Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2015, Article ID 638767, 4 pages
http://dx.doi.org/10.1155/2015/638767

Research Article

On 3-Regular Bipancyclic Subgraphs of Hypercubes

Y. M. Borse and S. R. Shaikh

Department of Mathematics, University of Pune, Pune 411 007, India

Correspondence should be addressed to Y. M. Borse; ymborse@math.unipune.ac.in

Received 31 July 2014; Accepted 15 April 2015

Academic Editor: Chris A. Rodger

The n-dimensional hypercube Q_n is *bipancyclic*; that is, it contains a cycle of every even length from 4 to 2^n. In this paper, we prove that Q_n $(n \geq 3)$ contains a 3-regular, 3-connected, bipancyclic subgraph with l vertices for every even l from 8 to 2^n except 10.

1. Introduction

The *cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$, and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1. A graph G with even number of vertices is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$. The *hypercube* Q_n of dimension n is a graph obtained by taking cartesian product of the complete graph K_2 on two vertices with itself n times; that is, $Q_n = K_2 \times K_2 \times \cdots \times K_2$ (n times). The hypercube Q_n is an n-regular, n-connected, bipartite, and bipancyclic graph with 2^n vertices. It is one of the most popular interconnection network topologies [1]. The bipancyclicity of a given network is an important factor in determining whether the network topology can simulate rings of various lengths. The connectivity of a network gives the minimum cost to disrupt the network. Regular subgraphs, bipancyclicity, and connectivity properties of hypercubes are well studied in the literature [2–6].

Since Q_n $(n \geq 2)$ is bipancyclic, it contains a 2-regular, 2-connected subgraph (cycle) with l vertices for every even integer l from 4 to 2^n. Suppose $3 \leq k \leq n$. Mane and Waphare [4] proved that Q_n contains a spanning k-regular, k-connected, bipancyclic subgraph. So the natural question arises; what are the other possible orders existing for k-regular, k-connected and bipancyclic subgraphs of Q_n? As $Q_n = Q_{n-k} \times Q_k$, Q_k can be regarded as a subgraph of Q_n. Hence Q_n has a k-regular, k-connected, bipancyclic subgraph with 2^k vertices. In this paper, we answer the question for $k = 3$. We prove that Q_n $(n \geq 3)$ contains a 3-regular, 3-connected, and bipancyclic subgraph with l vertices for every even integer l from 8 to 2^n except 10.

2. Proof

The cartesian product of a nontrivial path with the complete graph K_2 is a *ladder* graph. Let F be the graph obtained from a path $A_1, A_2, \ldots, A_m$ $(m \geq 4)$ by adding one extra edge A_1A_4. We call the graph $F \times K_2$ a *ladder type* graph on $2m$ vertices (see Figure 1).

Lemma 1. *A ladder graph is bipancyclic.*

Proof. Let L be a ladder graph with $2m$ vertices. Label the vertices of L by A_i's and B_i's so that L is the union of the paths $P_1 = A_1, A_2, \ldots, A_m$ and $P_2 = B_1, B_2, \ldots, B_m$ and the m edges A_iB_i for $i = 1, 2, \ldots, m$. Suppose $2 \leq l \leq m$. Let P_1' be the subpath of P_1 from A_1 to A_l and let P_2' be the subpath of P_2 from B_1 to B_l. Then $P_1' \cup P_2' \cup \{A_1B_1, A_lB_l\}$ is a cycle of length $2l$ in L. Hence L has a cycle of every even length from 4 to $|V(L)|$. □

The vertices of the hypercube Q_n can be labeled by the binary strings of length n so that two vertices are adjacent in Q_n if and only if their binary strings differ in exactly one coordinate. Denote by Q_{n-1}^j the subgraph of Q_n induced by the set of all vertices of Q_n each having first coordinate j for $j = 0, 1$. Then Q_{n-1}^0 and Q_{n-1}^1 are vertex-disjoint and each of them is isomorphic to Q_{n-1}. We can express Q_n as $Q_n = Q_{n-1}^0 \cup$

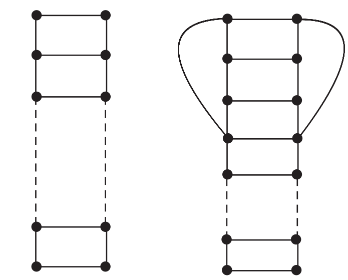

Figure 1

$Q^1_{n-1} \cup D$, where $D = \{XY \mid X \in V(Q^0_{n-1}) \text{ and } Y \in V(Q^1_{n-1})\}$. Note that D is a perfect matching in Q_n.

Lemma 2. *For every m with $4 \le m \le 2^{n-1}$, there exists a ladder type subgraph in Q_n ($n \ge 3$) with $2m$ vertices.*

Proof. We first prove that Q_n contains a Hamiltonian cycle C with a chord e which forms a 4-cycle with three edges of C. This is obvious for $n = 3$. Suppose $n \ge 4$. Write Q_n as $Q_n = Q^0_{n-1} \cup Q^1_{n-1} \cup D$. By induction, there exists a Hamiltonian cycle C_0 in Q^0_{n-1} with a chord e which forms a 4-cycle Z_0 with three edges of C_0. Let C_1 be the corresponding Hamiltonian cycle in Q^1_{n-1}. Let XY be any edge on C_0 which is not on Z_0 and let $X'Y'$ be the corresponding edge on C_1. Then XX' and YY' belong to D. Let $C = (C_0 - XY) \cup (C_1 - X'Y') \cup \{XX', YY'\}$. Then C is a Hamiltonian cycle in Q_n such that e is its chord which forms the 4-cycle Z_0 with three edges of C.

Now, we prove that Q_n contains a ladder type graph with $2m$ vertices. Obviously, Q_3 itself is a ladder type graph on 8 vertices. Suppose $n \ge 4$. By the above part, Q_{n-1} contains a Hamiltonian cycle C with a chord e which forms a 4-cycle with three edges of C. Label the vertices of C by A_i's so that $C = A_1, A_2, A_3, \ldots, A_{2^{n-1}}, A_1$ and $e = A_1A_4$. Let F be the subgraph of Q_{n-1} obtained by taking the union of the subpath $A_1, A_2, A_3, \ldots, A_m$ of C and the edge A_1A_4. Then $F \times K_2$ is a ladder type subgraph of $Q_{n-1} \times K_2 = Q_n$ with $2m$ vertices. □

As a consequence of a result of [7], we get the following lemma.

Lemma 3. *Let G_i be an n_i-regular, n_i-connected graph for $i = 1, 2$. Then the graph $G_1 \times G_2$ is $(n_1 + n_2)$-regular, $(n_1 + n_2)$-connected.*

It is well known that the hypercube Q_n does not contain the complete bipartite graph $K_{2,3}$ as a subgraph. The following result is the main theorem of this paper.

Theorem 4. *Let n be an integer such that $n \ge 3$. Then there exists a 3-regular, 3-connected, and bipancyclic subgraph of Q_n on l vertices if and only if l is an even integer with $8 \le l \le 2^n$ and $l \ne 10$.*

Proof. Suppose Q_n contains a 3-regular subgraph H with l vertices. By Handshaking Lemma, the sum of the degrees of all vertices of a graph is even. Hence $3l$ is even. Consequently, l is even. The minimum degree of H is three. Therefore H contains an even cycle. Since H is simple, $l \ge 4$. If $l = 4$, then H contains a triangle, a contradiction. Thus $l \ge 6$. Suppose $l = 6$. Then H must contain a cycle Z of length four. A vertex of H outside Z has at least two neighbours in Z giving a triangle or a $K_{2,3}$ in Q_n, which is a contradiction. Suppose $l = 10$. Let e be an edge of H. Without loss of generality, we may assume that the end vertices of e differ in the first coordinate. Write Q_n as $Q_n = Q^0_{n-1} \cup Q^1_{n-1} \cup D$. Then $e \in D$. Therefore H intersects with both Q^0_{n-1} and Q^1_{n-1}. Let H_j be a component of $H \cap Q^j_{n-1}$ for $j = 0, 1$. Then H_j is a subgraph of Q^j_{n-1} with minimum degree two and hence it contains a cycle. As Q_n is simple bipartite, H_j has at least four vertices. Since $|V(H)| = 10$, H_j is the only component of H in $H \cap Q^j_{n-1}$. We may assume that $|V(H_0)| \le |V(H_1)|$. Then $|V(H_0)| = 4$ or $|V(H_0)| = 5$. Let C_0 be an even cycle in H_0. Then $|C_0| = 4$. If H_0 has 5 vertices, then the vertex of H_0 which is not on C_0 is adjacent to at least two vertices of C_0 giving a triangle or a $K_{2,3}$ in Q^0_{n-1}, a contradiction. Consequently, H_0 has 4 vertices. Thus $H_0 = C_0$. Let C_1 be the cycle in Q^1_{n-1} corresponding to C_0. Since H is 3-regular, each vertex of C_0 has one neighbour in H_1 along an edge of D. Therefore all vertices of C_1 belong to H_1. As H_1 has six vertices, it has a vertex X which is not on C_1. Then X has no neighbour in H_0. Thus X has three neighbours in H_1. Therefore X has at least two neighbours in the 4-cycle C_1 giving a triangle or a $K_{2,3}$ in Q^1_{n-1}, a contradiction. Hence $l \ne 10$. Thus l is an even integer with $8 \le l \le 2^n$ and $l \ne 10$.

Now, we construct a 3-regular, 3-connected, bipancyclic subgraph of Q_n with l vertices for every even integer l with $8 \le l \le 2^n$ and $l \ne 10$. Suppose $l = 4m$ for some integer m with $2 \le m \le 2^{n-2}$. Write Q_n as $Q_n = Q_{n-1} \times K_2$. Since Q_{n-1} is a bipancyclic graph and $l/2$ is even, there is a cycle C of length $l/2$ in Q_{n-1}. By Lemma 3, $C \times K_2$ is a 3-regular, 3-connected subgraph of Q_n with l vertices. Let e be an edge of C. Then $(C - e) \times K_2$ is a ladder graph which spans $C \times K_2$. By Lemma 1, $C \times K_2$ is bipancyclic.

Suppose $l = 4m + 2$ with $3 \le m \le 2^{n-2} - 1$. Write Q_n as $Q_n = Q^0_{n-1} \cup Q^1_{n-1} \cup D$. As $4 \le m + 1 \le 2^{n-2}$, there exists a ladder type subgraph L_1 in Q^0_{n-1} on $2m + 2$ vertices by Lemma 2. Label the vertices of L_1 by A_i's and B_i's so that $A_1, A_2, \ldots, A_{m+1}$ and $B_1, B_2, \ldots, B_{m+1}$ are paths and A_iB_i is an edge of L_1 for $i = 1, 2, \ldots, m + 1$. Let L_2 be the ladder type subgraph of Q^1_{n-1} on $2m + 2$ vertices corresponding to L_1. Label the vertices of L_2 by $A'_1, A'_2, \ldots, A'_{m+1}$ and $B'_1, B'_2, \ldots, B'_{m+1}$, where the vertex A'_i corresponds to A_i, and the vertex B'_i corresponds to B_i for every $i = 1, 2, \ldots, m+1$. Let L'_1 be the graph obtained from L_1 by deleting the edges A_2B_2 and A_4B_4. Let L'_2 be the graph obtained from L_2 by deleting two vertices A'_1 and B'_1. Then L'_1 is a subgraph of Q^0_{n-1} with $2m + 2$ vertices and L'_2 is a ladder subgraph of Q^1_{n-1} with $2m$ vertices.

Let $H = L'_1 \cup L'_2 \cup D_2$, where $D_2 = \{A_2A'_2, B_2B'_2, A_{m+1}A'_{m+1}, B_{m+1}B'_{m+1}\} \subset D$ (see Figure 2). Then H is a 3-regular subgraph of Q_n with $4m + 2 = l$ vertices. We claim that H is bipancyclic and 3-connected. □

Claim 1. H is bipancyclic.

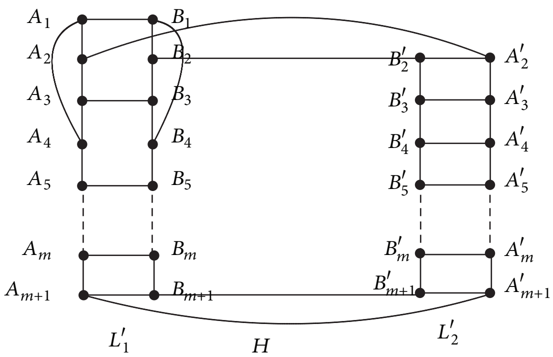

Figure 2

Clearly, $C = A_1, A_2, \ldots, A_{m+1}, A'_{m+1}, A'_m, \ldots, A'_2, B'_2, B'_3, \ldots, B'_{m+1}, B_{m+1}, B_m, \ldots, B_1, A_1$ is a Hamiltonian cycle in H. By deleting two vertices A'_2 and B'_2 and then adding the edge $A'_3B'_3$ to C, we get a cycle of length $4m$ in H. Similarly, we obtain a cycle of length $4m-2$ in H from C by deleting four vertices A'_2, A'_3, B'_2, B'_3 and then adding the edge $A'_4B'_4$. Now, by deleting six vertices $A_1, A_2, B_1, B_2, A'_2, B'_2$ from C adding the edges A_3B_3 and $A'_3B'_3$ gives a cycle of length $4m-4$ in H. Suppose $m = 3$. Then H has $4m + 2 = 14$ vertices. We get a cycle of length 4 and a cycle of length 6 in the ladder L'_2 as, by Lemma 1, it is a bipancyclic graph on six vertices. Thus H contains a cycle of every even length from 4 to 14. Suppose $m \geq 4$. Then L'_1 has at least 10 vertices. Let L be the ladder in H formed by two paths $A_5, A_6, \ldots, A_{m+1}, A'_{m+1}, A'_m, \ldots, A'_2$ and $B_5, B_6, \ldots, B_{m+1}, B'_{m+1}, B'_m, \ldots, B'_2$ and the matching A_iB_i and $A'_jB'_j$ for $i = 5, 6, \ldots, m+1$ and $j = 2, 3, \ldots, m+1$. By Lemma 1, L is bipancyclic. Hence L contains a cycle of every even length from 4 to $|V(L)| = 4m - 6$. Thus H contains a cycle of every even length from 4 to $|V(H)| = 4m + 2 = l$. Therefore H is bipancyclic.

Claim 2. H is 3-connected.

Since H contains a Hamiltonian cycle, it is 2-connected. It suffices to prove that deletion of any two vertices from H leaves a connected graph. Let $S \subset V(H)$ with $|S| = 2$. We prove that $H - S$ is connected. Let $S = \{X, Y\}$. Suppose S intersects both $V(L'_1)$ and $V(L'_2)$. We may assume that $X \in V(L'_1)$ and $Y \in V(L'_2)$. Being Hamiltonian graphs, both L'_1 and L'_2 are 2-connected. Hence $L'_1 - X$ and $L'_2 - Y$ are connected. There are at least two edges from the set D_2 which connects $L'_1 - X$ to $L'_2 - Y$ in $H - S$. Therefore $H - S$ is connected.

Suppose $S \subset V(L'_2)$. Then $S \cap V(L'_1) = \phi$ and $\{A'_2, B'_2, A'_{m+1}, B'_{m+1}\} \setminus S \neq \phi$. Obviously, L'_1 is connected. Suppose $L'_2 - S$ is connected. Then it is joined to L'_1 through at least two edges from the set D_2. This implies that $H - S$ is connected. Suppose $L'_2 - S$ is not connected. Then one vertex of S belongs the path $A'_2, A'_3, \ldots, A'_{m+1}$ and the other vertex belongs to the path $B'_2, B'_3, \ldots, B'_{m+1}$. Let $C = A'_2, A'_3, \ldots, A'_{m+1}, B'_{m+1}, B'_m, \ldots, B'_2, A'_2$ be a Hamiltonian cycle of L'_2. Then $C - S$ has exactly two components, say, T_1 and T_2 with vertex set $V(T_1)$ and $V(T_2)$. Note that T_1 or T_2 may have a single vertex. Therefore $L'_2 - S$ has two components one with vertex set $V(T_1)$ and the other with vertex set $V(T_2)$. It is easy to see that T_i contains a vertex from the set $\{A'_2, B'_2, A'_{m+1}, B'_{m+1}\} \setminus S$ and hence has a neighbour in L'_1 along an edge of the set D_2 for $i = 1, 2$. Consequently, each component of $L'_2 - S$ has a neighbour in L'_1 in the graph $H - S$. This implies that $H - S$ is connected.

Suppose $S \subset V(L'_1)$. Then L'_2 is connected. Let $\mathcal{F} = \{A_2, B_2, A_{m+1}, B_{m+1}\} \setminus S$. Then $\mathcal{F} \neq \phi$ and $\mathcal{F} \subset V(L'_1 - S)$. If each component of $L'_1 - S$ contains a vertex of the set $\mathcal{F}$, then all the components of $L'_1 - S$ are connected to L'_2 by the edges of the set D_2 giving $H - S$ connected. Therefore it suffices to prove that each component of $L'_1 - S$ contains a vertex of the set $\mathcal{F}$. If $L'_1 - S$ is connected, then we are done. Suppose $L'_1 - S$ is not connected. Consider the case when $m = 3$. Then L'_1 is the union of the two 4-cycles A_1, A_2, A_3, A_4, A_1 and B_1, B_2, B_3, B_4, B_1, and the two edges A_1B_1, A_3B_3. Each of the vertices A_2, B_2, A_4, B_4 has degree two in L'_1. If $S \cap \{A_2, B_2, A_4, B_4\} \neq \phi$, then $L'_1 - S$ is connected. Therefore $S \subset \{A_1, A_3, B_1, B_3\}$. Thus $S = \{A_1, A_3\}$, $\{A_1, B_1\}$, $\{A_1, B_3\}$, $\{A_3, B_1\}$, $\{A_3, B_3\}$ or $\{B_1, B_3\}$. In any case, each component of $L'_1 - S$ contains a vertex of the set $\mathcal{F}$. Suppose $m \geq 4$. Then $A_1, A_2, \ldots, A_{m+1}, B_{m+1}, B_m, \ldots, B_1, A_1$ is a Hamiltonian cycle in L'_1. Therefore $L'_1 - S$ has only two components. It follows that one component of $L'_1 - S$ contains a vertex from $\{A_2, B_2\} \setminus S$ and the other component contains a vertex from the set $\{A_{m+1}, B_{m+1}\} \setminus S$. Hence the vertex set of each component of $L'_1 - S$ intersects $\mathcal{F}$. Consequently, $H - S$ is connected. Therefore H is 3-connected.

Thus, from Claims 1 and 2, H is a 3-regular, 3-connected, bipancyclic subgraph of Q_n with l vertices.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank anonymous referees for their valuable suggestions. The first author is supported by the Department of Science and Technology, Government of India via Project no. SR/S4/MS: 750/12.

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