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Research Article

A Generalization of the Fuglede-Putnam Theorem to Unbounded Operators

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Let N, M be unbounded normal operators in a Hilbert space and let T be a closed operator whose domain $\mathscr{D}(T)$ contains the domain of N, and the domain $\mathscr{D}(T^*)$ contains the domain of M. It is shown that if $TN \subseteq MT$, then $TN^* \subseteq M^*T$.

1. Introduction

In this note we prove a generalization of the classical Fuglede-Putnam theorem to unbounded operators. A special case of this generalization is given in [1]. We begin with some preliminary results.

Let $\mathscr{H}$ be a complex Hilbert space and let $B(\mathscr{H})$ be the algebra of bounded linear operators in $\mathscr{H}$. Let $Op(\mathscr{H})$ denote the set of unbounded densely defined linear operators in $\mathscr{H}$. For $A \in Op(\mathscr{H})$ we denote the domain of A by $\mathscr{D}(A)$. Given $A, B \in Op(\mathscr{H})$, the operator B is called an *extension* of A, denoted by $A \subseteq B$, if $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $Ax = Bx$ for all $x \in \mathscr{D}(A)$. An operator $A \in Op(\mathscr{H})$ is called *closed* if $A = \overline{A}$ (the closure of A). A closed densely defined operator $A \in Op(\mathscr{H})$ is said to *commute* with the bounded operator $T \in B(\mathscr{H})$, if $TA \subseteq AT$. This means that for each $x \in \mathscr{D}(A)$, we have $Tx \in \mathscr{D}(A)$ and $TAx = ATx$. Let $\{A\}' = \{T \in B(\mathscr{H}) : TA \subseteq AT\}$. If $A \in B(\mathscr{H})$ this notion agrees with the usual notion of *commutant*. One sees $\{A\}'$ is a strogly closed subalgebra of $B(\mathscr{H})$, and $T \in \{A\}'$ if and only if $T^* \in \{A^*\}'$. Hence, $\{A\}' \cap \{A^*\}'$ is a von Neumann algebra.

Definition 1. Let $A \in Op(\mathscr{H})$ be closed and $\mathscr{A}$ a von Neumann algebra. If $\mathscr{A}' \subseteq \{A\}'$, the operator A is said to be affiliated with $\mathscr{A}$, denoted by $A\eta\mathscr{A}$.

The algebra $W^*(A) = \{\{A\}' \cap \{A^*\}'\}'$ is the smallest von Neumann algebra with which A is affiliated, and is referred to it as the *von Neumann algebra generated* by A.

Definition 2. Let $A \in Op(\mathscr{H})$. A bounding sequence for A is a non-decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of projections on $\mathscr{H}$ such that $\bigvee_{n=1}^{\infty} F_n = I$, $F_nA \subseteq AF_n$ and $AF_n \in B(\mathscr{H})$ for all $n \in \mathbb{N}$.

Lemma 3 (see [1]). *If $\mathscr{A}$ is an abelian von Neumann algebra and $A\eta\mathscr{A}$, then there is a bounding sequence $\{F_n\}$ for A such that $F_n \in \mathscr{A}$ and $AF_n \in \mathscr{A}$ for all $n \in \mathbb{N}$.*

A closed operator $N \in Op(\mathscr{H})$ is *normal* if $N^*N = NN^*$. This implies that $\mathscr{D}(N) = \mathscr{D}(N^*)$ and $\|Nx\| = \|N^*x\|$ for every $x \in \mathscr{D}(N)$ [2, page 51]. It turns out that the von Neumann algebra $W^*(N)$ is abelian, and $W^*(N) = \{N\}''$ [3]. Hence, from Lemma 3, there is a bounding sequence $\{F_n\}$ for N in $W^*(N)$. In fact, $F_n = E_n - E_{-n}$ for each $n \in \mathbb{N}$, where $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the selfadjoint operator N^*N [1].

2. Results

The Fuglede-Putnam theorem [4] in its classical form states the following.

Theorem 4 (Fuglede-Putnam). *Let N and M be normal operators in a Hilbert space. If T is any bounded operator satisfying $TN \subseteq MT$, then $TN^* \subseteq M^*T$.*

The following result from [2, page 97] is essential to our proof of the generalization of the Fuglede-Putnam theorem.

Lemma 5. *Let $A_1, A_2 \in Op(\mathscr{H})$ be self-adjoint operators and let $T \in B(\mathscr{H})$. Then $TA_1 \subseteq A_2T$ if and only if $TE_\lambda = P_\lambda T$ for all $\lambda \in \mathbb{R}$, where $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ are the spectral families of A_1 and A_2, respectively.*

Theorem 6. *Let $N, M \in Op(\mathscr{H})$ be normal operators and let $T \in Op(\mathscr{H})$ be a closed operator such that $\mathscr{D}(N) \subseteq \mathscr{D}(T)$ and $\mathscr{D}(M) \subseteq \mathscr{D}(T^*)$. If $TN \subseteq MT$, then $TN^* \subseteq M^*T$.*

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral families of the self-adjoint operators N^*N and M^*M, respectively. For $m, n \in \mathbb{N}$, consider the bounding sequences $F_n = E_n - E_{-n}$ and $G_m = P_m - P_{-m}$ for N and M, respectively. Since $\mathscr{D}(N) \subseteq \mathscr{D}(T)$, it follows $\mathscr{H} = \mathscr{D}(NF_n) \subseteq \mathscr{D}(TF_n)$. Since TF_n is closed, the closed graph theorem implies $TF_n \in B(\mathscr{H})$. Similarly, by the hypothesis on the domain of M and the closed graph theorem, we see $T^*G_m \in B(\mathscr{H})$.

From the hypothesis $TN \subseteq MT$, we have $TNF_n \subseteq MTF_n$. Moreover, since $F_nN \subseteq NF_n$, we also have $TF_nN \subseteq TNF_n$. Hence,

$$(TF_n)N \subseteq M(TF_n), \quad \forall n \in \mathbb{N}. \tag{1}$$

Since TF_n is bounded, the Fuglede-Putnam theorem implies

$$(TF_n)N^* \subseteq M^*(TF_n), \quad \forall n \in \mathbb{N}. \tag{2}$$

From (1), (2), we have $(TF_n)N^*N \subseteq M^*(TF_n)N \subseteq M^*M(TF_n)$. That is,

$$(TF_n)N^*N \subseteq M^*M(TF_n), \quad \forall n \in \mathbb{N}. \tag{3}$$

Consequently, from Lemma 5,

$$(TF_n)E_\lambda = P_\lambda(TF_n), \quad \forall \lambda \in \mathbb{R}. \tag{4}$$

Therefore

$$(TF_n)F_k = G_m(TF_n), \quad \forall k, n, m \in \mathbb{N}. \tag{5}$$

Taking adjoints in (5) we have

$$[G_m(TF_n)]^* = [(TF_n)F_k]^* = F_k(TF_n)^* \supseteq F_kF_nT^*. \tag{6}$$

But

$$[G_m(TF_n)]^* = (TF_n)^*G_m \supseteq F_nT^*G_m. \tag{7}$$

As $F_nT^*G_m \in B(\mathscr{H})$, we get

$$F_kF_nT^* \subseteq F_nT^*G_m. \tag{8}$$

Furthermore, since F_n and F_k commute,

$$F_nF_kT^* \subseteq F_nT^*G_m; \tag{9}$$

that is, for every $x \in \mathscr{D}(T^*)$, we have $G_mx \in \mathscr{D}(T^*)$ and

$$F_nF_kT^*x = F_nT^*G_mx. \tag{10}$$

Let $x \in \mathscr{D}(T^*)$ and fix $k, m < n$. Then since $F_n \to I$ (strongly) as $n \to \infty$, it follows

$$F_kT^* \subseteq T^*G_m, \quad \forall k, m \in \mathbb{N}. \tag{11}$$

Taking adjoints in (11) and using the closeness of T,

$$(F_kT^*)^* \supseteq (T^*G_m)^* \supseteq G_mT^{**} = G_m\overline{T} = G_mT. \tag{12}$$

But $(F_kT^*)^* = T^{**}F_k = \overline{T}F_k = TF_n$. Hence,

$$G_mT \subseteq TF_k, \quad \forall k, m \in \mathbb{N}. \tag{13}$$

Multiplying (2) by F_n, we get $(TF_n)N^*F_n \subseteq M^*(TF_n)F_n = M^*TF_n$. Since $(TF_n)(N^*F_n) = TN^*F_n$ and $(TF_n)(N^*F_n) \in B(\mathscr{H})$, we obtain

$$TN^*F_n = M^*TF_n \quad \forall n \in \mathbb{N}. \tag{14}$$

Now let $x \in \mathscr{D}(TN^*)$; that is, $x \in \mathscr{D}(N^*)$ and $N^*x \in \mathscr{D}(T)$. Fix $m > k$, and let $m \to \infty$. Then using (13) and the fact $G_m \to I$ (strongly), we have

$$TF_kx = G_mTx \longrightarrow Tx. \tag{15}$$

Moreover, from (14), the fact $F_nN^* \subseteq N^*F_n$, and (13), we have

$$M^*TF_kx = TN^*F_kx = TF_kN^*x = G_mTN^*x \longrightarrow TN^*x. \tag{16}$$

Since M^* is closed, it follows $x \in \mathscr{D}(M^*T)$ and $M^*Tx = TN^*x$. Therefore, $TN^* \subseteq M^*T$. □

As a special case for $M = N$, we obtain the following generalization of Fuglede's theorem [5].

Corollary 7. *Let $N \in Op(\mathscr{H})$ be normal and let $T \in Op(\mathscr{H})$ be a closed operator such that $\mathscr{D}(N) \subseteq \mathscr{D}(T) \cap \mathscr{D}(T^*)$. If $TN \subseteq NT$, then $TN^* \subseteq N^*T$.*

Corollary 8. *Let $N_1, N_2 \in Op(\mathscr{H})$ be normal operators. If $\mathscr{D}(N_1) \subseteq \mathscr{D}(N_2)$, then $N_2N_1 \subseteq N_1N_2 \Leftrightarrow N_2N_1^* \subseteq N_1^*N_2$.*

Corollary 9. *Let $N, N_1, N_2 \in Op(\mathscr{H})$ be normal operators. If $\mathscr{D}(N_i) \subseteq \mathscr{D}(N)$, for $i = 1, 2$, then $NN_1 \subseteq N_2N \Leftrightarrow NN_1^* \subseteq N_2^*N$.*

Remark 10. Recently in the article "An All-Unbounded-Operator Version of the Fuglede-Putnam Theorem," Complex Analysis and Operator Theory (2012) [6: 1269–1273], a similar result was offered, but its proof is incorrect. In fact, on the last page of this paper [page 1273] the proof is wrong; note that from the equality $P_{B_m}(M)AN^*P_{B_n}(N)x = P_{B_m}(M)M^*AP_{B_n}(N)x$, the fact $P_{B_m}(M) \to I$ (strongly) gives $AN^*P_{B_n}(N)x = M^*AP_{B_n}(N)x$; however, (dealing with unbounded operators, as is the case here) the fact (alone) that $P_{B_n}(N) \to I$ (strongly) does not give the equality $AN^*x = M^*Ax$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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