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Research Article

Generalized Malcev-Neumann Series Modules with the Beachy-Blair Condition

Mohamed A. Farahat[1,2]

[1]*Department of Mathematics and Statistics, Faculty of Science, Taif University, P.O. Box 888, Al-Hawiyah, Taif 21974, Saudi Arabia*
[2]*Mathematics Department, Faculty of Science, Al-Azhar University, P.O. Box 11884, Nasr City, Cairo, Egypt*

Correspondence should be addressed to Mohamed A. Farahat; m_farahat79@yahoo.com

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We introduce a new class of extension rings called the *generalized Malcev-Neumann series ring* $R((S;\sigma;\tau))$ with coefficients in a ring R and exponents in a strictly ordered monoid S which extends the usual construction of Malcev-Neumann series rings. Ouyang et al. in 2014 introduced the modules with the Beachy-Blair condition as follows: A right R-module satisfies the right Beachy-Blair condition if each of its faithful submodules is cofaithful. In this paper, we study the relationship between the right Beachy-Blair condition of a right R-module M_R and its Malcev-Neumann series module extension $M((S))_{R((S;\sigma;\tau))}$.

1. Introduction

Throughout this paper R denotes an associative ring with identity; $(S,\cdot,\leqslant)$ is a strictly ordered monoid (i.e., $(S,\leqslant)$ is an ordered monoid satisfying the conditions that if $s<s'$, then $st<s't$ and $ts<ts'$ for $s,s',t\in S$). Recall that a subset X of $(S,\leqslant)$ is said to be *artinian* if every strictly decreasing sequence of elements of X is finite and that X is *narrow* if every subset of pairwise order-incomparable elements of X is finite. Suppose the two maps $\sigma : S \to \mathrm{End}(R)$ and $\tau : S\times S \to U(R)$ (the group of invertible elements of R). Let $A = R((S;\sigma;\tau))$ denote the set of all formal sums $f=\sum_{x\in S} a_x\overline{x}$ such that $\mathrm{supp}(f)=\{x\in S \mid a_x\neq 0\}$ is an artinian and narrow subset of S, with componentwise addition and the multiplication rule is given by

$$\left(\sum_{x\in S} a_x\overline{x}\right)\left(\sum_{y\in S} b_y\overline{y}\right) = \sum_{z\in S}\left(\sum_{\{(x,y)\mid xy=z\}} a_x\sigma_x\left(b_y\right)\tau\left(x,y\right)\right)\overline{z}, \tag{1}$$

for each $\sum_{x\in S} a_x\overline{x}$ and $\sum_{y\in S} b_y\overline{y}\in A$. In order to ensure the associativity, it is necessary to impose two additional conditions on σ and τ: namely, for all $x,y,z\in S$,

(i) $\sigma_x(\tau(y,z))\tau(x,yz)=\tau(x,y)\tau(xy,z)$,

(ii) $\sigma_x\sigma_y=\eta(x,y)\sigma_{xy}$, where $\eta(x,y)$ denotes the automorphism of R defined by

$$\eta\left(x,y\right)\left(r\right)=\tau\left(x,y\right)r\tau\left(x,y\right)^{-1}\quad \forall r\in R. \tag{2}$$

It is now routine to check that $A=R((S;\sigma;\tau))$ is a ring which is called *the ring of generalized Malcev-Neumann series*. We can assume that the identity element of A is $\overline{1}$; this means that

$$\sigma_1=\mathrm{Id}_R,\qquad \tau\left(x,1\right)=\tau\left(1,x\right)=1. \tag{3}$$

In this case $r\mapsto r\overline{1}$ is an embedding of R as a subring into A.

For each $f\in A\setminus\{0\}$ we denote by $\pi(f)$ the set of minimal elements of $\mathrm{supp}(f)$. If $(S,\leq)$ is a strictly totally ordered monoid, then $\mathrm{supp}(f)$ is a nonempty well-ordered subset of S and $\pi(f)$ consists of only one element.

Clearly, the above construction generalizes the construction of Malcev-Neumann series rings, in case of $S=G$ (an ordered group), which was introduced independently by Malcev and Neumann (see [1, 2]).

If the order $\leq$ is the trivial order, then $A=R((S;\sigma;\tau))$ is the usual crossed product ring $R[S;\sigma;\tau]$. Also, if the monoid S has the trivial order and τ is trivial, then $A=R((S;\sigma;\tau))$ is the usual skew monoid ring $R[S;\sigma]$. However if the monoid S has the trivial order and σ is trivial, then $A = R((S;\sigma;\tau))$ is the usual twisted monoid ring $R[S;\tau]$. Finally, if the monoid S has the trivial order and σ and τ are trivial, then $A = R((S;\sigma;\tau))$ is the usual monoid ring $R[S]$ (see Sections 3.2 and 3.3 in [3]).

Moreover, if α is a ring endomorphism of R, set $S = \mathbb{Z}_{\geq 0}$ endowed with the trivial order. Define $\sigma : S \to \mathrm{End}(R)$ via $\sigma(x) = \alpha^x$ for every $x \in \mathbb{Z}_{\geq 0}$ and $\tau(x, y) = 1$ for any $x, y \in \mathbb{Z}$. We have $A = R((S;\sigma;\tau))$ is the usual skew polynomial ring $R[x,\alpha]$. However if $\leq$ is the usual order, then $A = R((S;\sigma;\tau))$ is the usual skew power series ring $R[[x,\sigma]]$. If α is a ring automorphism of R, $S = \mathbb{Z}$ and $\leq$ is the usual order, then $A = R((S;\sigma;\tau))$ is the usual ring of skew Laurent power series $R[[x, x^{-1}, \alpha]]$.

At the same time, if we set also $\sigma(s) = \mathrm{Id}_R \in \mathrm{End}(R)$ for all $s \in S$, then it is easy to check that polynomial rings, Laurent polynomial rings, formal power series rings, and Laurent power series rings are special cases of $A = R((S;\sigma;\tau))$.

If M_R is a unitary right R-module, then *the Malcev-Neumann series module* $B = M((S))$ is the set of all formal sums $\sum_{x\in S} m_x\overline{x}$ with coefficients in M and artinian and narrow supports, with pointwise addition and scalar multiplication rule is defined by

$$\left(\sum_{x\in S} m_x\overline{x}\right)\left(\sum_{y\in S} a_y\overline{y}\right) = \sum_{z\in S}\left(\sum_{\{(x,y)\mid xy=z\}} m_x\sigma_x\left(a_y\right)\tau\left(x,y\right)\right)\overline{z}, \tag{4}$$

where $\sum_{x\in S} m_x\overline{x} \in B$ and $\sum_{y\in S} a_y\overline{y} \in A$. One can easily check that (i) and (ii) ensure that $M((S))$ is a unitary right A-module. For each $\varphi \in B \setminus \{0\}$ we denote by $\pi(\varphi)$ the set of minimal elements of $\mathrm{supp}(\varphi)$. If $(S, \leq)$ is a strictly totally ordered monoid, then $\mathrm{supp}(\varphi)$ is a nonempty well-ordered subset of S and $\pi(\varphi)$ consists of only one element.

Recall from Faith [4] that a ring R is called a *right zip ring* and if the right annihilator $\mathrm{r}_R(X)$ of a subset $X \subseteq R$ is zero, then $\mathrm{r}_R(X_0) = 0$ for a finite subset X_0 of X. Although the concept of zip rings was initiated by Zelmanowitz [5] it was not called so at that time.

Recall from [6] that a right R-module M_R is called a *right zip* module provided that if the right annihilator of a subset X of M_R is zero, then there exists a finite subset $X_0 \subseteq X$ such that $\mathrm{r}_R(X_0) = 0$.

According to Rodríguez-Jorge [7], a ring R satisfies *the right Beachy-Blair condition* if its faithful right ideals are cofaithful; that is, if I is a right ideal of R such that $\mathrm{r}_R(I)$ vanishes, then $\mathrm{r}_R(I_0) = 0$ for a finite subset I_0 of I. Clearly, a right zip ring is a right Beachy-Blair ring.

Ouyang et al. in [8] generalized the right Beachy-Blair condition from rings into modules as follows: A right R-module M_R is called *module with the Beachy-Blair condition* provided that if the right annihilator of a submodule N_R of M_R is zero, then there exists a finite subset $N_0 \subseteq N$ such that $\mathrm{r}_R(N_0) = 0$.

The main aim of the present paper is to investigate conditions for the Malcev-Neumann series modules $M((S))_{R((S;\sigma;\tau))}$ to satisfy the right Beachy-Blair condition. The proofs of our results obtained here are very similar to those obtained by Ouyang et al. in [8] and by Salem et al. in [9].

2. Generalized Malcev-Neumann Series Modules with the Beachy-Blair Condition

We start this section with the following notions and definitions.

Let V be a subset of M_R; then

$$V((S)) = \left\{\varphi = \sum_{x\in S} m_x\overline{x} \in B \mid 0 \neq m_x \in V, x \in \mathrm{supp}(\varphi)\right\}. \tag{5}$$

Definition 1. A ring R is called S-compatible if, for all $a, b \in R$ and $x \in S$, $ab = 0$ if and only if $a\sigma_x(b) = 0$.

Definition 2. A right R-module M_R is called S-compatible if, for each $m \in M$, $a \in R$, and $x \in S$, $ma = 0$ if and only if $m\sigma_x(a) = 0$.

Definition 3. A ring R is called S-Armendariz if whenever $fg = 0$ implies $a_x\sigma_x(b_y) = 0$ for each $x \in \mathrm{supp}(f)$ and $y \in \mathrm{supp}(g)$, where $f = \sum_{x\in S} a_x\overline{x}$ and $g = \sum_{y\in S} b_y\overline{y}$ are elements of A.

We extend the S-Armendariz concept to modules as follows.

Definition 4. A right R-module M_R is called S-Armendariz if whenever $\varphi f = 0$ implies $m_x\sigma_x(a_y) = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$, where $\varphi = \sum_{x\in S} m_x\overline{x} \in B$ and $f = \sum_{y\in S} a_y\overline{y} \in A$.

It is clear that R is an S-Armendariz (S-compatible) ring if and only if R_R is an S-Armendariz (S-compatible) module.

For a subset U of M_R, we define $\mathrm{r}_A(U)$ as the set

$$\mathrm{r}_A(U) = \left\{f \in A \mid \left(u\overline{1}\right)f = 0 \text{ for each } u \in U\right\}. \tag{6}$$

Lemma 5. *Let M_R be a right R-module. Then $\mathrm{r}_A(U) = \mathrm{r}_R(U)((S;\sigma;\tau))$, for any subset U of M_R.*

Proof. Let $f = \sum_{s\in S} a_s\overline{s} \in \mathrm{r}_A(U)$. Then for each $u \in U$ we have $(u\overline{1})f = 0$. Thus

$$0 = \left(u\overline{1}\right)\left(\sum_{s\in S} a_s\overline{s}\right) = \sum_{s\in S} u\sigma_1\left(a_s\right)\tau\left(1,s\right)\overline{s} = \sum_{s\in S} ua_s\overline{s}, \tag{7}$$

which implies that $ua_s = 0$ for each $s \in \mathrm{supp}(f)$. Hence $a_s \in \mathrm{r}_R(U)$ for each $s \in \mathrm{supp}(f)$. So $f \in \mathrm{r}_R(U)((S;\sigma;\tau))$ and $\mathrm{r}_A(U) \subseteq \mathrm{r}_R(U)((S;\sigma;\tau))$.

On the other hand, suppose that $f = \sum_{s\in S} a_s\overline{s} \in \mathrm{r}_R(U)((S;\sigma;\tau))$; then $a_s \in \mathrm{r}_R(U)$ for each $s \in \mathrm{supp}(f)$. Thus $ua_s = 0$ for each $u \in U$, which implies that $u\sigma_1(a_s)\tau(1,s) = 0$ for each $u \in U$ and $s \in \mathrm{supp}(f)$. Hence $(u\overline{1})f = 0$ and $f \in \mathrm{r}_A(U)$. So $\mathrm{r}_R(U)((S;\sigma;\tau)) \subseteq \mathrm{r}_A(U)$. Therefore $\mathrm{r}_A(U) = \mathrm{r}_R(U)((S;\sigma;\tau))$. $\square$

When $M_R = R_R$ we have the following consequence of Lemma 5.

Corollary 6. *Consider* $\mathrm{r}_A(U) = \mathrm{r}_R(U)((S;\sigma;\tau))$, *for any subset* U *of* R.

Note the following: for $\varphi = \sum_{x\in S} m_x\overline{x} \in B$, let $\mathrm{C}_\varphi = \{m_x \mid x \in S\}$ and for a subset $V \subseteq M((S))$, we have $\mathrm{C}_V = \cup_{\varphi\in V}\mathrm{C}\varphi$.

Lemma 7. *Let* M_R *be an* S*-compatible and* S*-Armendariz* R*-module. Then*

$$\mathrm{r}_A(V) = \mathrm{r}_R(\mathrm{C}_V)((S;\sigma;\tau)) \tag{8}$$

for any $V \subseteq B$.

Proof. Let $V \subseteq B$ and $T = \mathrm{C}_V = \cup_{\varphi\in V}\mathrm{C}\varphi = \cup_{\varphi\in V}\{m_x \mid x \in S\}$. We show that $\mathrm{r}_A(V) = \mathrm{r}_R(T)((S;\sigma;\tau))$ and it is enough to show that $\mathrm{r}_A(\varphi) = \mathrm{r}_R(\mathrm{C}\varphi)((S;\sigma;\tau))$ for each $\varphi = \sum_{x\in S} m_x\overline{x} \in V$. In fact, let $f = \sum_{y\in S} a_y\overline{y} \in \mathrm{r}_A(\varphi)$. Then $\varphi f = 0$. Since M_R is an S -Armendariz module, $m_x a_y = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Then $a_y \in \mathrm{r}_R(\mathrm{C}_\varphi)$ for each $y \in \mathrm{supp}(f)$. Thus $f \in \mathrm{r}_R(\mathrm{C}\varphi)((S;\sigma;\tau))$ and $\mathrm{r}_A(\varphi) \subseteq \mathrm{r}_R(\mathrm{C}\varphi)((S;\sigma;\tau))$. Now, let $f = \sum_{y\in S} a_y\overline{y} \in \mathrm{r}_R(\mathrm{C}\varphi)((S;\sigma;\tau))$. Then $a_y \in \mathrm{r}_R(\mathrm{C}\varphi)$ for each $y \in \mathrm{supp}(f)$. Hence $m_x a_y = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Since M_R is S-compatible, it follows that $m_x\sigma_x(a_y) = 0$, which implies that $m_x\sigma_x(a_y)\tau(x,y) = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Consequently

$$0 = \sum_{z\in S}\left(\sum_{\{(x,y)\mid xy=z\}} m_x\sigma_x(a_y)\tau(x,y)\right)\overline{z} = \varphi f. \tag{9}$$

So $f \in \mathrm{r}_A(\varphi)$ and it follows that $\mathrm{r}_R(\mathrm{C}\varphi)((S;\sigma;\tau)) \subseteq \mathrm{r}_A(\varphi)$. So

$$\begin{aligned}\mathrm{r}_A(V) &= \bigcap_{\varphi\in V}\mathrm{r}_A(\varphi) = \bigcap_{\varphi\in V}\mathrm{r}_R(\mathrm{C}\varphi)((S;\sigma;\tau))\\ &= \left(\bigcap_{\varphi\in V}\mathrm{r}_R(\mathrm{C}\varphi)\right)((S;\sigma;\tau))\\ &= \mathrm{r}_R(T)((S;\sigma;\tau)) = \mathrm{r}_R(\mathrm{C}_V)((S;\sigma;\tau)).\end{aligned} \tag{10}$$

□

For a right R-module M_R, we define

$$\begin{aligned}\mathrm{r}_R(2^M) &= \{\mathrm{r}_R(U) \mid U \subseteq M\},\\ \mathrm{r}_A(2^B) &= \{\mathrm{r}_A(V) \mid V \subseteq B\}.\end{aligned} \tag{11}$$

Lemma 5 gives us the map $\Pi : \mathrm{r}_R(2^M) \to \mathrm{r}_A(2^B)$ defined by $\Pi(I) = I((S;\sigma;\tau))$ for every $I \in \mathrm{r}_R(2^M)$. Obviously Π is an injective map.

In the following lemma we show that Π is a bijective map if and only if M_R is S-Armendariz.

Lemma 8. *Let* M_R *be an* S*-compatible* R*-module. The following conditions are equivalent.*

(1) M_R *is an* S*-Armendariz* R*-module.*

(2) $\Pi : \mathrm{r}_R(2^M) \to \mathrm{r}_A(2^B)$ *defined by* $\Pi(I) = I((S;\sigma;\tau))$ *is a bijective map.*

Proof. (1)⇒(2).

It is only necessary to show that Π is surjective. Let $V \subseteq B$ and $T = \mathrm{C}_V$. Since $\Pi(\mathrm{r}_R(T)) = \mathrm{r}_R(T)((S;\sigma;\tau))$, the proof of this direction follows directly from Lemma 7.

(2)⇒(1).

Let $f = \sum_{y\in S} a_y\overline{y} \in A$ and $\varphi = \sum_{x\in S} m_x\overline{x} \in B$ such that $\varphi f = 0$. Then $f \in \mathrm{r}_A(\varphi)$. By assumption $\mathrm{r}_A(\varphi) = T((S;\sigma;\tau))$ for some right ideal T of R. Hence $f \in T((S;\sigma;\tau))$ which implies that $a_y \in T \subseteq \mathrm{r}_A(\varphi)$ for each $y \in \mathrm{supp}(f)$. So, $\varphi(a_y\overline{1}) = 0$ and we have that

$$0 = \left(\sum_{x\in S} m_x\overline{x}\right)(a_y\overline{1}) = \sum_{x\in S} m_x\sigma_x(a_y)\tau(x,1)\overline{x} \tag{12}$$

for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Thus $m_x\sigma_x(a_y) = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. So, M_R is an S-Armendariz module. □

Recall that a ring is reduced if it has no nonzero nilpotent elements. Reduced rings have been studied for over forty-eight years (see [10]). In 2004, the reduced ring concept was extended to modules by Lee and Zhou [11] as follows: a right R-module M_R is reduced if, for any $m \in M_R$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Clearly, if M_R is reduced, then, for all $m \in M_R$ and $a \in R$, $ma = 0$ implies $mRa = 0$. It is clear that R is a reduced ring if and only if R_R is a reduced module.

Now, we are able to prove the main result.

Theorem 9. *Let* M_R *be a reduced,* S*-compatible, and* S*-Armendariz right* R*-module. If* M_R *satisfies the right Beachy-Blair condition, then* B_A *satisfies the right Beachy-Blair condition.*

Proof. Suppose that a right R-module M_R satisfies the right Beachy-Blair condition and J is a right A-submodule of B such that $\mathrm{r}_A(J) = 0$.

From Lemma 8, we conclude that $\mathrm{r}_R(\mathrm{C}_J)((S;\sigma;\tau)) = \Pi(\mathrm{r}_R(\mathrm{C}_J)) = \mathrm{r}_A(J) = 0$. Thus $\mathrm{r}_R(\mathrm{C}_J) = 0$.

Let $\mathrm{C}_J R$ denote the right R-submodule of M_R generated by C_J. Since $\mathrm{C}_J \subset \mathrm{C}_J R$, we have $\mathrm{r}_R(\mathrm{C}_J R) \subset \mathrm{r}_R(\mathrm{C}_J) = 0$. Since M_R satisfies the right Beachy-Blair condition, there exists a finite subset

$$X = \left\{\sum_{i=1}^{n_t} q_i^t r_i^t \mid q_i^t \in \mathrm{C}_J, r_i^t \in R, 1 \le t \le k\right\} \subset \mathrm{C}_J R, \tag{13}$$

such that $\mathrm{r}_R(X) = 0$. Let

$$X_0 = \left\{q_1^1, q_2^1, \ldots, q_{n_1}^1, q_1^2, q_2^2, \ldots, q_{n_2}^2, q_1^k, q_2^k, \ldots, q_{n_k}^k\right\}. \tag{14}$$

Then X_0 is a finite subset of C_J. Now we will see that $\mathrm{r}_R(X_0) = 0$. Let $a \in \mathrm{r}_R(X_0)$; then $q_i^t a = 0$ for $1 \le i \le n_t$ and $1 \le t \le k$. Since M_R is a reduced R-module, then $q_i^t r_i^t a = 0$ for $1 \le i \le n_t$ and $1 \le t \le k$. Then for each $(\sum_{i=1}^{n_t} q_i^t r_i^t) \in X$, we have $(\sum_{i=1}^{n_t} q_i^t r_i^t)a = 0$. Therefore $a \in \mathrm{r}_R(X) = 0$, and so $\mathrm{r}_R(X_0) = 0$ is proved.

For each $q_i^t \in X_0$, there exists an element $\varphi_i^t \in J$ such that $q_i^t \in \mathrm{C}_{\varphi_i^t}$. Let V be a minimal subset of J such that $\varphi_i^t \in V$ for each $q_i^t \in X_0$; then V is a finite subset of J and $X_0 \subset \mathrm{C}_V$. Thus $\mathrm{r}_R(\mathrm{C}_V) \subset \mathrm{r}_R(X_0) = 0$. Now we show that $\mathrm{r}_A(V) = 0$. Let the contrary; that is, $\mathrm{r}_A(V) \neq 0$, and suppose that $f = \sum_{y \in S} b_y \overline{y} \in \mathrm{r}_A(V) \setminus \{0\}$; then $\varphi f = 0$ for each $\varphi = \sum_{x \in S} a_x \overline{x} \in V$. Let $y \in \mathrm{supp}(f)$; since M_R is an S-Armendariz and S-compatible module, we have $a_x b_y = 0$ for all $a_x \in \mathrm{C}_\varphi$ and each $\varphi \in V$. Hence $b_y \in \mathrm{r}_R(\mathrm{C}_V) = 0$, a contradiction. Hence $\mathrm{r}_A(V) = 0$ is proved. Thus B_A satisfies the right Beachy-Blair condition. □

When $M_R = R_R$ we have the following consequence of Theorem 9.

Corollary 10. *Suppose that R is a reduced, S-compatible, and S-Armendariz ring. If R satisfies the right Beachy-Blair condition, then A satisfies the right Beachy-Blair condition.*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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