zeta_3)^{*3}$ via the above homomorphism. It thus suffices to show that $\{D(a_i - 3b_i\zeta_3) \mid 1 \leq i \leq l\}$ is linearly independent in $L(\zeta_3)^*/L(\zeta_3)^{*3}$. Then the points $\{(\sqrt[3]{b_i}^{-2}, t_i) \mid 1 \leq i \leq l\}$ must be linearly independent in $\widehat{E_D}(L)/\phi(E_D(L))$. Assume that

$$\gamma := \prod_{1 \leq i \leq l} \{D(a_i - 3b_i\zeta_3)\}^{r_i} \in L(\zeta_3)^{*3} \tag{22}$$

for some integers $\{r_i\}_{1\le i\le l}$. For each i, let $\mathfrak{p}_i$ be a prime ideal of $L(\zeta_3)$ above the prime element $\pi_i \in \mathbb{Q}(\zeta_3)$. For any prime ideal $\mathfrak{p}$ of $L(\zeta_3)$, we denote by $\nu_{\mathfrak{p}} : L(\zeta_3)^*_{\mathfrak{p}} \to \mathbb{Z}$ the normalized (additive) valuation of the $\mathfrak{p}$-adic completion $L(\zeta_3)_{\mathfrak{p}}$. By Lemma 5, we have $\nu_{\overline{\mathfrak{p}_i}}(\overline{\pi_i}) = 1$. Then

$$\nu_{\overline{\mathfrak{p}_j}}(\gamma) = \begin{cases} \sum_{i=1}^{l} r_i \equiv 0 \pmod 3 & \text{if } j = 1 \\ \sum_{i=1}^{l} r_i + r_j \equiv 0 \pmod 3 & \text{if } j \neq 1, \end{cases} \tag{23}$$

and hence $r_i \equiv 0 \pmod 3$ for each i. This proves that $\{D(a_i - 3b_i\zeta_3) \mid 1 \le i \le l\}$ is linearly independent in $L(\zeta_3)^*/L(\zeta_3)^{*3}$. □

Let $\widehat{\phi} : \widehat{E_D} \to E_D$ be the dual isogeny to ϕ. Since the composition $\widehat{\phi}\circ\phi$ (resp., $\phi\circ\widehat{\phi}$) is the multiplication-by-3 map on E_D (resp., $\widehat{E_D}$), we have an exact sequence of $\mathbb{F}_3$-vector spaces

$$\begin{aligned} 0 \longrightarrow \frac{\widehat{E_D}(L)\left[\widehat{\phi}\right]}{\phi\left(E_D(L)[3]\right)} \longrightarrow \frac{\widehat{E_D}(L)}{\phi\left(E_D(L)\right)} \xrightarrow{\widehat{\phi}} \frac{E_D(L)}{[3]\,E_D(L)} \\ \longrightarrow \frac{E_D(L)}{\widehat{\phi}\left(\widehat{E_D}(L)\right)} \longrightarrow 0, \end{aligned} \tag{24}$$

and thus

$$\begin{aligned} \operatorname{rank} E_D(L) = \dim_{\mathbb{F}_3} \frac{E_D(L)}{\widehat{\phi}\left(\widehat{E_D}(L)\right)} + \dim_{\mathbb{F}_3} \frac{\widehat{E_D}(L)}{\phi\left(E_D(L)\right)} \\ - \dim_{\mathbb{F}_3} \frac{\widehat{E_D}(L)\left[\widehat{\phi}\right]}{\phi\left(E_D(L)[3]\right)} \\ - \dim_{\mathbb{F}_3} E_D(L)[3]. \end{aligned} \tag{25}$$

Since $\zeta_3 \notin L$, the group $\widehat{E_D}(L)[\widehat{\phi}]$ is trivial and the group $E_D(L)[3]$ is generated by the point $(0,0)$. We see that

$$\operatorname{rank} E_D(L) = \dim_{\mathbb{F}_3} \frac{E_D(L)}{\widehat{\phi}\left(\widehat{E_D}(L)\right)} + \dim_{\mathbb{F}_3} \frac{\widehat{E_D}(\mathrm{L})}{\phi\left(E_D(L)\right)} - 1. \tag{26}$$

Proof of Theorem 1. By Proposition 7, it suffices to show that $(0,0) \notin \widehat{\phi}(\widehat{E_D}(L))$. Then

$$\dim_{\mathbb{F}_3} \frac{E_D(L)}{\widehat{\phi}\left(\widehat{E_D}(L)\right)} \ge 1, \quad \text{and hence } \operatorname{rank} E_D(L) \ge l. \tag{27}$$

Assume that there exists some point $Q \in \widehat{E_D}(L)$ such that $\widehat{\phi}(Q) = (0,0)$. Then $Q \in \widehat{E_D}(L)[3]$. It follows from the 3-division polynomial $3x(x^3 - 27D^2)$ for $\widehat{E_D}$ that there is no nontrivial 3-torsion point in $\widehat{E_D}(L)$. Therefore, Q must be the trivial element in $\widehat{E_D}(L)[3]$. This contradicts the assumption, and hence $(0,0) \notin \widehat{\phi}(\widehat{E_D}(L))$. □

Proof of Corollary 2. By Theorem 1, we have only to consider the case $l > k$. (In the case where $l \le k$, take $d = 1$.) Take distinct prime numbers $\{q_1, \ldots, q_{l-k}\}$ not dividing D with $q_i \equiv 1 \pmod 3$. Let $d := q_1 \cdots q_{l-k}$. Then, applying Theorem 1 to the elliptic curve E_{dD}, there exists some subfield $L \subset \mathbb{Q}(\sqrt[3]{S_{dD}})$ so as to be rank $E_{dD}(L) \ge l$ with $[L : \mathbb{Q}] \le 3^l$. Let $L' := L(\sqrt[3]{d})$. Then, it is easily seen that E_{dD} is isomorphic over L' to E_D. Therefore, $\operatorname{rank} E_D(L') = \operatorname{rank} E_{dD}(L') \ge \operatorname{rank} E_{dD}(L) \ge l$. Here $[L' : \mathbb{Q}] \le 3 \cdot [L : \mathbb{Q}] \le 3^{l+1}$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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