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Research Article

On Ordinary, Linear q-Difference Equations, with Applications to q-Sato Theory

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The purpose of this paper is to develop the theory of ordinary, linear q-difference equations, in particular the homogeneous case; we show that there are many similarities to differential equations. In the second part we study the applications to a q-analogue of Sato theory. The q-Schur polynomials act as basis function, similar to q-Appell polynomials. The Ward q-addition plays a crucial role as operation for the function argument in the matrix q-exponential and for the q-Schur polynomials.

1. Introduction

We begin this paper with an introduction to q-difference equations. Since there is a well-known parallel approach to this theme, we quote some of the historical facts about this. Then we show an example of solutions to a q-difference equation with constant coefficients; the multiple root case can be solved in a similar way. When we know a solution to a homogeneous equation of order n, the equation can be transformed into another equation of order $n-1$; this is called reduction of order. The q-analogue of Euler's differential equation is of particular importance in q-calculus because of its operational form. In a previous article [1] we introduced the concept q-analogues of matrix formulas. In this paper we continue on this theme; the main content of this paper is q-Sato theory, which is only one way to treat the theory of q-deformed solitons. Previously, articles on the q-KdV equation and q-Schur polynomials, for example, [2], were published; in this paper we define a quite different q-Schur polynomial, which is connected to the Ward q-addition.

We now start with the definitions; many of these can be found in the book [3].

Definition 1. Assume that $0 < q < 1$, $a \in \mathbb{R}$. The power function is defined by $q^a \equiv e^{a\log(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \operatorname{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The variables $a, b, c, \ldots \in \mathbb{R}$ denote certain parameters. The variables i, j, k, l, m, n, p, and r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit.

The q-analogues of a real number a and the factorial function are defined by

$$\{a\}_q \equiv \frac{1-q^a}{1-q},$$
$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q, \quad \{0\}_q! \equiv 1. \tag{1}$$

The q-analogues of the derivate and the integral are given by

$$\left(D_q\varphi\right)(x) \equiv \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, \tag{2}$$

$$\int_0^a f(t,q)\, d_q(t) \equiv a(1-q)\sum_{n=0}^{\infty} f(aq^n,q)\, q^n,$$
$$0 < |q| < 1, \quad a \in \mathbb{R}. \tag{3}$$

The inverse q-derivate is accordingly defined by

$$D_q^{-1} : \varphi(x,q) \longmapsto \int_0^x \varphi(t,q)\, d_q(t). \tag{4}$$

Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\{n\}_q!}{\{k\}_q!\,\{n-k\}_q!}, \quad k = 0,1,\ldots,n. \tag{5}$$

If $|q| > 1$, or $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $E_q(z)$ is defined by

$$E_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{6}$$

Definition 2. Let ϵ denote the invertible operator $\mathbb{R}[x] \mapsto \mathbb{R}[x]$ defined by

$$\epsilon f(x) \equiv f(qx). \tag{7}$$

Definition 3. Let a and b be any elements with commutative multiplication. Then the NWA q-addition is given by

$$\left(a \oplus_q b\right)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \quad n = 0,1,2,\ldots. \tag{8}$$

There is a Ward number $\overline{n}_q$

$$\overline{n}_q \equiv 1 \oplus_q 1 \oplus_q \cdots \oplus_q 1, \tag{9}$$

where the number of 1 on the RHS is n. For instance,

$$\overline{3}_q \equiv 1 \oplus_q 1 \oplus_q \oplus_q 1. \tag{10}$$

The following theorem reminding of [4, page 258] shows how Ward numbers usually appear in applications.

Theorem 4. *Assume that $n, k \in \mathbb{N}$. Then*

$$\left(\overline{n}_q\right)^k = \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1,\ldots,m_n}_q, \tag{11}$$

where each partition of k is multiplied with its number of permutations.

A table of some $\overline{n}_q^k$ is given in [3, page 109].

Definition 5. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1,\ldots,m_n$ running over all nonnegative integer values.

Given an integer k, the formula

$$m_0 + m_1 + \cdots + m_j = k \tag{12}$$

determines a set $J_{m_0,\ldots,m_j} \in \mathbb{N}^{j+1}$.

Then if $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, its k'th NWA-power is given by

$$\begin{aligned}\left(\oplus_{q,l=0}^{\infty} a_l x^l\right)^k &\equiv \left(a_0 \oplus_q a_1 x \oplus_q \cdots\right)^k \\ &\equiv \sum_{|\vec{m}|=k} \prod_{m_l \in J_{m_0,\ldots,m_j}} \left(a_l x^l\right)^{m_l} \binom{k}{\vec{m}}_q.\end{aligned} \tag{13}$$

Difference equations are mathematical models describing real life situations in many applied sciences. For an excellent introduction to this subject see Nørlund 1924 [5].

Theorem 6. *Let y be a function of the continuous variable x. The homogeneous linear q-difference equation of order n is of the form*

$$\sum_{k=0}^{n} p_{n-k}(x)\, D_q^k y = 0, \tag{14}$$

or even

$$\sum_{k=0}^{n} p_{n-k}(x)\, x^k D_q^k y = 0. \tag{15}$$

Instead of studying (15), we can study an equation

$$\sum_{j=0}^{n} a_j(x)\, y\left(q^{n-j} x\right) = 0, \tag{16}$$

where the $a_j(x)$ are known functions of x, j.

Proof. Begin with formula (16), and use the formula [3, page 211 (6.101)]

$$f(q^n x) = \sum_{k=0}^{n} (q-1)^k x^k q^{\binom{k}{2}} \binom{n}{k}_q D_q^k(f)(x). \tag{17}$$

This gives

$$\begin{aligned}&\sum_{j=0}^{n} a_j(x) \sum_{k=0}^{n-j} (q-1)^k x^k q^{\binom{k}{2}} \binom{n-j}{k}_q D_q^k y \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_j(x)(q-1)^k x^k q^{\binom{k}{2}} \binom{n-j}{k}_q D_q^k y.\end{aligned} \tag{18}$$

The last expression is equivalent to the LHS of (15). □

The first steps from (16) to an investigation of the linear q-difference equation (15) were taken in two dissertations by Smith 1911 [6] and Nørlund's student Ryde 1921 [7], who generalized the method of Frobenius for solving linear differential equations.

Equation (16) was first studied by Carmichael 1912 [8]. He distinguished between the two cases $|q| \neq 1$ and $|q| = 1$ and, according to Trjitzinsky [9], treated the case $|q| = 1$ satisfactorily.

In 1915 Mason [10] proved two theorems about q-difference equations with entire function coefficients. He also introduced the notion of characteristic equation for a q-difference equation.

Equation (16) has also been studied by Adams [11], who generalized the results of Carmichael and Mason. He assumed the coefficient functions $a_j(x)$ to be analytic or to have poles of finite order at the origin. Adams also studied partial q-difference equations.

In 1933 Trjitzinsky [9] solved an inhomogeneous first order linear q-difference equation and studied the solutions of linear q-difference equations.

There is an alternative approach, called timescales; this is just another dialect of q-calculus, with completely different and more general definitions. This generality leads to many general theorems, but the q-analogues are far from easy to find. For instance, timescales have another q-Laplace transform than the one the author is going to use later.

2. The Ordinary, Linear Case

Some of the results in this section have previously occurred in a paper on internet by Bangerezako [12]. We refer to him in each case and to the page number.

A q-difference equation of order n, containing powers of operator (2), is said to be linear if it is linear in the dependent variable y and the q-difference $D_q y, D_q^2 y, \ldots, D_q^n y$. The most general linear nonhomogeneous q-difference equation of order n is of the form

$$\sum_{k=0}^{n} p_{n-k}(x) D_q^k y = Ly = f(x), \tag{19}$$

where L is a linear sum of q-differential operators.

We assume that since the equation is of order n, its general solution will depend on n distinct arbitrary constants and proceed to consider the mode of this dependence.

Suppose that two distinct particular solutions of (19) are known; say $y = y_1$ and $y = y_2$. Then

$$Ly_1 = f(x), \qquad Ly_2 = f(x); \tag{20}$$

that is,

$$Ly_1 - Ly_2 = 0. \tag{21}$$

Thus if u represents the difference between any two solutions of (19), u will satisfy the homogeneous equation

$$Lu = 0, \tag{22}$$

which contains no term free from u or a q-difference operator of u. The general solution of (19) will be the sum of two components:

(1) the general solution of the homogeneous equation involving n arbitrary constants and known as the complementary function,

(2) a particular solution involving no arbitrary constants.

Theorem 7 (see [12, page 38]). *The homogeneous linear nth-order q-difference equation*

$$\sum_{k=0}^{n} a_k D_q^k y(x) = 0, \quad a_k \in \mathbb{R}, \ 0 \le k \le n, \tag{23}$$

has the general solution

$$y(x) = \sum_{k=1}^{n} C_k E_q(r_k x), \tag{24}$$

where $\{r_i\}_{i=1}^n$ are solutions of the characteristic equation

$$\sum_{k=0}^{n} a_k x^k = 0. \tag{25}$$

We have assumed that (25) has no multiple roots.

Proof. Similar to the ordinary case, use the chain rule for D_q. □

Example 8. Compare with [12, page 17]. Consider the equation

$$D_q^2 y + 2D_q y + 5y = 0. \tag{26}$$

The corresponding differential equation has solutions $y = Ae^{-x}\sin 2x$ and $y = Be^{-x}\cos 2x$, and we find that (26) has the solutions

$$y = aE_q((-1+2i)x) + bE_q((-1-2i)x). \tag{27}$$

These solutions can be rephrased in the form

$$y = A\sum_{k=0}^{\infty} \frac{x^k}{\{k\}_q!} \sum_{m=0}^{[(k-1)/2]} \binom{k}{2m+1} (-1)^{k-m} 2^{2m+1} + B\sum_{k=0}^{\infty} \frac{x^k}{\{k\}_q!} \sum_{m=0}^{[k/2]} \binom{k}{2m} (-1)^{k-m} 2^{2m}. \tag{28}$$

It is obvious that we can continue this process to find q-analogues of any homogeneous, linear differential equation with constant coefficients, which has an exact solution in terms of sums of exponential functions.

2.1. The Multiple Root Case. For [12, page 38] we illustrate the general technique with an example.

Example 9. Try to find a q-difference equation satisfied by the homogeneous solution

$$y_h = (C_1 x + C_2) E_q(-x). \tag{29}$$

The differential equation (with multiple root characteristic equation)

$$y'' + 2y' + y = 0 \tag{30}$$

has solution $y_h = (C_1 x + C_2)e^{-x}$, and (29) is a q-analogue of this. Let $r \in \mathbb{R}$, and let $P(x) \in \mathbb{R}[x]$. Consider the space of functions

$$\Psi_{q,x}(r) \equiv E_q(rx) P(x), \tag{31}$$

and let ϵ_P denotes the invertible operator $\mathbb{R}[x] \mapsto \mathbb{R}[x]$ defined by

$$\epsilon_P P(x) = P(qx). \tag{32}$$

We find that $D_q f(x) = (-C_1 xq - C_2 + C_1)E_q(-x)$ and $D_q^2 f(x) = (C_1 xq^2 + C_2 - C_1 - C_1 q)E_q(-x)$.

We try with the equation

$$\epsilon_P^{-1} D_q^2 f + \{2\}_q D_q f + \epsilon_P q f = 0, \tag{33}$$

which indeed solves the problem.

In [12, page 31] for nonhomogeneous q-difference equations, the solution will be $y = y_h + y_p$, where y_p denotes a particular solution.

Example 10. Find a particular solution to

$$D_q^2 y + 3D_q y + 2y = 3x. \quad (34)$$

We try with $y_p(x) = Ax+B$. This gives $A = 3/2, B = -9/4$. The particular solution is the same as that in the ordinary case.

In general, we can find particular solutions very similar to the ordinary case $q = 1$ by replacing integers by q-integers and solving the resulting system of equations.

2.2. Reduction of Order. When any solution of a homogeneous equation of order n is known, the equation can be transformed into another (also linear and reduced) of order $n-1$. If the known solution is u_1, the transformation is

$$u = u_1 \int^x v(t)\, d_q(t), \quad (35)$$

where v is a new dependent variable. For simplicity consider an equation of the third order (the general proof is similar)

$$\sum_{k=0}^{3} p_{3-k}(x)\, D_q^k y = 0, \quad (36)$$

where the p_k are functions of x or constants. By substituting

$$\begin{aligned}
u(x) &= u_1(x) \int^x v(t)\, d_q(t), \\
D_q(u(x)) &= D_q(u_1(x)) \int^x v(t)\, d_q(t) + u_1(qx)\, v(x); \\
D_q^2 u(x) &= D_q^2(u_1(x)) \int^x v(t)\, d_q(t) \\
&\quad + (1+q)\, D_q(u_1(qx))\, v(x) \\
&\quad + u_1(q^2 x)\, D_q(v(x)); \\
D_q^3 u(x) &= D_q^3(u_1(x)) \int^x v(t)\, d_q(t) \\
&\quad + (1+q+q^2)\, D_q^2(u_1(qx))\, v(x) \\
&\quad + (1+q+q^2)\, D_q(u_1(q^2 x))\, D_q(v(x)) \\
&\quad + u_1(q^3 x)\, D_q^2 v(x)
\end{aligned} \quad (37)$$

and rearranging we have

$$\begin{aligned}
&\left(p_0 D_q^3(u_1(x)) + p_1 D_q^2(u_1(x)) + p_2 D_q(u_1(x)) \right. \\
&\quad \left. + p_3 u_1(x)\right) \int^x v(t)\, d_q(t) \\
&\quad + \left(\left(1+q+q^2\right) p_0 D_q^2(u_1(qx)) + (1+q)\, p_1 D_q(u_1(qx)) \right. \\
&\qquad \left. + p_2 u_1(qx)\right) v(x) \\
&\quad \times \left(\left(1+q+q^2\right) p_0 D_q\left(u_1\left(q^2 x\right)\right) \right. \\
&\qquad \left. + p_1 u_1\left(q^2 x\right)\right) D_q v(x) + p_0 u_1\left(q^3 x\right) D_q^2 v(x) = 0.
\end{aligned} \quad (38)$$

Since u_1 is a solution of (36) the first term disappears, leaving a homogeneous linear equation of the second order in v.

2.3. A q-Analogue of the Euler Equation. The q-difference operator

$$\theta_q \equiv \frac{1-\epsilon}{1-q} \quad (39)$$

is a q-analogue of $x(d/dx)$. The operator θ_q maps the polynomial x^n to $\{n\}_q x^n$; θ_q keeps the degree of a polynomial and is very important in q-calculus.

This implies that the equation

$$\sum_{k=0}^{n} \theta_q^k y(x) = b(x) \quad (40)$$

is a q-analogue of the Euler equation. Our investigations show that the regularity theorems of Adams [11], Carmichael [8], and Mason [10] are also valid for the regularity of solutions to the generalized Euler equation (15).

The following two formulas from [3, page 179] are of particular interest in this context:

$$\begin{aligned}
\theta_q^n &= \sum_{k=0}^{n} S(n,k)_q\, q^{\binom{k}{2}} x^k D_q^k, \\
q^{\binom{n}{2}} x^n D_q^n &= \sum_{k=1}^{n} s(n,k)_q\, \theta_q^k,
\end{aligned} \quad (41)$$

where $S(n,k)_q$ and $s(n,k)_q$ are q-Stirling numbers, inverse to each other.

3. First Matrix Calculations

We now come to the main content of this paper, which is a continuation of [1]. We start with a short repetition. The definition of letters in an alphabet and the corresponding linear functional is found in [1]. In our case, the alphabet is the reals.

Definition 11. Matrix elements will always be denoted (i,j). Here i denotes the row and j denotes the column. The matrix elements range from 0 to $n-1$. This holds both for real numbers (linear functional) and for the letters in the matrix. Juxtaposition of matrices (like in (53)) will always be interpreted as matrix multiplication. If A and B are commuting matrices of the same dimension (belonging to the alphabet), one defines $A \oplus_q B$ as a matrix with matrix elements (i.e., letters) $A(i,j)\oplus_q B(i,j)$. If A and B are commuting matrices of the same dimension, one defines $A\boxplus_q B$ as a matrix with matrix elements $A(i,j)\boxplus_q B(i,j)$.

Definition 12. Let A be an $n\times n$ matrix, $0<|q|<1$, and $\|A\|<|1-q|^{-1}$. Then

$$E_q(A)\equiv\sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}A^k,\qquad E_{1/q}(A)\equiv\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{\{k\}_q!}A^k. \tag{42}$$

3.1. q-Sato Theory. In Sato theory, infinite-dimensional matrices and pseudodifferential operators are used to solve differential equations, with applications to soliton theory and the KdV equation. The following polynomial is used in the computations.

Definition 13. Given an integer n, the formula

$$k_1+2k_2+3k_3+\cdots+mk_m=n \tag{43}$$

determines a set $M_{k_1,\ldots,k_m}\in\mathbb{N}^m$.

Then the elementary Schur polynomial p_n is defined by the following equation:

$$p_n(x_1,x_2,\ldots)\equiv\sum_{\substack{k_1+2k_2+3k_3+\cdots=n\\ k_1,k_2,\ldots\geq 0}}\prod_{k_l\in M_{k_1,\ldots,k_m}}\frac{x_l^{k_l}}{k_l!}. \tag{44}$$

These polynomials satisfy the equation

$$\frac{\partial p_n}{\partial x_m}=p_{n-m},\quad (p_n=0 \text{ for } n<0). \tag{45}$$

We now begin with the q-deformations. The following definition is slightly different from [13, page 213], where it was assumed that $w_k\in\mathbb{R}[[x]]$ (formal power series).

Definition 14 (see [14, page 60]). Define the following pseudo-q-differential operator

$$W_{m,q}\equiv 1+\sum_{k=1}^{m}w_kD_q^{-k},\quad w_k\in\mathbb{R}, \tag{46}$$

where D_q^{-k} is defined by iterating (4).

Theorem 15. *The homogeneous, linear q-difference equation*

$$W_{m,q}D_q^m f(x)=\left(D_q^m+\sum_{k=1}^{m}w_kD_q^{m-k}\right)f(x)=0 \tag{47}$$

has m linearly independent solutions $\{f_q^{(k)}\}_{k=1}^m$, *which are all analytic; that is,*

$$f_q^{(k)}(x)=\sum_{l=0}^{\infty}\frac{\xi_{l,q}^{(k)}x^l}{\{l\}_q!},\quad k=1,2,\ldots,m. \tag{48}$$

The constants $\xi_{l,q}^{(k)}$ *are uniquely determined by the initial values of the function* f. *The solutions form an m-dimensional vector space.*

Proof. According to the fundamental theorem of algebra, the corresponding characteristic equation has m complex roots. This gives m solutions like in (27). When there are multiple roots, we multiply by a suitable polynomial, like in formula (33). □

The rank of the $\infty\times m$ Wronskian matrix

$$\Xi=\begin{pmatrix}\xi_{0,q}^{(1)} & \xi_{0,q}^{(2)} & \cdots & \xi_{0,q}^{(m)}\\ \xi_{1,q}^{(1)} & \xi_{1,q}^{(2)} & \cdots & \xi_{1,q}^{(m)}\\ \vdots & \vdots & \cdots & \vdots\end{pmatrix} \tag{49}$$

is m and we have

$$W_{m,q}D_q^m\left(1,\frac{x}{\{1\}_q!},\frac{x^2}{\{2\}_q!},\ldots\right)\Xi=0. \tag{50}$$

The shift operator Λ (not to be confused with the Polya-Vein matrix from [1]) is defined by

$$\Lambda\equiv\begin{pmatrix}0 & 1 & \cdots & 0 & 0 & 0\\ 0 & 0 & 1 & & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & \cdots & \cdots & \cdots & 1\\ 0 & \cdots & 0 & 0 & \cdots & \cdots\end{pmatrix}. \tag{51}$$

This implies

$$E_q(x\Lambda)=\begin{pmatrix}1 & x & \frac{x^2}{\{2\}_q!} & \frac{x^3}{\{3\}_q!} & \cdots\\ & 1 & x & \frac{x^2}{\{2\}_q!} & \cdots\\ & & 1 & x & \cdots\\ & 0 & & 1 & \cdots\\ \ddots & & & & \end{pmatrix}. \tag{52}$$

Introduce the following notation $H_q(x)$:

$$H_q(x)\equiv E_q(x\Lambda)\,\Xi=\begin{pmatrix}f_q^{(1)} & f_q^{(2)} & \cdots & f_q^{(m)}\\ D_qf_q^{(1)} & D_qf_q^{(2)} & \cdots & D_qf_q^{(m)}\\ D_q^2f_q^{(1)} & D_q^2f_q^{(2)} & \cdots & D_q^2f_q^{(m)}\\ \vdots & \vdots & \cdots & \vdots\end{pmatrix}. \tag{53}$$

We will now try to determine $W_{m,q}$ from the m solutions $f_q^{(k)}$. By (47)

$$\begin{aligned}
&\left(D_q^{m-1} f_q^{(1)}\right) w_1 + \left(D_q^{m-2} f_q^{(1)}\right) w_2 + \cdots + f_q^{(1)} w_m \\
&\quad = -D_q^m f_q^{(1)} \\
&\vdots \\
&\left(D_q^{m-1} f_q^{(m)}\right) w_1 + \left(D_q^{m-2} f_q^{(m)}\right) w_2 + \cdots + f_q^{(m)} w_m \\
&\quad = -D_q^m f_q^{(m)}.
\end{aligned} \tag{54}$$

Theorem 16. *A formula for the pseudo-q-differential operator $W_{m,q}$ as a quotient of determinants is*

$$W_{m,q} = \frac{\begin{vmatrix} f_q^{(1)} & \cdots & f_q^{(m)} & D_q^{-m} \\ \vdots & \cdots & \vdots & \vdots \\ D_q^{m-1} f_q^{(1)} & \cdots & D_q^{m-1} f_q^{(m)} & D_q^{-1} \\ D_q^m f_q^{(1)} & \cdots & D_q^m f_q^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} f_q^{(1)} & \cdots & f_q^{(m)} \\ \vdots & \cdots & \vdots \\ D_q^{m-1} f_q^{(1)} & \cdots & D_q^{m-1} f_q^{(m)} \end{vmatrix}}. \tag{55}$$

The entries of the matrices are functions, except for the last column of the numerator, which consists of pseudo-q-differential operators.

Proof. By Cramer's rule we have

$$\begin{aligned}
w_1 &= \frac{\begin{vmatrix} -D_q^m f_q^{(1)} & \cdots & D_q^{m-2} f_q^{(1)} & \cdots & f_q^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ -D_q^m f_q^{(m)} & \cdots & D_q^{m-2} f_q^{(m)} & \cdots & f_q^{(m)} \end{vmatrix}}{\begin{vmatrix} D_q^{m-1} f_q^{(1)} & \cdots & f_q^{(1)} \\ \vdots & \cdots & \vdots \\ D_q^{m-1} f_q^{(m)} & \cdots & f_q^{(m)} \end{vmatrix}}, \\
w_j &= \frac{\begin{vmatrix} D_q^{m-1} f_q^{(1)} & \cdots & -D_q^m f_q^{(1)} & \cdots & f_q^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_q^{m-1} f_q^{(m)} & \cdots & -D_q^m f_q^{(m)} & \cdots & f_q^{(m)} \end{vmatrix}}{\begin{vmatrix} D_q^{m-1} f_q^{(1)} & \cdots & D_q^{m-j} f_q^{(1)} & \cdots & f_q^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_q^{m-1} f_q^{(m)} & \cdots & D_q^{m-j} f_q^{(m)} & \cdots & f_q^{(m)} \end{vmatrix}}.
\end{aligned} \tag{56}$$

By combining (46) and the above two equations we obtain a formula for $W_{m,q}$. An expansion of the numerator of (55) along the last column completes the proof. □

4. Time Evolution

We now assume that w_j also depend on an infinite number of time variables t_i. This implies that the solutions of (47), $f_q^{(k)}(x)$, also depend on t_i:

$$f_q^{(k)}(x;t) = f_q^{(k)}(x;t_1,t_2,\ldots), \tag{57}$$

and $H_q(x)$ given by (53) can be written as $H(x,t,q)$. We assume that $H(x,t,q)$ evolves in time as

$$H(x,t,q) = QE(x\Lambda)\,QE(\eta(t,\Lambda))\,\Xi, \tag{58}$$

where

$$\eta(t,\Lambda) \equiv \left(\oplus_{q,n=1}^{\infty} t_n \Lambda^n\right), \quad t_1 \equiv x \oplus_q t_1. \tag{59}$$

We find that the q-Schur polynomial $p_{n,q}$ is defined by the following equation:

$$p_{n,q}\left(x\oplus_q t_1, t_2, t_3, \ldots\right) \equiv \sum_{\substack{k_1+2k_2+3k_3+\cdots=n \\ k_1,k_2,\ldots\geq 0}} \frac{\left(x\oplus_q t_1\right)^{k_1}}{\{k_1\}_q!} \prod_{k_l\in M_{k_1,\ldots,k_m}} \frac{t_l^{k_l}}{\{k_l\}_q!}, \tag{60}$$

where $M_{k_1,\ldots,k_m}$ is defined by (44). Or equivalently

$$\sum_{n=0}^{\infty} p_{n,q} z^n = E_q(\eta(t,z)). \tag{61}$$

The first $p_{n,q}$ are

$$\begin{aligned}
p_{0,q} &= 1, \\
p_{1,q} &= x + t_1, \\
p_{2,q} &= \frac{\left(x\oplus_q t_1\right)^2}{\{2\}_q!} + t_2, \\
p_{3,q} &= \frac{\left(x\oplus_q t_1\right)^3}{\{3\}_q!} + (x+t_1)\,t_2 + t_3.
\end{aligned} \tag{62}$$

Remark 17. These q-Schur polynomials are completely different than those in [2, 15] and give richer q-differential properties, due to the NWA q-addition.

Theorem 18. *These polynomials satisfy the equations*

$$\begin{aligned}
D_{q,t_m} p_{n,q} &= p_{n-m,q}, \quad \left(p_{n,q} = 0 \text{ for } n < 0\right), \\
D_{q,x} p_{n,q} &= p_{n-1,q}.
\end{aligned} \tag{63}$$

Proof. Operate with D_{q,t_m} on (61), and write the right hand side as a product of q-exponentials. After performing the q-differentiation to the right, multiply both sides by z^{-m}. □

We can express $H(x,t,q)$ by means of the q-Schur polynomials as follows:

$$H(x,t,q) = \begin{pmatrix} 1 & p_{1,q} & p_{2,q} & p_{3,q} & \cdots \\ & 1 & p_{1,q} & p_{2,q} & \cdots \\ & & 1 & p_{1,q} & \cdots \\ & & \cdots & & \\ & 0 & & \cdots & \cdots \end{pmatrix}$$

$$\cdot\begin{pmatrix} \xi_{0,q}^{(1)} & \xi_{0,q}^{(2)} & \cdots & \xi_{0,q}^{(m)} \\ \xi_{1,q}^{(1)} & \xi_{1,q}^{(2)} & \cdots & \xi_{1,q}^{(m)} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

$$\equiv \begin{pmatrix} h_{0,q}^{(1)} & h_{0,q}^{(2)} & \cdots & h_{0,q}^{(m)} \\ h_{1,q}^{(1)} & h_{1,q}^{(2)} & \cdots & h_{1,q}^{(m)} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}. \tag{64}$$

We have the following theorem for the entries of $H(x, t, q)$.

Theorem 19. *Consider*

$$h_{0,q}^{(j)}(x; 0) = f_q^{(j)}(x). \tag{65}$$

$$h_{n,q}^{(j)}(x; t) = D_{q,t_n} h_{0,q}^{(j)}(x; t) = D_{q,x}^n h_{0,q}^{(j)}(x; t). \tag{66}$$

This means that the function $h_{0,q}^{(j)}(x;t)$ is the solution of the partial q-difference equation

$$\left(D_{q,t_n} - D_{q,x}^n\right) h(q, x, t) = 0, \tag{67}$$

with initial value

$$h(q, x, 0)^{(j)} = f_q^{(j)}(x). \tag{68}$$

Proof. We have $h_{m,q}^{(j)} = \sum_{k=m}^{\infty} p_{k-m,q} \xi_{k,q}^{(j)}$. Then

$$D_{q,t_m} h_{0,q}^{(j)} = \sum_{n=0}^{\infty} D_{q,t_m} p_{n,q} \xi_{n,q}^{(j)} = \sum_{n=0}^{\infty} p_{n-m,q} \xi_{n,q}^{(j)} = h_{m,q}^{(j)},$$
$$D_{q,x}^m h_{0,q}^{(j)} = \sum_{n=0}^{\infty} D_{q,x}^m p_{n,q} \xi_{n,q}^{(j)} = \sum_{n=0}^{\infty} p_{n-m,q} \xi_{n,q}^{(j)} = h_{m,q}^{(j)}. \tag{69}$$

The two expressions are equal. □

The operators $W_{m,q}$ and w_k in (46) now also depend on t and

$$W_{m,q}(q, x, t) D_{q,x}^m h_{0,q}^{(j)}(x; t)$$
$$= \left(D_{q,x}^m + \sum_{k=1}^{m} w_k D_q^{m-k}\right) h_{0,q}^{(j)}(x; t) = 0, \tag{70}$$
$$j = 1, 2, \ldots, m.$$

By formula (66) we find

$$w_j(q, x, t) = \frac{\begin{vmatrix} h(m-1,q)^{(1)} & \cdots & -h(m,q)^{(1)} & \cdots & h(0,q)^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \cdots & \vdots \\ h(m-1)^{(m)} & \cdots & -h(m,q)^{(m)} & \cdots & h(0,q)^{(m)} \end{vmatrix}}{\begin{vmatrix} h(m-1)^{(1)} & \cdots & h(m-j,q)^{(1)} & \cdots h(0,q)^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ h(m-1)^{(m)} & \cdots & h(m-j,q)^{(m)} & \cdots h(0,q)^{(m)} \end{vmatrix}},$$
$$W_{m,q}(x, t) = \frac{\begin{vmatrix} h(0,q)^{(1)} & \cdots & h(0,q)^{(m)} & D_q^{-m} \\ \vdots & \cdots & \vdots & \vdots \\ h(m-1)^{(1)} & \cdots & h(m-1,q)^{(m)} & D_q^{-1} \\ h(m),q^{(1)} & \cdots & h(m,q)^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} h(0)^{(1)} & \cdots & h(0,q)^{(m)} \\ \vdots & \cdots & \vdots \\ h(m-1,q)^{(1)} & \cdots & h(m-1,q)^{(m)} \end{vmatrix}}. \tag{71}$$

By applying the operator D_{q,t_n} to (70) and employing (66), we obtain

$$\left(D_{q,t_n} W_{m,q} D_q^m + \left(\epsilon_{t_n} W_{m,q}\right) D_q^{m+n}\right) h_{0,q}^{(j)}(x; t) = 0, \tag{72}$$

which is a q-difference equation of order $m + n$ with the same linearly independent solutions as (70). The q-difference operators in (72) can be factorized as

$$D_{q,t_n} W_{m,q} D_q^m + \left(\epsilon_{t_n} W_{m,q}\right) D_q^{m+n} = B_{n,q} W_{m,q} D_q^m, \tag{73}$$

where $B_{n,q}$ is a certain q-difference operator. After applying $D_q^{-m} W_{m,q}^{-1}$ from the right, we obtain

$$B_{n,q} = D_{q,t_n} W_{m,q} W_{m,q}^{-1} + \epsilon_{t_n} W_{m,q} D_q^n W_{m,q}^{-1}. \tag{74}$$

By a similar reasoning as in the case $q = 1$, we have

$$B_{n,q} = \left(W_{m,q} D_q^n W_{m,q}^{-1}\right)^+, \tag{75}$$

where $()^+$ denotes the q-difference part of the operator. This implies that the time evolution of $W_{m,q}(x; t)$ is governed by

$$D_{q,t_n} W_q = B_{n,q} W_q - \epsilon_{t_n} W_q D_q^n,$$
$$B_{n,q} = \left(W_q D_q^n W_q^{-1}\right)^+, \tag{76}$$

which we will call the q-Sato equation.

5. Conclusion

We have found a q-analogue of a simplified and more mathematical form of Sato theory. We hope that this paper will have many applications for q-difference equations and in soliton theory. A further paper on q-Laplace transformations is in preparation.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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